# MT845: ALGEBRAIC CURVES

DAWEI CHEN

## Contents



## 1. Sheaves and cohomology

1.1. **Sheaves.** Let $X$ be a topological space. A *sheaf* $\mathscr{F}$ on $X$ associates to each open set $U$ an abelian group $\mathscr{F}(U)$, called the *sections* of $\mathscr{F}$ over $U$, along with a *restriction map* $r_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ for any open sets $U \subset V$ (for a section $\sigma \in \mathscr{F}(V)$, we often write $\sigma|_U$ to denote $r_{V,U}(\sigma)$), satisfying the following conditions:

(1) For any open sets $U \subset V \subset W$, $r_{V,U} \circ r_{W,V} = r_{W,U}$;

(2) For a collection of open sets $\{U_i\}_{i\in I}$ and sections $\alpha_i \in \mathscr{F}(U_i)$, if $\alpha_i|_{U_i\cap U_j} = \alpha_j|_{U_i\cap U_j}$ for any $i, j \in I$, then there exists a unique $\alpha \in \mathscr{F}(\cup_i U_i)$ such that $\alpha|_{U_i} = \alpha_i$ for any $i$.

**Remark 1.1.** If $\mathscr{F}$ satisfies (1) only, we call it a *presheaf*. One can perform *sheafification* for a presheaf to make it become a sheaf. For many sheaves we consider, the restriction maps are the obvious ones by restricting functions from bigger subsets to smaller ones.

**Exercise 1.2.** Show that $\mathscr{F}(\emptyset)$ consists of exactly one element.

**Example 1.3.** Let $G$ be an abelian group. We have the *sheaf of locally constant functions* $\mathbb{G}$ on a topological space $X$, where $\mathbb{G}(U)$ is the group of locally constant maps $f : U \to G$ on a non-empty open set $U \subset X$ and $\mathbb{G}(\emptyset) = 0$.

**Exercise 1.4.** Show that for the sheaf $\mathbb{G}$ of locally constant functions, we have $\mathbb{G}(U) = G$ for any non-empty connected open set $U$.

**Exercise 1.5.** Suppose we define $\mathbb{G}(U) = G$ as the set of constant functions on a non-empty open set $U$ with the natural restriction maps. If $G$ contains at least two distinguished elements and if $X$ has two disjoint non-empty open subsets, show that $\mathbb{G}$ is a sheaf.

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**Example 1.6.** Let $X$ be a complex manifold and $U \subset X$ an open set.

(1) *Sheaf $\mathscr{O}$ of holomorphic functions*:

$$\mathscr{O}(U) = \{\text{holomorphic functions on } U\}.$$

(2) *Sheaf $\mathscr{O}^*$ of everywhere nonzero holomorphic functions*:

$$\mathscr{O}^*(U) = \{\text{holomorphic functions } f \text{ on } U : f(p) \neq 0 \text{ for any } p \in U\}.$$

The group law is given by multiplication.

(3) *Sheaf $\mathscr{M}$ of meromorphic functions*: strictly speaking, a meromorphic function is not a function, even we take $\infty$ into account. If $X$ is compact, we cannot take a meromorphic function as a quotient of two holomorphic functions, since any globally defined holomorphic function is a constant on $X$. Instead, we define $f \in \mathscr{M}(U)$ as local quotient of holomorphic functions. Namely, there exists an open covering $\{U_i\}$ of $U$ such that on each $U_i$, $f$ is given by $g_i/h_i$ for some $g_i, h_i \in \mathscr{O}(U_i)$ satisfying $g_i/h_i = g_j/h_j$, i.e. $g_i h_j = g_j h_i \in \mathscr{O}(U_i \cap U_j)$, hence these local quotients can be glued together over $U$.

(4) *Sheaf $\mathscr{M}^*$ of meromorphic functions not identically zero*: this is defined similarly and the group law is given by multiplication.

1.2. **Maps between sheaves.** Let $\mathscr{E}$ and $\mathscr{F}$ be two sheaves on a topological space $X$. A *map* $f : \mathscr{E} \to \mathscr{F}$ is a collection of group homomorphisms

$$\{f_U : \mathscr{E}(U) \to \mathscr{F}(U)\}$$

such that they commute with the restriction maps, i.e. for open sets $U \subset V$ and $\sigma \in \mathscr{E}(V)$ we have

$$f_V(\sigma)|_U = f_U(\sigma|_U).$$

Define the *sheaf of kernel* $\ker(f)$ as

$$\ker(f)(U) = \{\ker(f_U : \mathscr{E}(U) \to \mathscr{F}(U))\}.$$

**Exercise 1.7.** Prove that in the above definition $\ker(f)$ is a sheaf.

**Example 1.8.** Let $X$ be a complex manifold. Define the *exponential map*

$$\exp : \mathscr{O} \to \mathscr{O}^*$$

by $\exp(h) = e^{2\pi\sqrt{-1}h}$ for any open set $U \subset X$ and section $h \in \mathscr{O}(U)$. It is easy to see that $\ker(\exp)$ is the locally constant sheaf $\mathbb{Z}$.

The *sheaf of cokernel* is harder to define. Naively, one would like to define $\operatorname{coker}(f)(U) = \operatorname{coker}(f_U : \mathscr{E}(U) \to \mathscr{F}(U))$, but this is problematic. For instance, consider the exponential map $\exp : \mathscr{O} \to \mathscr{O}^*$ on the punctured plane $\mathbb{C}\backslash\{0\}$. The section $z \in \mathscr{O}^*(\mathbb{C}\backslash\{0\})$ is not in the image of $f$, hence it would define a section in the cokernel. Nevertheless, restricted to any contractible open set $U \subset \mathbb{C}\backslash\{0\}$, $z$ lies in the image of $f$. Now cover $\mathbb{C}\backslash\{0\}$ by contractible open sets. By the gluing property of sheaves, $z$ would be zero everywhere, leading to a contradiction.

Instead, we want a section of $\operatorname{coker}(f)(U)$ to be a collection of sections $\sigma_\alpha \in \mathscr{F}(U_\alpha)$ for an open covering $\{U_\alpha\}$ of $U$ such that for all $\alpha, \beta$ we have

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in f_{U_\alpha \cap U_\beta}(\mathscr{E}(U_\alpha \cap U_\beta)).$$

Here the definition still depends on the choice of an open covering. To get rid of this, we pass to the direct limit. More precisely, we further identify two collections

$\{(U_\alpha, \sigma_\alpha)\}$ and $\{(V_\beta, \sigma_\beta\}$ if for all $p \in U_\alpha \cap V_\beta$, there exists an open set $W$ satisfying $p \in W \subset U_\alpha \cap V_\beta$ such that

$$\sigma_\alpha|_W - \sigma_\beta|_W \in f_W(\mathscr{E}(W)).$$

This identification yields an equivalence relation and correspondingly we define $\operatorname{coker}(f)(U)$ as the group of equivalence classes of the above sections.

**Exercise 1.9.** Prove that in the above definition $\operatorname{coker}(f)$ is a sheaf.

Consider the following sequence of maps between sheaves:

$$0 \longrightarrow \mathscr{E} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{G} \longrightarrow 0.$$

We say that it is a *short exact sequence* if $\mathscr{E} = \ker(\beta)$ and $\mathscr{G} = \operatorname{coker}(\alpha)$. In this case we also say that $\mathscr{E}$ is a *subsheaf* of $\mathscr{F}$ and $\mathscr{G}$ is the *quotient sheaf* $\mathscr{F}/\mathscr{E}$.

**Example 1.10.** Let $X$ be a complex manifold. We have the exact *exponential sequence*:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathscr{O} \xrightarrow{\exp} \mathscr{O}^* \longrightarrow 0,$$

where $i$ is the natural inclusion and $\exp(f) = e^{2\pi\sqrt{-1}f}$ for $f \in \mathscr{O}(U)$.

**Exercise 1.11.** Prove that the exponential sequence is exact.

**Example 1.12.** Let $X$ be a complex manifold and $Y \subset X$ a submanifold. Define the *ideal sheaf* $\mathscr{I}_{Y/X}$ of $Y$ in $X$ (or simply $\mathscr{I}_Y$ if there is no confusion) by

$$\mathscr{I}_Y(U) = \{\text{holomorphic functions on } U \text{ vanishing on } Y \cap U\}.$$

We have the exact sequence:

$$0 \longrightarrow \mathscr{I}_Y \xrightarrow{i} \mathscr{O}_X \xrightarrow{r} i_*\mathscr{O}_Y \longrightarrow 0,$$

where $i$ is the natural inclusion and $r$ is defined by the natural restriction map. Here $i_*\mathscr{O}_Y$ is the extension of $\mathscr{O}_Y$ by zero outside $Y$, as a sheaf defined on $X$.

**Exercise 1.13.** Prove that the above sequence is exact.

1.3. **Stalks and germs.** Let $\mathscr{F}$ be a sheaf on a topological space $X$ and $p \in X$ a point. Suppose $U$ and $V$ are two open subsets, both containing $p$, with two sections $\alpha \in \mathscr{F}(U)$ and $\beta \in \mathscr{F}(V)$. Define an equivalence relation $\alpha \sim \beta$, if there exists an open subset $W$ satisfying $p \in W \subset U \cap V$ such that $\alpha|_W = \beta|_W$. Define the *stalk* $\mathscr{F}_p$ as the union of all sections in open neighborhoods of $p$ modulo this equivalence relation. Namely, $\mathscr{F}_p$ is the direct limit

$$\mathscr{F}_p := \varinjlim_{U \ni p} \mathscr{F}(U) = \Big(\bigsqcup_{U \ni p} \mathscr{F}(U)\Big)/\sim.$$

Note that $\mathscr{F}_p$ is also a group, by adding representatives of two equivalence classes. There is a group homomorphism $r_U : \mathscr{F}(U) \to \mathscr{F}_p$ mapping a section $\alpha \in \mathscr{F}(U)$ to its equivalence class. The image is called the *germ* of $\alpha$.

**Example 1.14** (Skyscraper sheaf)**.** Let $p \in X$ be a point on a topological space $X$. Define the *skyscraper sheaf* $\mathscr{F}$ at $p$ by $\mathscr{F}(U) = \{0\}$ for $p \notin U$ and $\mathscr{F}(U) = A$ for $p \in U$, where $A$ is an abelian group. The restriction maps are either the identity map $A \to A$ or the zero map. For $q \neq p$, the stalk $\mathscr{F}_q = \{0\}$. At $p$, we have $\mathscr{F}_p = A$. Note that $\mathscr{F}$ can also be obtained by extending the constant sheaf $A$ at $p$ by zero to $X$.

**Exercise 1.15.** Let $X$ be a Riemann surface and $p \in X$ a point. Let $\mathscr{I}_p$ be the ideal sheaf of $p$ in $X$ parameterizing holomorphic functions vanishing at $p$. We have the exact sequence

$$0 \longrightarrow \mathscr{I}_p \xrightarrow{\ i\ } \mathscr{O}_X \xrightarrow{\ r\ } \mathscr{O}_p \longrightarrow 0.$$

Show that the quotient sheaf $\mathscr{O}_p$ is isomorphic to the skyscraper sheaf with stalk $\mathbb{C}$ at $p$.

It is more convenient to verify injections and surjections for maps of sheaves by the language of stalks.

**Proposition 1.16.** *Let $\phi : \mathscr{E} \to \mathscr{F}$ be a map for sheaves $\mathscr{E}$ and $\mathscr{F}$ on a topological space $X$.*

*(1) $\phi$ is injective if and only if the induced map $\phi_p : \mathscr{E}_p \to \mathscr{F}_p$ is injective for the stalks at every point $p$.*

*(2) $\phi$ is surjective if and only if the induced map $\phi_p : \mathscr{E}_p \to \mathscr{F}_p$ is surjective for the stalks at every point $p$.*

*(3) $\phi$ is an isomorphism if and only if the induced map $\phi_p : \mathscr{E}_p \to \mathscr{F}_p$ is an isomorphism for the stalks at every point $p$.*

*Proof.* The claim (3) follows from (1) and (2). Let us prove (1) only, and one can easily find the proof of (2) in many books, e.g. Hartshorne.

Suppose $\phi$ is injective. Take a section $\sigma \in \mathscr{E}(U)$ on an open subset $U$. If $\phi([\sigma]) = 0 \in \mathscr{F}_p$, there exists a smaller open subset $V \subset U$ such that $\phi_V(\sigma) = 0 \in \mathscr{F}(V)$, hence $\sigma|_V = 0 \in \mathscr{E}(V)$. Consequently the equivalence class $[\sigma] = 0 \in \mathscr{E}_p$ and we conclude that $\phi_p$ is injective.

Conversely, suppose $\phi_p$ is injective for every point $p$. Take a section $\sigma \in \mathscr{E}(U)$. If $\phi(\sigma) = 0 \in \mathscr{F}(U)$, then for every point $p \in U$, $[\phi(\sigma)] = 0 \in \mathscr{F}_p$. Since $\phi_p$ is injective, it implies that $[\sigma] = 0 \in \mathscr{E}_p$ i.e. there exists an open subset $U_p \ni p$ such that $\sigma|_{U_p} = 0 \in \mathscr{E}(U_p)$. Applying the gluing property to the open covering $\{U_p\}$ of $U$, we conclude that $\sigma = 0 \in \mathscr{E}(U)$. $\square$

**Remark 1.17.** The image of $\phi$ does not automatically form a sheaf. In general, it is only a *presheaf*. If the sheafification of $\mathrm{Im}(\phi)$ equals $\mathscr{F}$, we say that $\phi$ is surjective. In particular, it does not mean $\mathscr{E}(U) \to \mathscr{F}(U)$ is surjective for every open set $U$. Sometimes one has to pass to a smaller open set in order to obtain a surjection between sections.

**Example 1.18.** Consider the exponential map $\exp : \mathscr{O} \to \mathscr{O}^*$ on the punctured plane $\mathbb{C}\backslash\{0\}$. This map is surjective, but the section $z$ over the total space does not have an inverse image. It does have an inverse over any contractible open subset.

1.4. **Cohomology of sheaves.** Let $\mathscr{F}$ be a sheaf on a topological space $X$. Take an open covering $\underline{U} = \{U_\alpha\}$ of $X$. Define the $k$-th *cochain group*

$$C^k(\underline{U}, \mathscr{F}) := \prod_{\alpha_0, \ldots, \alpha_k} \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}).$$

An element $\sigma$ of $C^k(\underline{U}, \mathscr{F})$ consists of a section $\sigma_{\alpha_0,\ldots,\alpha_k} \in \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$ for every $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$.

Define a *coboundary map* $\delta : C^k(\underline{U}, \mathscr{F}) \to C^{k+1}(\underline{U}, \mathscr{F})$ by

$$(\delta\sigma)_{\alpha_0,\ldots,\alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \sigma_{\alpha_0,\ldots,\hat{\alpha}_j,\ldots,\alpha_{k+1}}|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_{k+1}}}.$$

**Example 1.19.** Consider $\underline{U} = \{U_1, U_2, U_3\}$ as an open covering of $X$. Take a cochain element $\sigma \in C^0(\underline{U}, \mathscr{F})$, i.e. $\sigma$ is a collection of a section $\sigma_i \in \mathscr{F}(U_i)$ for every $i$. Then we have

$$(\delta\sigma)_{ij} = (\sigma_j - \sigma_i)|_{U_i \cap U_j} \in \mathscr{F}(U_i \cap U_j).$$

Now take $\tau \in C^1(\underline{U}, \mathscr{F})$, i.e. $\tau$ is a collection of a section $\tau_{ij} \in \mathscr{F}(U_i \cap U_j)$ for every pair $i, j$. Then we have

$$(\delta\tau)_{123} = (\tau_{23} - \tau_{13} + \tau_{12})|_{U_1 \cap U_2 \cap U_3} \in \mathscr{F}(U_1 \cap U_2 \cap U_3).$$

A cochain $\sigma \in C^k(\underline{U}, \mathscr{F})$ is called a *cocyle* if $\delta\sigma = 0$. We say that $\sigma$ is a *coboundary* if there exists $\tau \in C^{k-1}(\underline{U}, \mathscr{F})$ such that $\delta\tau = \sigma$.

**Lemma 1.20.** *A coboundary is a cocycle, i.e.* $\delta \circ \delta = 0$.

*Proof.* Let us prove it for the above example. The same idea applies in general with messier notation. Under the above setting, we have

$$\begin{aligned} ((\delta \circ \delta)\sigma)_{123} &= (\delta\sigma)_{23} - (\delta\sigma)_{13} + (\delta\sigma)_{12} \\ &= (\sigma_3 - \sigma_2) - (\sigma_3 - \sigma_1) + (\sigma_2 - \sigma_1) \\ &= 0 \in \mathscr{F}(U_1 \cap U_2 \cap U_3). \end{aligned}$$

Here we omit the restriction notation, since it is obvious. □

**Exercise 1.21.** Prove in full generality that $\delta \circ \delta = 0$.

For the coboundary map $\delta_k : C^k(\underline{U}, \mathscr{F}) \to C^{k+1}(\underline{U}, \mathscr{F})$, define the *k-th cohomology group* (respect to $\underline{U}$) by

$$H^k(\underline{U}, \mathscr{F}) := \frac{\ker(\delta_k)}{\operatorname{Im}(\delta_{k-1})}.$$

This is well-defined due to the above lemma.

**Example 1.22.** For $k = 0$, we have $H^0(\underline{U}, \mathscr{F}) = \ker(\delta_0)$. Take an element $\{\sigma_i \in \mathscr{F}(U_i)\}$ in this group. Because it is a cocycle, it satisfies

$$\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j} \in \mathscr{F}(U_i \cap U_j).$$

By the gluing property of sheaves, there exists a global section $\sigma \in \mathscr{F}(X)$ such that $\sigma|_{U_i} = \sigma_i$. Conversely, if $\sigma$ is a global section, then define $\sigma_i = \sigma|_{U_i} \in \mathscr{F}(U_i)$. In this way we obtain a cocycle in $C^1(\underline{U}, \mathscr{F})$. From the discussion we see that $H^0(\underline{U}, \mathscr{F}) = \mathscr{F}(X)$, which is independent of the choice of an open covering. Hence $H^0(\underline{U}, \mathscr{F})$ is called the *group of global sections* of $\mathscr{F}$ and we often denote it by $H^0(X, \mathscr{F})$ or simply $H^0(\mathscr{F})$.

In general, we would like to define cohomology independent of open coverings. Take two open coverings $\underline{U} = \{U_\alpha\}_{\alpha \in I}$ and $\underline{V} = \{V_\beta\}_{\beta \in J}$. We say that $\underline{U}$ is a refinement of $\underline{V}$ if for every $U_\alpha$ there exists a $V_\beta$ such that $U_\alpha \subset V_\beta$ and we write it as $\underline{U} < \underline{V}$. Then we also have an index map $\phi : I \to J$ sending $\alpha$ to $\beta$. It induces a map

$$\rho_\phi : C^k(\underline{V}, \mathscr{F}) \to C^k(\underline{U}, \mathscr{F})$$

given by

$$\rho_\phi(\sigma)_{\alpha_0, \ldots, \alpha_k} = \sigma_{\phi(\alpha_0), \ldots, \phi(\alpha_p)}|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}}.$$

One checks that it commutes with the coboundary map $\delta$, i.e. $\delta \circ \rho_\phi = \rho_\phi \circ \delta$. Moreover, it induces a map

$$\rho : H^k(\underline{V}, \mathscr{F}) \to H^k(\underline{U}, \mathscr{F}),$$

which is independent of the choice of $\phi$. Finally, we define the $k$-th (Čech) cohomology group by passing to the direct limit:

$$H^k(X, \mathscr{F}) := \varinjlim H^k(\underline{U}, \mathscr{F}).$$

The definition involves direct limit, which is inconvenient to use in practice. Nevertheless, we can simplify the situation once the open covering $\underline{U}$ is fine enough. We say that $\underline{U} = \{U_i\}_{i\in I}$ is *acyclic* respect to $\mathscr{F}$, if for any $k > 0$ and $i_1, \dots, i_l \in I$ we have

$$H^k(U_{i_1} \cap \cdots \cap U_{i_l}, \mathscr{F}) = 0.$$

**Theorem 1.23** (Leray's Theorem)**.** *If the open covering $\underline{U}$ is acyclic respect to $\mathscr{F}$, then $H^*(\underline{U}, \mathscr{F}) \cong H^*(X, \mathscr{F})$.*

**Remark 1.24.** In the context of complex manifolds, if $U_i$'s are contractible, then $\underline{U}$ is acyclic respect to the sheaves we will consider. While for varieties, if $U_i$'s are affine, then $\underline{U}$ is acyclic.

**Example 1.25.** Let us compute the cohomology of the structure sheaf $\mathscr{O}$ on $\mathbb{P}^1_{\mathbb{C}}$. It is clear that $H^0(\mathbb{P}^1, \mathscr{O}) = \mathbb{C}$, since any global holomorphic function on a compact complex manifold is constant. For higher cohomology, use $[X, Y]$ to denote the coordinates of $\mathbb{P}^1$. Take the standard open covering $U = \{[X, Y] : X \neq 0\}$ and $V = \{[X, Y] : Y \neq 0\}$. It is acyclic respect to the structure sheaf $\mathscr{O}$ (morally because $U \cap V = \mathbb{C}^*$ is contractible). Let $s = Y/X$ and $t = X/Y$ as affine coordinates of $U$ and $V$, respectively. Suppose $h$ is an element in $C^1(\{U, V\}, \mathscr{O})$, i.e. $h \in \mathscr{O}(U \cap V)$. We can write

$$h = \sum_{i=-\infty}^{\infty} a_i s^i.$$

Now take

$$f = -\sum_{i=0}^{\infty} a_i s^i \in \mathscr{O}(U),$$

$$g = \sum_{i=-\infty}^{-1} a_i s^i = \sum_{i=-\infty}^{-1} a_i t^{-i} \in \mathscr{O}(V).$$

Then we have $(f, g) \in C^0(\{U, V\}, \mathscr{O})$ and $\delta((f, g)) = g - f = h$. It implies that $H^1(\mathbb{P}^1, \mathscr{O}) = 0$. All the other $H^k(\mathbb{P}^1, \mathscr{O}) = 0$ for $k > 1$, since there are only two open subsets in the covering.

**Example 1.26.** Let $\Omega$ denote the *sheaf of holomorphic one-forms* on a Riemann surface, i.e. locally a section of $\Omega$ can be expressed as $f(z)dz$, where $z$ is local coordinate and $f(z)$ a holomorphic function. Let us compute the cohomology of $\Omega$ on $\mathbb{P}^1$. Take the above open covering. Suppose $\omega$ is a global holomorphic one-form. Then on the open chart $U$, it can be written as

$$\Big(\sum_{i=0}^{\infty} a_i s^i\Big) ds.$$

Using the relation $s = 1/t$ and $ds = -dt/t^2$, on $V$ it can be expressed as

$$-\Big(\sum_{i=0}^{\infty} a_i t^{-i-2}\Big)dt,$$

which is holomorphic if and only if $a_i = 0$ for all $i$. Hence $w$ is the zero one-form and $H^0(\mathbb{P}^1, \Omega) = 0$. Now take $\omega \in C^1(\{U, V\}, \Omega)$, i.e. $\omega \in \Omega(U \cap V) = \Omega(\mathbb{C}^*)$, we express it as

$$\omega = \Big(\sum_{i=-\infty}^{\infty} a_i t^i\Big)dt.$$

Note that any $\alpha \in \Omega(U)$ and $\beta \in \Omega(V)$ can be written as

$$\alpha = \Big(\sum_{i=0}^{\infty} b_i s^i\Big)ds,$$

$$\beta = \Big(\sum_{i=0}^{\infty} c_i t^i\Big)dt.$$

Hence on $U \cap V$ we have

$$\delta((\alpha, \beta)) = \beta - \alpha = -\Big(\sum_{i=0}^{\infty} b_i t^{-i-2}\Big)dt + \Big(\sum_{i=0}^{\infty} c_i t^i\Big)dt.$$

Note that only the term $t^{-1}$ is missing from the expression. We conclude that $H^1(\mathbb{P}^1, \Omega) = \{a_{-1}t^{-1}dt\} \cong \mathbb{C}$.

**Remark 1.27.** If $\mathbb{P}^1$ is defined over an algebraically closed field $K$, replacing holomorphic functions by regular functions, we have $H^0(\mathbb{P}^1, \mathscr{O}) \cong K$ and $H^1(\mathbb{P}^1, \mathscr{O}) = 0$. Indeed, we have seen that the coordinate ring $A(U \cap V)$ is given by $K[s, 1/s]$, hence the above argument goes word by word. Similarly replacing holomorphic one-forms by *regular differentials*, i.e. in the expression $f(z)dz$, $f(z)$ is regular and $dz$ satisfies the usual differentiation rules, we have $H^0(\mathbb{P}^1, \Omega) = 0$ and $H^1(\mathbb{P}^1, \Omega) \cong K$. In general, the rank of $H^1(X, \mathscr{O}) \cong H^0(X, \Omega)$ (by Serre Duality) is called the *genus* of a Riemann surface (or an algebraic curve) $X$.

**Exercise 1.28.** Let $D = p_1 + \cdots + p_n$ be a collection of $n$ points in $\mathbb{P}^1$. We say that $D$ is an *effective divisor of degree* $n$. Define the sheaf $\mathscr{O}(D)$ on $\mathbb{P}^1$ by

$$\mathscr{O}(D)(U) = \{f \in \mathscr{M}(U) : f \in \mathscr{O}(U \backslash \{p_1, \ldots, p_n\}) \text{ with at worst a simple pole at each } p_i\}.$$

Assume that the standard covering of $\mathbb{P}^1$ is acyclic respect to $\mathscr{O}(D)$. Use it to calculate the cohomology groups $H^*(\mathbb{P}^1, \mathscr{O}(D))$.

As many other homology/cohomology theories, one can associate a *long exact sequence of cohomology* to a short exact sequence. Suppose we have a short exact sequence of sheaves

$$0 \longrightarrow \mathscr{E} \xrightarrow{\ \alpha\ } \mathscr{F} \xrightarrow{\ \beta\ } \mathscr{G} \longrightarrow 0.$$

Then $\alpha$ and $\beta$ induce maps

$$\alpha : C^k(\underline{U}, \mathscr{E}) \to C^k(\underline{U}, \mathscr{F}), \quad \beta : C^k(\underline{U}, \mathscr{F}) \to C^k(\underline{U}, \mathscr{G}).$$

Since the coboundary map $\delta$ is given by alternating sums of restrictions, $\alpha$ and $\beta$ commute with $\delta$, hence they send a cocycle to cocycle and a coboundary to coboundary. Consequently they induce maps for colomology

$$\alpha_*: H^k(X,\mathscr{E}) \to H^k(X,\mathscr{F}), \quad \beta_*: H^k(X,\mathscr{F}) \to H^k(X,\mathscr{G}).$$

Next we define the coboundary map

$$\delta_*: H^k(X,\mathscr{G}) \to H^{k+1}(X,\mathscr{E}).$$

For $\sigma \in C^k(\underline{U},\mathscr{G})$ satisfying $\delta\sigma = 0$, after refining $\underline{U}$ (still denoted by $\underline{U}$) such that there exists $\tau \in C^k(\underline{U},\mathscr{F})$ satisfying $\beta(\tau) = \sigma$, because $\beta$ is surjective. Then $\beta(\delta\tau) = \delta(\beta(\tau)) = \delta\sigma = 0$, hence after refining further there exists $\mu \in C^{k+1}(\underline{U},\mathscr{E})$ satisfying $\alpha(\mu) = \delta\tau$. Note that $\mu$ is a cocycle. It is because $\alpha(\delta\mu) = \delta(\alpha(\mu)) = \delta\delta(\tau) = 0$ and $\alpha$ is injective, hence $\delta\mu = 0$ and $\mu \in \ker(\delta)$. We thus take $\delta_*\sigma := [\mu] \in H^{k+1}(X,\mathscr{E})$. One checks that this is independent of the choice of $\tau$ and $\mu$.

We say that a sequence of maps

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \longrightarrow \cdots$$

is *exact* if $\mathrm{Im}(\alpha_{n-1}) = \ker(\alpha_n)$.

**Proposition 1.29.** *The long sequence of cohomology associated to a short exact sequence of sheaves is exact.*

*Proof.* We prove it under an extra assumption that there exists an acyclic open covering $\underline{U}$ such that for any $U = U_{i_1} \cap \cdots \cap U_{i_k}$ we have the short exact sequence:

$$0 \to \mathscr{E}(U) \to \mathscr{F}(U) \to \mathscr{G}(U) \to 0.$$

At least for sheaves considered in this course, this assumption always holds. It further implies the following sequence is exact:

$$0 \to C^k(\underline{U},\mathscr{E}) \to C^k(\underline{U},\mathscr{F}) \to C^k(\underline{U},\mathscr{G}) \to 0.$$

Let us prove that

$$H^k(\underline{U},\mathscr{F}) \xrightarrow{\beta_*} H^k(\underline{U},\mathscr{G}) \xrightarrow{\delta_*} H^{k+1}(\underline{U},\mathscr{E})$$

is exact. The other cases are easier.

Consider $\tau \in Z^k(\underline{U},\mathscr{F})$. In the definition of $\delta_*$, take $\sigma = \beta(\tau)$. Then there exists $\mu \in C^k(\underline{U},\mathscr{E})$ such that $\alpha(\mu) = \delta\tau = 0$. Then we have $\mu = 0$ since $\alpha$ is injective. Consequently $\delta_*\beta_*(\tau) = \delta_*(\sigma) = \mu = 0$, hence $\delta_*\beta_* = 0$ and $\mathrm{Im}(\beta_*) \subset \ker(\delta_*)$.

Conversely, suppose $\delta_*\sigma = 0$ for $\sigma \in Z^k(\underline{U},\mathscr{G})$. In the definition of $\delta_*$, it implies that $\mu = 0 \in H^{k+1}(\underline{U},\mathscr{E})$, hence there exists $\gamma \in C^k(\underline{U},\mathscr{E})$ such that $\delta\gamma = \mu$. Since $\alpha(\mu) = \delta\tau$, we have $\delta\tau = \delta\alpha(\gamma)$ and $\tau - \alpha(\gamma) \in Z^k(\underline{U},\mathscr{F})$ is a cocycle. Moreover, $\beta(\tau - \alpha(\gamma)) = \beta(\tau) = \sigma$, hence $\beta_*(\tau - \alpha(\gamma)) = \sigma$. We conclude that $\ker(\delta_*) \subset \mathrm{Im}(\beta_*)$. $\square$

**Exercise 1.30.** Prove in general the cohomology sequence is exact.

**Example 1.31.** Consider the short exact sequence

$$0 \longrightarrow \mathscr{I}_p \xrightarrow{i} \mathscr{O}_{\mathbb{P}^1} \xrightarrow{r} \mathscr{O}_p \longrightarrow 0.$$

Its long exact sequence of cohomology is as follows:

$$0 \to H^0(\mathscr{I}_p) \to H^0(\mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathscr{O}_p) \to H^1(\mathscr{I}_p) \to H^1(\mathscr{O}_{\mathbb{P}^1}) \to 0.$$

The last term is zero because $p$ is a point so it does not have higher cohomology. We have $H^0(\mathscr{O}_{\mathbb{P}^1}) = K$ because any global regular function on $\mathbb{P}^1$ is constant. Note that $H^0(\mathscr{I}_p) = 0$, because vanishing at $p$ forces such a constant function to be zero. Moreover we have seen that $H^1(\mathscr{O}_{\mathbb{P}^1}) = 0$. Altogether it implies $H^1(\mathscr{I}_p) = 0$, because $H^0(\mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathscr{O}_p)$ is an isomorphism by evaluating at $p$.

**Exercise 1.32.** Let $D$ be an effective divisor of degree $n$ on $\mathbb{P}^1$. We have the short exact sequence

$$0 \longrightarrow \mathscr{I}_D \xrightarrow{\ i\ } \mathscr{O}_{\mathbb{P}^1} \xrightarrow{\ r\ } \mathscr{O}_D \longrightarrow 0.$$

Use the associated long exact sequence to calculate the cohomology $H^*(\mathbb{P}^1, \mathscr{I}(D))$.

## 2. Vector bundles, line bundles and divisors

2.1. **Holomorphic vector bundles.** Let $k$ be a positive integer. Consider $\pi : E \to X$ a holomorphic map between complex manifolds, such that for every $x \in X$ the fiber $E_x = \pi^{-1}(x)$ is isomorphic to $\mathbb{C}^k$ and there exists an open neighborhood $U$ of $x$ along with an isomorphism

$$\phi_U : E_U = \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^k$$

mapping $E_x$ to $\{x\} \times \mathbb{C}^k$ which is a linear isomorphism between vector spaces. Then $E$ is called a *holomorphic vector bundle of rank $k$ on $X$* and has a *trivialization* $\{(U, \phi_U)\}$. If $E$ is of rank one, we say that $E$ is a *line bundle*.

We give another characterization of vector bundles based on transition functions. Suppose $\underline{U} = \{U_\alpha\}$ is an open covering of $X$. Given holomorphic functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}^k),$$

we can construct a vector bundle $E$ by gluing $U_\alpha \times \mathbb{C}^k$ together. More precisely,

$$E = \sqcup (U_\alpha \times \mathbb{C}^k)/\sim$$

as a complex manifold is defined by identifying $(x, v)$ with $(x, g_{\alpha\beta}(v))$ for $x \in U_\alpha \cap U_\beta$ and $v \in \mathbb{C}^k$ and $E \to X$ is given by projection to the bases $U_\alpha$. Call $\{g_{\alpha\beta}\}$ the transition functions of $E$. They have to satisfy the following compatibility conditions:

$$g_{\alpha\beta}(x) \cdot g_{\beta\alpha}(x) = I, \quad \text{for all } x \in U_\alpha \cap U_\beta,$$
$$g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = I, \quad \text{for all } x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

**Exercise 2.1.** Let $E$ and $F$ be two vector bundles on $X$ of rank $k$ and $l$, respectively. Define the *direct sum* $E \oplus F$, the *tensor product* $E \otimes F$, the *dual* $E^*$, and the *wedge product* $\wedge^r E$ for $r \leq k$. Calculate the ranks of these bundles and represent their transition functions in terms of the transition functions of $E$ and $F$.

A *map* between two vector bundles $E$ and $F$ on $X$ is given by a holomorphic map $f : E \to F$ such that $f(E_x) \subset F_x$ and $f_x = f|_{E_x} : E_x \to F_x$ is linear. Note that if $f(E_x)$ has the same rank for every $x$, then $\ker(f)$ and $\mathrm{Im}(f)$ are naturally subbundles of $E$ and $F$, respectively. We say that $E$ and $F$ are *isomorphic* if $f_x$ is a linear isomorphism for every $x$. A vector bundle is called *trivial* if it is isomorphic to $X \times \mathbb{C}^k$.

**Exercise 2.2.** Give an example of a map between vector bundles $f : E \to F$ on $X$ such that the the image of $f$ is not a vector bundle.

**Exercise 2.3.** Let $L$ be a line bundle on $X$. Prove that $L \otimes L^*$ is a trivial line bundle.

Define a *section* $\sigma$ as a holomorphic map $\sigma : X \to E$ such that $\sigma(x) \in E_x$ for every $x \in X$, i.e. $\pi \circ \sigma$ is identity. If $\sigma(x) = 0 \in E_x$, we say that $\sigma$ is vanishing on $x$.

**Exercise 2.4.** Let $L$ be a line bundle on $X$. Prove that $L$ is trivial if and only if it possesses a nowhere vanishing section.

**Example 2.5** (Holomorphic tangent bundles)**.** Let $X$ be an $n$-dimensional complex manifold. Suppose $\phi_U : U \to \mathbb{C}^n$ are coordinate charts of $X$. Define the (holomorphic) *tangent bundle* $T_X$ by setting $T_X = \sqcup T_x$ with

$$T_x = \mathbb{C}\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\} \cong \mathbb{C}^n$$

as well as transition functions $g_{UV} = J(\phi_V \phi_U^{-1})$, where $J$ denotes the Jacobian matrix $(\frac{\partial w_i}{\partial z_j})$ for $1 \le i, j \le n$. The dual bundle $T_X^*$ is called the *cotangent bundle* of $X$. The determinant $\det(T_X^*)$ is called the *canonical line bundle* of $X$.

**Remark 2.6.** Alternatively, one can define vector bundles on a topological space, a differential manifold and an algebraic variety. The above definitions and properties go through word by word once replacing "holomorphic map" by "homomorphism", "smooth map" or "regular map".

2.2. **Vector bundles and locally free sheaves.** There is a *one-to-one correspondence* between isomorphism classes of vector bundles of rank $n$ and isomorphism classes of locally free sheaves of rank $n$ on a variety $X$. Here we briefly explain the idea. The reader can refer to Hartshorne II 5, especially Ex. 5.18 for more details.

Let $\mathscr{O}_X$ be the structure sheaf of a variety $X$. Note that $\mathscr{O}_X(U)$ has a *ring* structure (not only a group) for any open set $U$. A *sheaf of $\mathscr{O}_X$-modules* is a sheaf $\mathscr{F}$ on $X$ such that for each open set $U$, the group $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module. An $\mathscr{O}_X$-module $\mathscr{F}$ is called *free* if it is isomorphic to a direct sum of $\mathscr{O}_X$. It is called *locally free* if there is an open covering $\underline{U} = \{U_\alpha\}$ such that for each open subset $U_\alpha$, $\mathscr{F}|_{U_\alpha}$ is a free $\mathscr{O}_X|_U$-module. The *rank* of $\mathscr{F}$ on $U$ is the number of copies of $\mathscr{O}$ in the summation. In this course we only consider the rank being finite (such sheaves are called *coherent sheaves*). If $X$ is connected, the rank of $\mathscr{F}$ does not vary with the open subsets. In particular, a locally free sheaf of rank 1 is also called an *invertible sheaf.*

Roughly speaking, if $\mathscr{F}$ is locally free of rank $n$, we can choose a set of $n$ generators $x_1, \dots, x_n$ for the $\mathscr{O}_X(U)$-module $\mathscr{F}(U)$. They span an $n$-dimensional affine space $A[x_1, \dots, x_n]$ over $U$, where $A$ is the coordinate ring of $U$. By changing to a different set of generators over another open subset, one can write down the transition functions, hence it associates to $\mathscr{F}$ a vector bundle structure. Conversely if $F$ is a vector bundle on $X$, locally we have $F|_U \cong U \times \mathbb{A}^n$ with $x_1, \dots, x_n$ a basis (i.e. $n$ linearly independent sections) of $\mathbb{A}^n$ over $U$. Then we can associate to $F|_U$ an $\mathscr{O}_X(U)$-module of rank $n$ using $x_1, \dots, x_n$ as generators.

**Example 2.7.** Let $X \subset \mathbb{P}^n$ be a (smooth) variety and $Y \subset X$ a hypersurface, i.e. $Y$ is cut out (transversely) by a hypersurface $F$ in $\mathbb{P}^n$ with $X$. We have the short exact sequence

$$0 \to \mathscr{I}_{Y/X} \to \mathscr{O}_X \to \mathscr{O}_Y \to 0.$$

The ideal sheaf $\mathscr{I}_{Y/X}$ is an invertible sheaf. Indeed, for an open subset $U \subset X$, $\mathscr{I}_{Y/X}(U)$ can be expressed as $(F|_U) \cdot \mathscr{O}_X(U)$, hence is locally free of rank 1. The sheaf $\mathscr{O}_Y$ (extended to $X$ by zero) is *not* locally free. For $U \cap Y = \emptyset$, $\mathscr{O}_Y(U) = 0$ and for $U \cap Y \neq \emptyset$, $\mathscr{O}_Y(U)$ is non-zero. Later we will see how to construct a line bundle corresponding to $\mathscr{I}_{Y/X}$.

2.3. **Divisors.** Let $X$ be a variety. Suppose $Y \subset X$ is an irreducible subvariety of codimension one. We say that $Y$ is an *irreducible divisor* of $X$. More precisely, for every $p \in Y$ there exists an open neighborhood $U \subset X$ of $p$ such that $U \cap Y$ is cut out by a (holomorphic or regular) function $f$. We call $f$ a local defining equation for $Y$ near $p$. A *divisor* $D$ on $X$ is a formal linear combination of irreducible divisors:

$$D = \sum_{i=i}^{n} a_i Y_i,$$

where $a_i \in \mathbb{Z}$ (or $\mathbb{Q}$, $\mathbb{R}$ depending on the context). If $a_i \geq 0$ for all $i$, we say that $D$ is *effective* and denote it by $D \geq 0$. The divisors on $X$ form an additive group $\mathrm{Div}(X)$.

Suppose $f$ is a local defining equation of an irreducible divisor $Y \subset X$ on an open subset $U \subset X$. For another function $g$ on $X$, locally we can write

$$g = f^a \cdot h$$

such that the regular function $h$ is coprime with $f$ in $\mathscr{O}_X(U)$. We say that $a$ is the *vanishing order* of $g$ along $Y \cap U$. Note that the vanishing order is locally a constant, hence is independent of $U$. We use

$$\mathrm{ord}_Y(g) = a$$

to denote the vanishing order of $g$ along $Y$.

For two regular functions $g, h$ on $X$, we have

$$\mathrm{ord}_Y(gh) = \mathrm{ord}_Y(g) + \mathrm{ord}_Y(h).$$

For a function $f = g/h$, we define

$$\mathrm{ord}_Y(f) = \mathrm{ord}_Y(g) - \mathrm{ord}_Y(h).$$

If $\mathrm{ord}_Y(f) > 0$, we say that $f$ has a *zero* along $Y$. If $\mathrm{ord}_Y(f) < 0$, we say that $f$ has a *pole* along $Y$. We also define the *divisor associated to* $f$ by

$$(f) = \sum_Y \mathrm{ord}_Y(f),$$

as well as the *divisor of zeros*

$$(f)_0 = \sum_Y \mathrm{ord}_Y(g)$$

and the *divisor of poles*

$$(f)_\infty = \sum_Y \mathrm{ord}_Y(h).$$

They satisfy

$$(f) = (f)_0 - (f)_\infty.$$

If $D = (f)$ is the associated divisor of a global meromorphic function $f$, $D$ is called a *principal divisor*.

Let $\mathscr{M}^*$ be the multiplicative sheaf of (not identically zero) meromorphic functions and $\mathscr{O}^*$ the multiplicative sheaf of nowhere vanishing regular functions, which is a subsheaf of $\mathscr{M}^*$.

**Proposition 2.8.** *We have a correspondence* $\mathrm{Div}(X) \cong H^0(X, \mathscr{M}^*/\mathscr{O}^*)$.

*Proof.* Suppose $\{f_\alpha\}$ represents a global section of $\mathscr{M}^*/\mathscr{O}^*$ with respect to an open covering $\underline{U} = \{U_\alpha\}$. Associate to it a divisor $D_\alpha = (f_\alpha)$ in $U_\alpha$. We claim that $D_\alpha = D_\beta$ in $U_\alpha \cap U_\beta$. This is due to

$$\frac{f_\alpha}{f_\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta),$$

hence $f_\alpha$ and $f_\beta$ define the same divisor. Consequently $\{D_\alpha\}$ defines a global divisor. Moreover, if $\{f_\alpha\}$ and $\{g_\alpha\}$ define the same divisor, then $f_\alpha/g_\alpha \in \mathscr{O}^*(U_\alpha)$, hence $\{f_\alpha\}$ and $\{g_\alpha\}$ represent the same section of $\mathscr{M}^*/\mathscr{O}^*$. This shows an injection

$$H^0(X, \mathscr{M}^*/\mathscr{O}^*) \hookrightarrow \mathrm{Div}(X).$$

Conversely, suppose $D = \sum a_i Y_i$ is a divisor on $X$ with $a_i \in \mathbb{Z}$ and $Y_i$ effective. We can choose an open covering $\underline{U} = \{U_\alpha\}$ such that $Y_i$ is locally defined by $g_{i\alpha} \in \mathscr{O}(U_\alpha)$. Consider

$$f_\alpha = \prod_i (g_{i\alpha})^{a_i} \in \mathscr{M}^*(U_\alpha).$$

Then we have

$$\frac{f_\alpha}{f_\beta} = \prod_i \Big(\frac{g_{i\alpha}}{g_{i\beta}}\Big)^{a_i}.$$

Both $g_{i\alpha}$ and $g_{i\beta}$ cut out the same divisor $Y_i|_{U_\alpha \cap U_\beta}$ in $U_\alpha \cap U_\beta$, hence we conclude that

$$\frac{g_{i\alpha}}{g_{i\beta}} \in \mathscr{O}^*(U_\alpha \cap U_\beta), \quad \frac{f_\alpha}{f_\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta).$$

Then $\{f_\alpha\}$ defines a global section of $\mathscr{M}^*/\mathscr{O}^*$. Finally if $D$ determines the zero section of $\mathscr{M}^*/\mathscr{O}^*$ (which is 1 since the group structure is multiplicative), it means locally $f_\alpha \in \mathscr{O}^*(U_\alpha)$ (after refining the open covering). Then it does not have zeros or poles, hence $D|_{U_\alpha} = 0$ for each $U_\alpha$ and $D$ is globally zero. This shows the other injection

$$\mathrm{Div}(X) \hookrightarrow H^0(X, \mathscr{M}^*/\mathscr{O}^*).$$

□

2.4. **Line bundles.** Recall that a line bundle $L$ on $X$ is a vector bundle of rank 1. Equivalently, it is a locally free sheaf of rank 1. Define the *Picard group* $\mathrm{Pic}(X)$ parameterizing isomorphism classes of line bundles on $X$. The group law is given by tensor product. We can interpret $\mathrm{Pic}(X)$ as a cohomology group.

**Proposition 2.9.** *There is a one-to-one correspondence between the isomorphism classes of line bundles on* $X$ *and* $H^1(X, \mathscr{O}^*)$, *i.e.*

$$\mathrm{Pic}(X) \cong H^1(X, \mathscr{O}^*).$$

*Proof.* Take an open covering $\underline{U} = \{U_\alpha\}$ of $X$ with respect to the trivialization of a line bundle $L$. The transition function

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$$

can be regarded as a section of $\mathscr{O}^*(U_\alpha \cap U_\beta)$, satisfying

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = 1,$$

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1.$$

Therefore, $\{g_{\alpha\beta}\}$ is a cocycle in $C^1(\underline{U}, \mathscr{O}^*)$, hence represents a cohomology class in $H^1(X, \mathscr{O}^*)$.

Suppose $M$ is another line bundle with transition functions $\{h_{\alpha\beta}\}$. If $M$ and $L$ are isomorphic, then $L \otimes M^*$ is trivial, i.e. $\{g_{\alpha\beta}/h_{\alpha\beta}\}$ are transition functions of $L \otimes M^*$, which has a nowhere vanishing section $\sigma$. Suppose on $U_\alpha$ we have $\sigma_\alpha : U_\alpha \to \mathbb{C}^*$ as the restriction of $\sigma$. Then on $U_\alpha \cap U_\beta$ we have

$$\frac{g_{\alpha\beta}}{h_{\alpha\beta}} \cdot \sigma_\alpha = \sigma_\beta.$$

Therefore we conclude that

$$\frac{g_{\alpha\beta}}{h_{\alpha\beta}} = \frac{\sigma_\beta}{\sigma_\alpha} \in \delta C^0(\underline{U}, \mathscr{O}^*).$$

□

Now we describe another important correspondence between line bundles and divisors. Suppose $D$ is a divisor on $X$ with local defining equations $\{f_\alpha\}$ such that $f_\alpha \in \mathscr{M}^*(U_\alpha)$. Define

$$g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

Then we have $g_{\alpha\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta)$. Moreover, $\{g_{\alpha\beta}\}$ satisfy the assumptions imposed to transition functions, hence they define a line bundle, denoted by $L = [D]$ or $L = \mathscr{O}_X(D)$. We have a group homomorphism

$$\mathrm{Div}(X) \to \mathrm{Pic}(X)$$

induced by

$$D + D' \mapsto [D] \otimes [D'].$$

We say that $D$ and $D'$ are *linearly equivalent*, if $[D]$ and $[D']$ are isomorphic line bundles. We denote linear equivalence by

$$D \sim D'.$$

The following result says that the kernel of the above map consists of principal divisors. In other words, two divisors $D \sim D'$ if and only if $D - D'$ is a principal divisor.

**Proposition 2.10.** *The associated line bundle $[D]$ is trivial if and only if $D$ is a principal divisor, i.e. $D = (f)$ for some $f \in \mathscr{M}^*(X)$.*

*Proof.* Suppose $D = (f)$ is the associated divisor of a meromorphic function $f$ on $X$. Then $D$ has local defining equations $\{f_\alpha = f|_{U_\alpha}\}$. The transition functions associated to $[D]$ are all equal to 1, hence $[D]$ is a trivial line bundle. Conversely, suppose $[D]$ is trivial. Then it has a nowhere vanishing section $\sigma$ whose restriction to $U_\alpha$ is denoted by $\sigma_\alpha$. The transition functions $g_{\alpha\beta} = f_\beta/f_\alpha$ defined above satisfy

$$g_{\alpha\beta} \cdot \sigma_\alpha = \sigma_\beta,$$

hence we have

$$\frac{f_\alpha}{\sigma_\alpha} = \frac{f_\beta}{\sigma_\beta} \in \mathscr{M}^*(U_\alpha \cap U_\beta).$$

We can glue $\{f_\alpha/\sigma_\alpha\}$ to form a global function $f \in \mathscr{M}^*(X)$. Since $\sigma$ is nowhere vanishing, we obtain that $(f) = D$. □

Let us summarize, using the short exact sequence

$$0 \to \mathscr{O}^* \to \mathscr{M}^* \to \mathscr{M}^*/\mathscr{O}^* \to 0.$$

Recall that

$$\mathrm{Div}(X) \cong H^0(X, \mathscr{M}^*/\mathscr{O}^*), \quad \mathrm{Pic}(X) \cong H^1(X, \mathscr{O}^*).$$

Then we have the long exact sequence

$$0 \to H^0(X, \mathscr{O}^*) \to H^0(X, \mathscr{M}^*) \xrightarrow{(\cdot)} \mathrm{Div}(X) \xrightarrow{[\cdot]} \mathrm{Pic}(X) \to \cdots$$

which encodes all the information in the above discussions.

2.5. **Sections of a line bundle.** Let $L$ be a line bundle on $X$ with transition functions $\{g_{\alpha\beta}\}$. A *holomorphic section* $s$ of $L$ has restriction $s_\alpha \in \mathscr{O}(U_\alpha)$, satisfying

$$g_{\alpha\beta} s_\alpha = s_\beta.$$

Conversely, a collection $\{s_\alpha \in \mathscr{O}(U_\alpha)\}$ such that $s_\beta/s_\alpha = g_{\alpha\beta}$ determines a section of $L$.

Similarly, we define a *meromorphic section* $s$ to be a collection

$$\{s_\alpha \in \mathscr{M}(U_\alpha)\}$$

such that $g_{\alpha\beta} s_\alpha = s_\beta$. Suppose $t \not\equiv 0$ is another meromorphic section with collection $\{t_\alpha\}$. We have

$$\frac{s_\beta}{t_\beta} = \frac{s_\alpha}{t_\alpha},$$

hence the quotient $s/t$ is a global meromorphic function. Conversely, if $f$ is a global meromorphic function, then $\{f \cdot s_\alpha\}$ defines another meromorphic section of $L$.

For a meromorphic section $s \not\equiv 0$, consider the divisor $(s_\alpha)$ associated to the local section $s_\alpha$ in $U_\alpha$. Since

$$\frac{s_\beta}{s_\alpha} = g_{\alpha\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta),$$

$\{(s_\alpha)\}$ form a *global* divisor $(s)$ on $X$. Conversely, suppose $D$ is a divisor. Consider the construction of the associated line bundle $[D]$. Suppose the local defining equations of $D$ are given by $\{s_\alpha\}$. Then the transition functions of $[D]$ are $\{g_{\alpha\beta} = s_\beta/s_\alpha\}$ and consequently the collection $\{s_\alpha\}$ gives rise to a meromorphic section of $[D]$. Note that for a section $s$ of $L$, the divisor $(s)$ is effective if and only if $s$ is a holomorphic section. We thus obtain the following result.

**Proposition 2.11.** *For any section $s$ of $L$, we have $L \cong [(s)]$. A line bundle $L$ is associated to a divisor $D$ if and only if it has a meromorphic section $s$ such that $(s) = D$. In particular, $L$ has a holomorphic section if and only if it is associated to an effective divisor.*

Now we treat a line bundle as a locally free sheaf of rank 1 and reinterpret the above correspondence. Let $D$ be a divisor on $X$. Define a sheaf $\mathscr{O}_X(D)$ or simply $\mathscr{O}(D)$ by

$$\mathscr{O}(D)(U) = \{f \in \mathscr{M}(U) : (f) + D|_U \geq 0\}.$$

It has a vector space structure since $(f) + D|_U \geq 0$ and $(g) + D|_U \geq 0$ implies that $(af + bg) + D|_U \geq 0$ for any $a, b$ in the base field.

**Proposition 2.12.** *The space of holomorphic sections of $[D]$ can be identified with $H^0(X, \mathscr{O}(D))$.*

*Proof.* A global section $s \in H^0(X, \mathscr{O}(D))$ is a meromorphic function satisfying

$$(s) + D \geq 0.$$

Suppose $D$ is locally defined by $\{f_\alpha\}$. The associated line bundle $[D]$ has transition functions

$$\{g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}\}.$$

Then the collection $\{s \cdot f_\alpha\}$ defines a section $\sigma$ of $[D]$. Since

$$(s) + (f_\alpha) \geq 0$$

in every $U_\alpha$, $\sigma$ is a holomorphic section of $[D]$. Moreover, the associated divisor $D' = (s) + D$ of the section is linearly equivalent to $D$, since $D' - D = (s)$ is principal.

Conversely, given a holomorphic section $\sigma$ of $[D]$, i.e. a collection $\{h_\alpha \in \mathscr{O}(U_\alpha)\}$ such that

$$\frac{h_\beta}{h_\alpha} = g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

Then $\{h_\alpha / f_\alpha\}$ defines a global meromorphic function $g$. Since $(h_\alpha) \geq 0$ in every $U_\alpha$, we have

$$(g|_{U_\alpha}) + (f_\alpha) = (h_\alpha) \geq 0,$$

hence $(g) + D \geq 0$ globally on $X$ and $g \in H^0(X, \mathscr{O}(D))$. □

**Remark 2.13.** Replacing $X$ by any open subset $U$, the proposition implies that the sheaf $\mathscr{O}(D)$ can be regarded as gathering local holomorphic sections of the line bundle $[D]$. If $D \sim D'$, i.e. $D' - D = (f)$ for a global meromorphic function $f$, then for any $g \in \mathscr{O}(D')(U)$, we have

$$0 \leq (g) + D' = (g) + (f) + (D) = (fg) + (D)$$

restricted to $U$. So we obtain an isomorphism

$$\mathscr{O}(D')(U) \xrightarrow{\cdot f} \mathscr{O}(D)(U)$$

for any open subset $U$, compatible with the sheaf restriction maps. In this sense, the sheaf $\mathscr{O}(D)$ and the line bundle $[D]$ have a one-to-one correspondence up to isomorphism and linear equivalence, respectively, assuming that every line bundle can be associated to a divisor.

Let $|D|$ be the set of effective divisors that are linearly equivalent to $D$. We call $|D|$ the *linear system* associated to $D$.

**Proposition 2.14.** *Let $X$ be compact and $D$ a divisor on $X$. Then we have*

$$\mathbb{P}H^0(X, \mathscr{O}(D)) = |D|,$$

*i.e. an effective divisor in $|D|$ and a holomorphic section of $[D]$ (up to scalar) determine each other.*

*Proof.* For any $D' \in |D|$, by definition $D' - D = (f)$ is principal for some $f \in \mathscr{M}(X)$, hence $(f) + D = D' \geq 0$ and $f \in H^0(X, \mathscr{O}(D))$. Since $X$ is compact, if $g$ is another function such that $D' - D = (g)$, then $(f/g) = 0$, i.e. $f/g$ is holomorphic, hence it is a constant.

Conversely, any $f \in H^0(X, \mathscr{O}(D))$ defines an effective divisor $D' = (f) + D$. If $(f) + D = (g) + D$, then $(f/g) = 0$ and $f/g$ is a constant, since $X$ is compact. □

**Exercise 2.15.** Let $D = \sum a_i p_i$ be a divisor on $\mathbb{P}^1$ with $a_i \in \mathbb{Z}$ and $p_i \in \mathbb{P}^1$. Define the *degree* of $D$ by $\deg(D) = \sum a_i$.

(1) Prove that $D \sim D'$ if and only if $\deg(D) = \deg(D')$.

(2) Calculate the cohomology $H^*(\mathbb{P}^1, \mathscr{O}(D))$ in terms of $\deg(D)$.

## 3. Preliminaries on curves

In this section, we discuss some basic properties of algebraic curves.

3.1. **The Riemann-Roch formula.** Let $X$ be a compact Riemann surface or a closed algebraic curve. Define its *arithmetic genus* by

$$g := h^1(\mathscr{O}_X) = \dim_{\mathbb{C}} H^1(\mathscr{O}_X).$$

**Theorem 3.1** (Riemann-Roch Formula)**.** *Let $D$ be a divisor on $X$ and $\mathscr{O}(D)$ the associated line bundle or invertible sheaf. Then we have*

$$h^0(\mathscr{O}(D)) - h^1(\mathscr{O}(D)) = 1 - g + \deg(D).$$

**Remark 3.2.** Define the (holomorphic) *Euler characteristic* of a sheaf $\mathscr{F}$ by

$$\chi(\mathscr{F}) := \sum_{i \geq 0} (-1)^i h^i(\mathscr{F}).$$

The Riemann-Roch formula can be written as

$$\chi(\mathscr{O}(D)) - \chi(\mathscr{O}_X) = \deg(D).$$

*Proof.* Let us first prove it for effective divisors of degree $\geq 0$. Do induction on $n$. The formula obviously holds for $\mathscr{O}_X$. Suppose it is true for $\deg(D) < n$. Consider $D = p + D'$ with $D'$ an effective divisor of degree $n-1$. We have the short exact sequence

$$0 \to \mathscr{O}(D') \to \mathscr{O}(D) \to \mathbb{C}_p \to 0,$$

where $\mathbb{C}_p$ is the skyscraper sheaf with one-dimensional stalk supported at $p$. The exactness can be easily checked. The map $\mathscr{O}(D') \to \mathscr{O}(D)$ is an inclusion, since

$$(f) + D' \geq 0$$

implies that

$$(f) + D = (f) + D' + p \geq 0.$$

The quotient corresponds to germs of functions $f$ at $p$ such that

$$(f)|_U + D'|_U = -p$$

in arbitrarily small neighborhoods $U$ of $p$. In other words, if $\mathrm{ord}_p(D') = m \geq 0$, we can write

$$f = z^{-m-1} h(z),$$

where $h \in \mathscr{O}^*(U)$. So the quotient sheaf is given by $\mathbb{C} \cdot \{z^{-m-1}\} \cong \mathbb{C}$ supported at $p$. Since the associated cohomology sequence is long exact, we have

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}(D')) + 1 = 1 - g + (n-1) + 1 = 1 - g + n.$$

In general, write a divisor $D = D_1 - D_2$, where $D_1$ and $D_2$ are both effective divisors of degree $d_1$ and $d_2$, respectively, and $d_1 - d_2 = \deg(D)$. By the same token, we have the short exact sequence

$$0 \to \mathscr{O}(D) \to \mathscr{O}(D_1) \to \mathbb{C}^{d_2} \to 0.$$

Then we obtain that

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}(D_1)) - d_2 = 1 - g + d_1 - d_2 = 1 - g + \deg(D).$$

□

**Remark 3.3.** Assuming the Serre duality

$$H^1(\mathscr{O}(D)) \cong H^0(K \otimes \mathscr{O}(-D)),$$

where $K$ is the canonical line bundle of $X$, then we can rewrite the Riemann-Roch formula as

$$h^0(L) - h^0(K \otimes L^*) = 1 - g + \deg(L),$$

where $L$ is a line bundle on $X$. Note that $K$ is a degree $2g-2$ line bundle (to be discussed later). We conclude that

$$h^0(K) = g, \quad h^1(K) = h^0(\mathscr{O}) = 1.$$

It implies that the space of holomorphic one-forms on a genus $g$ Riemann surface is $g$-dimensional.

3.2. **The Riemann-Hurwitz formula.** A *branched cover* $\pi : X \to Y$ between two (compact, connected) Riemann surfaces is a (surjective) regular morphism. For a general point $q \in Y$, $\pi^{-1}(q)$ consists of $d$ distinct points. Call $d$ the *degree* of $\pi$. Locally around $p \mapsto q$, if the map is given by

$$x \mapsto y = x^m,$$

where $x, y$ are local coordinates of $p, q$, respectively, call $m$ the *vanishing order* of $\pi$ at $p$ and denote it by

$$\mathrm{ord}_p(\pi) = m.$$

If $\mathrm{ord}_p(\pi) > 1$, we say that $p$ is a *ramification point*. If $\pi^{-1}(q)$ contains a ramification point, then $q$ is called a *branch point*. Define the *pullback*

$$\pi^*(q) = \sum_{p \in \pi^{-1}(q)} \mathrm{ord}_p(\pi) \cdot p.$$

Note that $\pi^*(q)$ is a degree $d$ effective divisor on $X$.

**Theorem 3.4** (Riemann-Hurwitz Formula)**.** *Let $\pi : X \to Y$ be a branched cover between two Riemann surfaces. Then we have*

$$K_X \sim \pi^* K_Y + \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1) \cdot p,$$

*where $K_X$ and $K_Y$ are canonical divisors on $X$ and $Y$, respectively.*

*Proof.* Take a one-form $\omega$ on $Y$ locally expressed as $f(w)dw$ around a point $q = \pi(p)$. Suppose the covering at $p$ is given by

$$z \mapsto w = z^m,$$

then we have

$$\pi^*(f(w)dw) = mf(z^m)z^{m-1}dz.$$

Namely, the associated divisors satisfy the relation

$$(\pi^*\omega)|_U = (\pi^*(\omega))|_U + (\mathrm{ord}_p(\pi) - 1) \cdot p$$

in a local neighborhood $U$ of $p$. So globally it implies that

$$(\pi^*\omega) = \pi^*(\omega) + \sum_{p\in X}(\mathrm{ord}_p(\pi) - 1)\cdot p.$$

Since $\pi^*\omega$ is a one-form on $X$, $(\pi^*\omega)$ is a canonical divisor of $X$ and the claimed formula follows. $\square$

We can interpret the (numerical) Riemann-Hurwitz formula from a topological viewpoint. Let $\chi(X)$ denote the *topological Euler characteristic* of $X$. If $X$ is a Riemann surface of genus $g$, take a triangulation of $X$ and suppose the number of $k$-dimensional edges is $c_k$ for $k = 0, 1, 2$. Then we have

$$\chi(X) = c_0 - c_1 + c_2 = 2 - 2g.$$

**Proposition 3.5.** *Let $\pi : X \to Y$ be a degree $d$ branched cover between two Riemann surfaces. Then we have*

$$\chi(X) = d\cdot\chi(Y) - \sum_{p\in X}(\mathrm{ord}_p(\pi) - 1).$$

*Proof.* Take a triangulation of $Y$ such that every branch point is a vertex. Pull it back as a triangulation of $X$. Note that it pulls back a face to $d$ faces, an edge to $d$ edges and a vertex $v$ to $|\pi^{-1}(v)|$ vertices. Note that if

$$\pi^{-1}(v) = \sum_{i=1}^{k} m_i p_i$$

for distinct points $p_i$, then $|\pi^{-1}(v)| = m$. In other words, we have

$$|\pi^{-1}(v)| = d - \sum_{p\in\pi^{-1}(v)}(\mathrm{ord}_p(\pi) - 1).$$

Then the claimed formula follows right away. $\square$

**Corollary 3.6** (Numerical Riemann-Hurwitz)**.** *Let $\pi : X \to Y$ be a degree $d$ branched cover between two Riemann surfaces of genus $g$ and $h$, respectively. Then we have*

$$2g - 2 = d(2h - 2) + \sum_{p\in X}(\mathrm{ord}_p(\pi) - 1).$$

*In particular, if $g < h$, such branched covers do not exist.*

**Corollary 3.7.** *The canonical line bundle of a genus $g$ Riemann surface $X$ has degree equal to $2g - 2$.*

*Proof.* Every Riemann surface $X$ possesses a nontrivial meromorphic function, say by the Riemann-Roch formula. It induces a branched cover $\pi : X \to \mathbb{P}^1$ of degree $d$. By the Riemann-Hurwitz Formula we know

$$\deg(K_X) = d(-2) + \sum_{p\in X}(\mathrm{ord}_p(\pi) - 1),$$

since we have seen that $\deg(K_{\mathbb{P}^1}) = -2$. By the Numerical Riemann-Hurwitz we have

$$2 - 2g = 2d - \sum_{p\in X}(\mathrm{ord}_p(\pi) - 1).$$

Then the claim follows immediately. $\square$

**Exercise 3.8.** Let $X$ be a Riemann surface or algebraic curve of genus $g$. If $X$ admits a branched cover of degree 2 to $\mathbb{P}^1$, we say that $X$ is a *hyperelliptic curve*. Prove that every $g \leq 2$ curve is hyperelliptic. For $g \geq 2$, can you calculate the dimension of the parameter space of genus $g$ hyperelliptic curves?

**Remark 3.9.** The moduli space of genus $g$ curves has dimension $3g-3$, which is bigger than $2g-1$ for $g > 2$. Hence a general $g > 2$ curve is *not* hyperelliptic.

3.3. **Genus formula of plane curves.** Suppose $F(Z_0, Z_1, Z_2)$ is a general degree $d$ homogeneous polynomial whose vanishing locus is a plane curve $C \subset \mathbb{P}^2$. Since $F$ is general, $C$ is smooth. In other words, the singularities of $C$ locate at the common zeros of $F = 0$ and $\partial F/\partial Z_i = 0$ for all $i$, which are empty for a general $F$.

**Theorem 3.10.** *In the above setting, the genus $g$ of $C$ is given by*

$$g = \frac{(d-1)(d-2)}{2}.$$

*Proof.* We give two proofs. The first one is more algebraic. Note that all degree $d$ curves are linearly equivalent in $\mathbb{P}^2$. Hence it makes sense to use $\mathscr{O}_{\mathbb{P}^2}(d)$ to denote the associated line bundle. In particular, $\mathscr{O}(1)$ is the associated line bundle of a line $L$. Then we have the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_L \to 0.$$

Tensor it with $\mathscr{O}_{\mathbb{P}^2}(1-m)$. We obtain that

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-m) \to \mathscr{O}_{\mathbb{P}^2}(-(m-1)) \to \mathscr{O}_{\mathbb{P}^2}(1-m)|_L \to 0.$$

Since $\mathscr{O}_{\mathbb{P}^2}(1-m)|_L$ is the line bundle associated to a degree $1-m$ divisor on $L$ and $L \cong \mathbb{P}^1$, we conclude that

$$\chi(\mathscr{O}_{\mathbb{P}^2}(-(m-1))) - \chi(\mathscr{O}_{\mathbb{P}^2}(m)) = \chi(\mathscr{O}_{\mathbb{P}^1}(1-m)) = 2-m,$$

where we apply the Riemann-Roch formula to $\mathbb{P}^1$ in the last equality. Then we obtain that

$$\chi(\mathscr{O}_{\mathbb{P}^2}) - \chi(\mathscr{O}_{\mathbb{P}^2}(-d)) = \sum_{m=1}^{d}(2-m) = -\frac{d(d-3)}{2}.$$

Now by the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-d) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_C \to 0,$$

we have

$$1 - g = \chi(\mathscr{O}_C) = -\frac{d(d-3)}{2},$$

hence the genus formula follows.

The other proof is an application of the Riemann-Hurwitz formula. Without loss of generality, suppose $o = [0,0,1] \notin C$. Let $L$ be the line $Z_2 = 0$ and project $C$ to $L$ from $o$, i.e.

$$[Z_0, Z_1, Z_2] \mapsto [Z_0, Z_1].$$

In affine coordinates $x = Z_1/Z_0$ and $y = Z_2/Z_0$, this map is given by

$$(x, y) \mapsto x,$$

i.e. we project $C$ vertically to the $x$-axis. This yields a degree $d$ branched cover

$$\pi : C \to L \cong \mathbb{P}^1.$$

A point $p$ is a ramification point of $\pi$ if and only if there exists a vertical line tangent to $C$ at $p$, i.e. $p$ is a common zero of $F$ and $\partial F/\partial Z_2$. Since $F$ and $\partial F/\partial Z_2$ have degree $d$ and $d-1$, respectively, they intersect at $d(d-1)$ points. By Riemann-Hurwitz, we have

$$2g - 2 = d(-2) + d(d-1),$$

hence the genus formula follows. In order to make sure all the ramifications are simple, we may choose a general projection direction such that it is different from those of the (finitely many) lines with higher tangency to $C$. $\square$

**Remark 3.11.** In the first proof, indeed we did not use the smoothness of $C$. So the (arithmetic) genus formula holds for an arbitrary plane curve, even if it is singular, reducible or non-reduced. In the second proof, even if the projection has higher ramification points, a detailed local study plus Riemann-Hurwitz still gives rise to the desired formula.

3.4. **Base point free and very ample line bundles.** Let $L$ be a line bundle on a complete variety $X$. We say that $L$ has a *base point* at $p \in X$ if $p$ belongs to the vanishing locus of every regular section of $L$. If the base locus of $L$ is empty, then $L$ is called *base point free.*

For a base point free line bundle $L$, let $\sigma_0, \ldots, \sigma_n$ be a basis of the space $H^0(X, L)$ of regular sections. Locally around a point $p \in X$, consider $\sigma_i$ as a regular function and associate to $p$ the point

$$[\sigma_0(p), \ldots, \sigma_n(p)] \in \mathbb{P}^n.$$

This is well-defined, since if we take a different chart, then we get $\sigma_i'(p) = g_{\alpha\beta}\sigma_i(p)$ where $\{g_{\alpha\beta}\}$ are transition functions of $L$. Therefore, we obtain a regular map

$$\phi_L : X \to \mathbb{P}^n.$$

We can give a more conceptual and coordinate free description of $\phi_L$. Since $L$ is base point free, the space of regular sections $\sigma$ vanishing at $p$ forms a hyperplane $H_p \subset H^0(X, L) \cong \mathbb{C}^{n+1}$. Then one can define $\phi_L(p) = [H_p] \in (\mathbb{P}^n)^*$ in the dual projective space parameterizing hyperplanes.

**Proposition 3.12.** *In the above setting, there is a one-to-one correspondence between (the pullback of ) hyperplane sections of $X$ and effective divisors in the linear system $|L|$.*

*Proof.* This is just a reformulation of the one-to-one correspondence

$$|L| = \mathbb{P}H^0(X, L),$$

which we proved before. In other words, an effective divisor in $|L|$ uniquely determines a regular section $\sigma = \sum a_i\sigma_i$ mod scalar. $\square$

**Example 3.13.** If $L = \mathscr{O}$, then we have $\phi_{\mathscr{O}}$ maps $X$ to a single point.

**Example 3.14.** Let $X = \mathbb{P}^1$ and $L = \mathscr{O}(2p)$ where $p = [0, 1]$. Then $H^0(\mathbb{P}^1, L)$ is 3-dimensional and we can choose a basis by

$$1, \quad \frac{Y}{X}, \quad \frac{Y^2}{X^2}.$$

Recall that around $p$ the sections of $L$ are given by $f \cdot (X/Y)^2$. Hence we obtain that

$$\phi_L([X, Y]) = [X^2, XY, Y^2],$$

which is a smooth conic in $\mathbb{P}^2$. The genus formula for a plane curve of degree 2 also implies that the image has $g = 0$.

**Exercise 3.15.** A variety $X \subset \mathbb{P}^n$ is called *non-degenerate* if it is not contained in any hyperplane.

(1) Show that any non-degenerate smooth rational curve in $\mathbb{P}^n$ has degree $\geq n$.

(2) For $d \geq n \geq 3$, show that there exist non-degenerate smooth degree $d$ rational curves in $\mathbb{P}^n$.

**Example 3.16.** Let $E$ be an elliptic curve and $L = \mathscr{O}(2p)$. By Riemann-Roch, $h^0(E, L) = 2$. Moreover, $L$ is base point free. Otherwise if $q$ is a base point, then $q$ has to be $p$ and there exists another effective divisor $p+r \in |2p|$ such that $p+r \sim 2p$. But this implies $r - p$ is principal and $E \cong \mathbb{P}^1$, leading to a contradiction. Now $\phi_L : E \to \mathbb{P}^1$ is a branched cover of degree 2. Two points $s$ and $t$ lie in the same fiber if and only if $s + t \sim 2p$.

The above example indicates that $\phi_L$ is *not* always an embedding. We say that $L$ is *very ample* if $\phi_L$ is an embedding and that $L$ is *ample* if $L^{\otimes m}$ is very ample for some $m > 0$.

**Example 3.17.** The line bundle $\mathscr{O}(d)$ is very ample on $\mathbb{P}^1$ if and only if $d > 0$. The induced map $\phi$ embeds $\mathbb{P}^1$ into $\mathbb{P}^d$ as a degree $d$ smooth rational curve, i.e. a rational normal curve.

Let us give a criterion when $L$ is base point free or very ample on an algebraic curve.

**Proposition 3.18.** *Let $L$ be a line bundle on a curve $X$.*

*(1) $L$ is base point free if and only if*

$$h^0(X, L \otimes \mathscr{O}(-p)) = h^0(X, L) - 1$$

*for any $p \in X$.*

*(2) $L$ is very ample if and only $L$ is base point free and for any $p, q \in X$ (not necessarily distinct)*

$$h^0(X, L \otimes \mathscr{O}(-p-q)) = h^0(X, L \otimes \mathscr{O}(-p)) - 1 = h^0(X, L \otimes \mathscr{O}(-q)) - 1.$$

*Proof.* Treat $L$ as a locally free sheaf of rank 1. By the short exact sequence

$$0 \to L \otimes \mathscr{O}(-p) \to L \to \mathbb{C}_p \to 0,$$

we have

$$h^0(X, L) - 1 \leq h^0(X, L \otimes \mathscr{O}(-p)) \leq h^0(X, L).$$

Then $L$ has a base point at $p$ if and only if all sections of $L$ vanish at $p$, i.e. $H^0(X, L \otimes \mathscr{O}(-p)) = H^0(X, L)$. This proves (1).

For (2), a very ample line bundle is necessarily base point free by definition. If $p \neq q \in X$ have the same image under $\phi_L$, it is equivalent to saying that the subspace of sections vanishing at $p$ is the same as the subspace of sections vanishing at $q$, which is further equivalent to

$$h^0(X, L \otimes \mathscr{O}(-p)) = h^0(X, L \otimes \mathscr{O}(-p-q)) = h^0(X, L \otimes \mathscr{O}(-q)).$$

Moreover, $\phi_L$ induces an injection restricted to the tangent space $T_p(X)$ if and only if there exists a hyperplane such that it cuts out $X$ locally a simple point at $p$, namely, if and only if there is a section vanishing at $p$ with multiplicity 1, i.e.

$$h^0(X, L \otimes \mathscr{O}(-2p)) < h^0(X, L \otimes \mathscr{O}(-p)).$$

But we have

$$h^0(X, L \otimes \mathscr{O}(-2p)) \geq h^0(X, L \otimes \mathscr{O}(-p)) - 1.$$

Hence (2) follows by combining the two cases. $\square$

**Remark 3.19.** In (2), for $p \neq q$ the condition geometrically means the sections of $L$ separate any two points. When $p = q$, it says that the sections of $L$ separate tangent vectors at $p$.

**Example 3.20.** Let $E$ be an elliptic curve, i.e. a torus as a Riemann surface of genus 1. Fix a point $p \in E$. The morphism

$$\tau : E \to J(E) \cong \mathrm{Pic}^0(E)$$

by $\tau(q) = [q - p]$ is an isomorphism. This defines a group law on $E$ with respect to $p$, i.e. $q + r = s$, where $s \in E$ is the unique point satisfying

$$(q - p) + (r - p) \sim s - p.$$

Now consider the linear system $|3p|$ on $E$. Since

$$h^0(E, \mathscr{O}(3p)) = 3, \quad h^0(E, \mathscr{O}(2p)) = 2, \quad h^0(E, \mathscr{O}(p)) = 1,$$

$\mathscr{O}(3p)$ is very ample. It induces an embedding of $E$ into $\mathbb{P}^2$ as a plane cubic. A line cuts out a divisor of degree 3 in $E$, say, $q + r + s$ (not necessarily distinct) if and only if

$$q + r + s \sim 3p.$$

Note that the tangent line $L$ of $E$ at $p$ is a *flex line*, i.e. the tangency multiplicity $(L \cdot E)_p = 3$. Such $p$ is called a *flex point*.

**Exercise 3.21.** Show that there are in total 9 flex points on a smooth plane cubic.

Let $V \subset |L|$ be a linear subspace. We say that $V$ is a *linear series* of $L$. The linear system $|L|$ is also called a *complete linear series*. The above definitions and properties go through similarly for the induced map $\phi_V$.

**Exercise 3.22.** Write down a linear series of $|\mathscr{O}(3)|$ on $\mathbb{P}^1$ such that it maps $\mathbb{P}^1$ into $\mathbb{P}^2$ as a *singular* plane cubic. How many different types of such singular plane cubics can you describe?

3.5. **Canonical maps.** Let $K$ be the canonical line bundle a curve $X$. If $X$ is $\mathbb{P}^1$, $\deg(K) = -2$ and $K$ is not effective. If $X$ is an elliptic curve, then $K \cong \mathscr{O}$ and the induced map $\phi_K$ is onto a point. From now on, assume that the genus of $X$ satisfies $g \geq 2$. We say that $X$ is *hyperelliptic* if it admits a degree 2 branched cover of $\mathbb{P}^1$. Two points $p, q \in X$ are called *conjugate* if they have the same image in $\mathbb{P}^1$. A ramification point of the double cover is called a *Weierstrass point* of $X$, i.e. it is self conjugate. By Riemann-Hurwitz, a genus $g \geq 2$ hyperelliptic curve possesses $2g + 2$ Weierstrass points.

**Lemma 3.23.** *If $X$ is a hyperelliptic curve of genus $\geq 2$, then $X$ admits a unique double cover of $\mathbb{P}^1$.*

*Proof.* Otherwise suppose $h^0(X, \mathscr{O}(p + q)) = 2$ and $h^0(X, \mathscr{O}(p + r)) = 2$ for $q \neq r$. Note that $h^0(X, \mathscr{O}(p + q + r)) < 3$, since $X$ cannot be a plane cubic by the genus formula. Then we conclude that

$$H^0(X, \mathscr{O}(p + q)) = H^0(X, \mathscr{O}(p + r)) = H^0(X, \mathscr{O}(p + q + r)),$$

which implies that both $q, r$ are base points of $|p+q+r|$ and $h^0(X, \mathscr{O}(p)) = 2$, $X \cong \mathbb{P}^1$, leading to a contradiction. $\square$

**Proposition 3.24.** *Let $X$ be a curve of genus $g \geq 2$. Then the canonical line bundle $K$ is base point free. The induced map*

$$\phi_K : X \to \mathbb{P}^{g-1}$$

*is an embedding if and only if $X$ is not hyperelliptic. If $X$ is hyperelliptic, $\phi_K$ is a double cover of a rational normal curve in $\mathbb{P}^{g-1}$.*

*Proof.* First, let us show that $K$ is base point free. For any point $p \in X$, by Riemann-Roch we have

$$h^0(X, K \otimes \mathscr{O}(-p)) - h^0(X, \mathscr{O}(p)) = 1 - g + (2g-3),$$

$$h^0(X, K \otimes \mathscr{O}(-p)) = g - 1 = h^0(X, K) - 1.$$

Hence $K$ satisfies the criterion of base point freeness.

Next, $K$ fails to separate $p, q$ (not necessarily distinct) if and only if

$$h^0(X, K \otimes \mathscr{O}(-p-q)) = h^0(X, K \otimes \mathscr{O}(-p)) = g - 1,$$

which is equivalent to, by Riemann-Roch again, that

$$h^0(X, \mathscr{O}(p+q)) = 2.$$

In other words, the linear system $|p+q|$ induces a double cover $X \to \mathbb{P}^1$.

Finally, if $X$ is hyperelliptic of genus $\geq 2$, it admits a unique double cover of $\mathbb{P}^1$. By the above analysis, two points $p, q$ have the same image under the canonical map if and only if $h^0(X, \mathscr{O}(p+q)) = 2$, i.e. $p, q$ are conjugate. Then the canonical map is a double cover of a rational curve of degree $\deg(K)/2 = g-1$ in $\mathbb{P}^{g-1}$, i.e. a rational normal curve. A hyperplane section of $\phi_K(X)$ pulls back to $X$ a divisor

$$\sum_{i=1}^{g-1} (p_i + q_i),$$

where $p_i, q_i$ are conjugate or $p_i = q_i$ a Weierstrass point. $\square$

**Remark 3.25.** For a non-hyperelliptic curve $X$, $\phi_K$ is called the *canonical embedding* of $X$ and its image is called a *canonical curve*.

**Example 3.26.** Let $X$ be a genus 2 curve. Then $h^0(X, K) = 2$, hence $X$ is hyperelliptic and the double cover of $\mathbb{P}^1$ is induced by the canonical line bundle, as we have seen.

**Example 3.27.** A genus 3 non-hyperelliptic curve admits a canonical embedding to $\mathbb{P}^2$ as a plane quartic. An effective canonical divisor corresponds to a line section of the quartic. By the genus formula, any smooth plane quartic also has genus equal to 3. Moreover, a smooth plane quartic $X$ gives rise to a line bundle $L$ of degree 4 on $X$ by restricting $\mathscr{O}_{\mathbb{P}^2}(1)$. By Riemann-Roch, $h^0(X, K \otimes L^*) \geq 1$, but $\deg(K \otimes L^*) = 0$, hence $L = K$. So any plane quartic is a canonical embedding of a genus 3 non-hyperelliptic curve.

**Example 3.28.** Let $X$ be a genus 4 non-hyperellipic curve. Then its canonical embedding is a degree 6 curve in $\mathbb{P}^3$. Let $\mathscr{O}_{\mathbb{P}^3}(1)$ denote the line bundle on $\mathbb{P}^3$

associated to a hyperplane class. Its restriction to $X$ is the canonical line bundle $K_X$. We have the exact sequence

$$0 \to \mathscr{I}_X \otimes \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_X(2) \to 0.$$

Since $h^0(\mathbb{P}^3, \mathscr{O}(2)) = 10 > h^0(X, \mathscr{O}(2)) = 9$, we conclude that $X$ is contained in a quadric surface $Q$ in $\mathbb{P}^3$. Indeed $Q$ is unique, because otherwise $\deg(X) \leq 4$.

If $Q$ is smooth, it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Up to projective equivalence, it has coordinates

$$[XZ, XW, YZ, YW],$$

i.e. its equation is

$$Z_0Z_3 - Z_1Z_2 = 0.$$

Note that

$$Z_0 - aZ_1 = Z_2 - aZ_3 = 0$$

defines a family $A$ of lines in $Q$ parameterized by the value of $a$. Any two lines in $A$ are disjoint. Similarly,

$$Z_0 - bZ_2 = Z_1 - bZ_3 = 0$$

defines another family $B$ of lines in $Q$ parameterized by $b$, and any two lines in $B$ are disjoint. Moreover, a line $L_1$ in $A$ and a line $L_2$ in $B$ intersect at a unique point $[ab, b, a, 1]$. Hence they span a plane $H$ in $\mathbb{P}^3$. Suppose $H \cdot X = D_1 + D_2$, where $D_i$ is a divisor in $L_i$. Since $L_i$ varies in $Q$ in a $\mathbb{P}^1$ family, we have $h^0(X, \mathscr{O}(D_i)) \geq 2$ for $i = 1, 2$. Moreover, because $D_1 + D_2 \sim K_X$, we have $\deg(D_1) + \deg(D_2) = 6$. Since $X$ is not hyperelliptic, we conclude that $\deg(D_i) = 3$ for $i = 1, 2$. This can also be seen since $X$ is a complete intersection of $Q$ with a cubic surface, hence its class in $Q$ is linearly equivalent to $3L_1 + 3L_2$. Therefore, we obtain two distinct line bundles of degree 3 given by $\mathscr{O}_X(D_1)$ and $\mathscr{O}_X(D_2)$ such that $h^0(\mathscr{O}_X(D_i)) = 2$. In other words, $X$ admits two triple covers of $\mathbb{P}^1$. This can be seen by projecting $X$ along the direction of $L_i$ for $i = 1, 2$.

If $Q$ is singular, since $X$ is non-degenerate, $Q$ is a quadric cone whose equation is given by

$$Z_0^2 - Z_1Z_2 = 0.$$

It has a unique singular point $v = [0, 0, 0, 1]$ as the vertex and $Q$ can be viewed as the cone over a plane conic $C$ defined by the same equation. Every line $L$ contained in $Q$ passes through $v$, hence $C$ parameterizes a $\mathbb{P}^1$-family of lines. Similarly, one checks that $L \cdot X = 3$ and $X$ admits a unique triple cover of $\mathbb{P}^1$ by projecting from $v$ to $C$.

**Exercise 3.29.** Suppose $g \geq 3$. Let $X \subset \mathbb{P}^{g-1}$ be a non-degenerate smooth genus $g$ curve of degree $2g - 2$. Show that $X$ is a canonical curve.

**Exercise 3.30.** Let $X$ be a genus 4 non-hyperelliptic curve. If $D = p + q + r$ is a degree 3 effective divisor on $X$ such that $h^0(X, \mathscr{O}(D)) = 2$, show that $p$, $q$ and $r$ are collinear in the canonical embedding of $X$ in $\mathbb{P}^3$.

**Exercise 3.31.** A hyperelliptic curve $X$ of genus $g$ can be explicitly written as the locus of $(z, w)$ satisfying

$$w^2 = (z - a_1) \cdots (z - a_{2g+2}).$$

One can treat $(w,z) \mapsto z$ as the double cover of $\mathbb{P}^1$ which is branched at $a_1, \ldots, a_{2g+2}$. Prove that a basis for the space $H^0(X, K)$ of holomorphic one-forms on $X$ is

$$\frac{dz}{w},\ z\frac{dz}{w}, \ldots,\ z^{g-1}\frac{dz}{w}.$$

3.6. **Dimension of linear systems.** Let $D = p_1 + \cdots + p_d$ be an effective divisor of degree $d$ on a genus $g$ curve $X$. Recall that the linear system $|D|$ can be identified with $\mathbb{P}H^0(X, \mathscr{O}(D))$ parameterizing effective divisors linearly equivalent to $D$. Suppose as a projective space

$$r = \dim|D| = h^0(X, \mathscr{O}(D)) - 1.$$

By Riemann-Roch, we have

$$\dim|K \otimes \mathscr{O}(-D)| = r + g - d - 1.$$

Note that $|K \otimes \mathscr{O}(-D)|$ is the linear system of canonical divisors that contain $D$. By the canonical map

$$\phi_K : X \to \mathbb{P}^{g-1},$$

it says that the space of hyperplanes of $\mathbb{P}^{g-1}$ that contain $\phi_K(p_1), \ldots, \phi_K(p_d)$ is $(r+g-d-1)$-dimensional. In other words, the linear span of $\phi_K(p_1), \ldots, \phi_K(p_d)$ is a

$$(g-2) - (r+g-d-1) = (d-1) - r$$

dimensional subspace in $\mathbb{P}^{g-1}$. Since we expect $d$ points to span a $(d-1)$-dimensional linear subspace, geometrically it says that $\phi_K(D)$ fails to impose

$$r = \dim|D|$$

independent conditions. We summarize the discussion as a geometric version of the Riemann-Roch formula.

**Theorem 3.32** (Geometric Riemann-Roch)**.** *In the above setting, let $\overline{\phi_K(D)}$ be the linear span of the image of $D$ under the canonical map. Then we have*

$$\dim|D| = \deg(D) - 1 - \dim\overline{\phi_K(D)}.$$

**Remark 3.33.** Even if $D$ contains points with higher multiplicity, the formulation still holds. Say, if $D$ contains $2p$, then $2p$ span the tangent line at $p$. If $D$ contains $3p$, then $3p$ spans an osculating 2-plane at $p$, and etc.

**Example 3.34.** Let us revisit the canonical embedding of a genus 4 non-hyperelliptic curve $X$ in $\mathbb{P}^3$. Recall that $X$ is contained in a unique quadric surface $Q$ and we showed that $X$ admits a triple cover of $\mathbb{P}^1$ corresponding to a family of lines in $Q$. Conversely, if $D = p + q + r$ induces a triple cover $X \to \mathbb{P}^1$, i.e. if $\dim|D| = 1$, by Geometric Riemann-Roch, we have

$$\dim\overline{\phi_K(D)} = 3 - 1 - 1 = 1,$$

i.e. $p$, $q$ and $r$ are collinear in a line $L$ in $\mathbb{P}^3$. Because $L \cdot Q = 2$ unless $L$ is contained in $Q$, any triple cover of $\mathbb{P}^1$ on $X$ corresponds to a family of lines in $Q$. We have seen that if $Q$ is smooth, there are two such families of lines, i.e. $X$ admits two distinct triple covers of $\mathbb{P}^1$, while if $Q$ is singular, such a triple cover is unique.

Let us study in detail the dimension of a linear system.

**Lemma 3.35.** *Let $D$ be a divisor on a curve $X$. Then $\dim |D| \geq k$ if and only if for every $k$ points $p_1, \ldots, p_k \in X$ there exists an effective divisor in $|D|$ containing all of them.*

*Proof.* Suppose for every $k$ points $p_1, \ldots, p_k \in X$ there exists an effective divisor in $|D|$ containing all of them. Since $\sum_{i=1}^k p_i$ varies in a $k$-dimensional family, then $\dim |D| \geq k$ is obvious. Alternatively, we can prove it by induction. Suppose it holds for $\leq k$. Assume for every $p_1, \ldots, p_{k+1}$, there exists $D' \in |D|$ containing all of them. Then we conclude that $\dim |D - p| \geq k$ for any $p \in X$. Choose a point $p$ not in the base locus of $|D|$. Consequently we have

$$\dim |D| = \dim |D - p| + 1 \geq k + 1.$$

Now suppose $\dim |D| \geq k$. Then we have

$$h^0(X, \mathscr{O}(D - \sum_{i=1}^k p_i)) \geq h^0(X, \mathscr{O}(D)) - k \geq 1.$$

It implies that there exists a non-zero meromorphic function $f$ such that

$$(f) + D - \sum_{i=1}^k p_i \geq 0,$$

hence $(f) + D = D'$ is an effective divisor in $|D|$ containing $p_1, \ldots, p_k$. □

**Corollary 3.36.** *For any two effective divisors $D_1$ and $D_2$ on $X$, we have*

$$\dim |D_1| + \dim |D_2| \leq \dim |D_1 + D_2|.$$

*Proof.* Suppose $\dim |D_i| = k_i$ for $i = 1, 2$. Take any $k_1 + k_2$ points

$$p_1, \ldots, p_{k_1}, q_1, \ldots, q_{k_2}$$

in $X$. By the above lemma, there exist $D_1' \in |D_1|$ and $D_2' \in |D_2|$ such that $D_1'$ contains all the $p_i$ and $D_2'$ contains all the $q_j$. Then $D_1' + D_2' \in |D_1 + D_2|$ contains all the $p_i$, $q_j$, hence we obtain

$$\dim |D_1 + D_2| \geq k_1 + k_2$$

by the lemma again. □

Note that if $h^0(X, K \otimes (-D)) = 0$, then Riemann-Roch determines that

$$h^0(X, \mathscr{O}(D)) = 1 - g + \deg(D).$$

Some subtlety may occur if

$$h^0(X, K \otimes (-D)) > 0$$

and we call such a divisor $D$ a *special divisor* and the associated linear system $|D|$ a *special linear system*. By Riemann-Roch, any $D$ with $\deg(D) > 2g - 2$ is non-special. By Geometric Riemann-Roch, $D$ is non-special if and only if the linear span of $\phi_K(D)$ is the entire space $\mathbb{P}^{g-1}$.

**Theorem 3.37** (Clifford's Theorem)**.** *Let $D$ be an effective divisor such that* $\deg(D) \leq$ $2g - 2$ *on $X$. Then we have*

$$\dim |D| \leq \frac{1}{2} \cdot \deg(D).$$

*Proof.* If $D$ is non-special, we have

$$\dim|D| = \deg(D) - g < \frac{1}{2}\deg(D).$$

If $D$ is special, there exists an effective divisor $D'$ such that $D + D' \sim K$. By the above lemma we have

$$\dim|D| + \dim|D'| \leq \dim|K| = g - 1.$$

By Riemann-Roch, we have

$$\dim|D| - \dim|D'| = 1 - g + \deg(D).$$

The desired inequality follows by combining the two relations. □

**Remark 3.38.** Indeed, the equality holds only if $D = 0$, $D = K$ or $X$ is hyperelliptic. If $D = 0$ or $D = K$, one easily checks that the equality holds. If $X$ is hyperelliptic, we can take $D = p + q$, where $p, q$ are conjugate and $\dim|p + q| = 1$. To prove that these are the only possibilities, we need the *uniform position theorem* regarding a general hyperplane section of a non-degenerate space curve, see e.g. [Griffiths-Harris, p. 249].

**Exercise 3.39.** Let $X$ be a genus $\geq 2$ hyperelliptic curve. For $0 < 2k \leq g$, find an effective divisor $D$ of degree $2k$ on $X$ such that $\dim|D| = k$. Classify all such divisors up to linear equivalence.

## 4. Geometry of Weierstrass points

We want to generalize the concept of Weierstrass points on a hyperelliptic curve to an arbitrary curve. Let $X$ be a curve of genus $g \geq 2$ and $p \in X$ a point.

### 4.1. Weierstrass semigroups and gap sequences.

Set $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$ and define $H_p \subset \mathbb{N}$ by

$$H_p = \{n : \exists f \in \mathscr{M}(X) \text{ such that } (f)_\infty = np\}.$$

Note that if $(f)_\infty = np$ and $(h)_\infty = mp$, then $(fh)_\infty = (m + n)p$. We say that $H_p$ is the *Weierstrass semigroup* of $p$. Define $G_p \subset \mathbb{N}$ as the complement of $H_p$, i.e.

$$G_p = \{n : \nexists f \in \mathscr{M}(X) \text{ such that } (f)_\infty = np\}.$$

We say that $G_p$ is the *Weierstrass gap sequence* of $p$.

**Lemma 4.1.** *We have $n \in H_p$ if and only if*

$$h^0(X, \mathscr{O}(np)) = h^0(X, \mathscr{O}((n-1)p)) + 1$$

*and $n \in G_p$ if and only if*

$$h^0(X, \mathscr{O}(np)) = h^0(X, \mathscr{O}((n-1)p)).$$

*In other words, $n \in G_p$ (resp. $H_p$) if and only if $p$ is (resp. is not) a base point of the linear system $|np|$. Moreover, $G_p$ is a subset of $\{1, \ldots, 2g-1\}$ with cardinality $g$.*

*Proof.* By the exact sequence

$$0 \to \mathscr{O}((n-1)p) \to \mathscr{O}(np) \to \mathbb{C}_p \to 0,$$

we conclude that

$$h^0(X, \mathscr{O}((n-1)p)) \le h^0(X, \mathscr{O}(np)) \le h^0(X, \mathscr{O}((n-1)p)) + 1.$$

The right hand side equality holds if and only if $p$ is not a base point of $|np|$, namely, if and only if there exists $f \in \mathscr{M}(X)$ such that $(f) + np \ge 0$ but $(f) + (n-1)p \not\ge 0$, which is equivalent to saying that $(f)_\infty = np$, i.e. if and only if $n \in H_p$.

For any $n \ge 2g$, by Riemann-Roch we have

$$h^0(X, \mathscr{O}(np)) = h^0(X, \mathscr{O}(n-1)p) + 1.$$

Hence we conclude that $G_p \subset \{1, \ldots, 2g-1\}$. Moreover, we have

$$\begin{aligned} g - 1 &= h^0(X, \mathscr{O}((2g-1)p)) - h^0(X, \mathscr{O}) \\ &= \sum_{n=0}^{2g-1} \Big( h^0(X, \mathscr{O}(np)) - h^0(X, \mathscr{O}((n-1)p)) \Big) \end{aligned}$$

So there are $g-1$ elements of $\{1, \ldots, 2g-1\}$ belonging to $H_p$. In other words, the cardinality of $G_p$ is

$$(2g-1) - (g-1) = g.$$

$\square$

Let us give an alternative interpretation of the above result.

**Lemma 4.2.** *We have $n \in G_p$ if and only if there exists a section of the canonical line bundle, i.e. a holomorphic one-form $\omega$, such that $\mathrm{ord}_p(\omega) = n-1$. Consequently we conclude that $G_p$ is a subset of $\{1, \ldots, 2g-1\}$ of cardinality $g$.*

*Proof.* By Riemann-Roch, $h^0(X, \mathscr{O}(np)) = h^0(X, \mathscr{O}((n-1)p))$ if and only if

$$h^0(X, K \otimes \mathscr{O}(-(n-1)p)) = h^0(X, K \otimes \mathscr{O}(-np)) + 1.$$

Note that $H^0(X, K \otimes \mathscr{O}(-mp))$ parameterizes holomorphic one-forms $\omega$ such that $(\omega) \ge mp$. So the above equality holds if and only if there exists $\omega$ such that $(\omega) \ge (n-1)p$ but $(\omega) \not\ge np$, namely, if and only if $\mathrm{ord}_p(\omega) = (n-1)p$.

Since $h^0(X, K) = g$, one can choose a basis $\omega_1, \ldots, \omega_g$ such that $\mathrm{ord}_p(\omega_i) = a_i$ and

$$a_1 < a_2 < \cdots < a_g.$$

Then we obtain that

$$G_p = \{a_1 + 1, a_2 + 1, \ldots, a_g + 1\}.$$

Because $\deg(K) = 2g-2$, we have $0 \le a_i \le 2g-2$ for all $i$, hence $G_p$ is a subset of $\{1, \ldots, 2g-1\}$. $\square$

4.2. **Weierstrass points.** We say that $p$ is a *Weierstrass point* of $X$ if $G_p \neq \{1, 2, \dots, g\}$ and $p$ is a *normal Weierstrass point* if $G_p = \{1, 2, \dots, g-1, g+1\}$. Define the *weight* of $p$ by

$$\begin{aligned} w(p) &= \Big(\sum_{n \in G_p} n\Big) - (1 + 2 + \cdots + g) \\ &= \Big(\sum_{n \in G_p} n\Big) - \frac{g(g+1)}{2}. \end{aligned}$$

Then $p$ is a Weierstrass point if and only if $w(p) > 0$ and $p$ is a normal Weierstrass point if and only if $w(p) = 1$.

Now we introduce some basic properties of a semigroup of $\mathbb{N}$. In general, if $H \subset \mathbb{N}$ is a semigroup whose complement $G = \mathbb{N} - H$ consists of $g$ elements, define the *weight* of $H$ by

$$w(H) = \Big(\sum_{n \in G} n\Big) - \frac{g(g+1)}{2}.$$

**Lemma 4.3.** *In the above setting, we have*

$$w(H) \leq \frac{(g-1)g}{2}.$$

*The equality holds if and only if* $2 \in H$.

*Proof.* Let $H = \{a_1, a_2, \dots,\}$ such that $a_1 < a_2 < \cdots$. Suppose $b \in G$ is the smallest element such that $b > a_n$. Then for $1 \leq i \leq n$, we have

$$b - a_i \in G, \quad b - a_i \leq a_n.$$

The two disjoint sets

$$\{b - a_n, b - a_{n-1}, \dots, b - a_1\}, \quad \{a_1, \dots, a_n\}$$

are both contained in $\{1, 2, \dots, a_n\}$. Hence we conclude that

$$a_n \geq n + n = 2n.$$

Now suppose $G = \{b_1, \dots, b_g\}$. Then $H$ can be expressed as

$$\{1, \dots, b_1 - 1; b_1 + 1, \dots, b_2 - 1; b_2 + 1, \dots, b_3 - 1; \dots\}.$$

We have $b_1 - 1 = a_{b_1 - 1}$, $b_2 - 1 = a_{b_2 - 2}$, $\dots$, and in general, that is,

$$b_k - 1 = a_{b_k - k} \geq 2(b_k - k)$$

by the above inequality. Hence we conclude that $b_k \leq 2k - 1$.

Now by definition, we have

$$\begin{aligned} w(H) &= \sum_{k=1}^{g} (b_k - k) \\ &\leq \sum_{k=1}^{g} (k - 1) \\ &= \frac{(g-1)g}{2}. \end{aligned}$$

Moreover, the equality holds if and only if $b_k = 2k - 1$ for $1 \leq k \leq g$, hence $G = \{1, 3, \dots, 2g - 1\}$ and $2 \in H$. $\square$

**Proposition 4.4.** *Let $X$ be a curve of genus $g \geq 2$. Then we have*

$$\sum_{p \in X} w(p) = (g-1)g(g+1).$$

*In particular, $X$ has finitely many Weierstrass points and the number of distinct Weierstrass points is at least $2g+2$. This lower bound can be attained if and only if $X$ is hyperelliptic.*

*Proof.* Choose a basis $\omega_1, \ldots, \omega_g$ of $H^0(X, K)$ and write $\omega_i = f_i(z)dz$ for $i = 1, \ldots, g$ in terms of a local coordinate $z$ around $p$. Consider the Wronskian

$$W(z) = \det \begin{pmatrix} f_1(z) & \cdots & f_g(z) \\ f_1'(z) & \cdots & f_g'(z) \\ \vdots & & \vdots \\ f_1^{(g-1)}(z) & \cdots & f_g^{(g-1)}(z) \end{pmatrix}$$

Since $f_1, \ldots, f_g$ form a basis, $W(z)$ is non-zero.

Suppose $\tilde{z}$ is another coordinate around $p$ such that $\omega_i = \tilde{f}_i(\tilde{z})d\tilde{z}$. Let

$$\psi = \frac{d\tilde{z}}{dz} \in \mathscr{O}^*(U \cap \tilde{U}).$$

Since $f_i(z)dz = \tilde{f}_i(\tilde{z})d\tilde{z}$, we have $f = \psi \tilde{f}$, and hence

$$\begin{aligned} \frac{df}{dz} &= \psi \frac{d\tilde{f}}{d\tilde{z}} \frac{d\tilde{z}}{dz} + \frac{d\psi}{dz} \tilde{f} \\ &= \psi^2 \frac{d\tilde{f}}{d\tilde{z}} + \frac{d\psi}{dz} \tilde{f} \end{aligned}$$

In general, we have

$$\frac{d^n f}{dz^n} = \psi^{n+1} \frac{d^n \tilde{f}}{d\tilde{z}^n} + \cdots$$

for higher derivatives. Denote by

$$N = 1 + 2 + \cdots + g = g(g+1)/2.$$

Then we conclude that

$$W(z) = \psi^N W(\tilde{z}) = \Big(\frac{d\tilde{z}}{dz}\Big)^N W(\tilde{z}).$$

For example, if $g = 2$, we have

$$\begin{aligned} \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} &= \det \begin{pmatrix} \psi \tilde{f}_1 & \psi \tilde{f}_2 \\ \psi^2 \tilde{f}_1' + \psi' \tilde{f}_1 & \psi^2 \tilde{f}_2' + \psi' \tilde{f}_2 \end{pmatrix} \\ &= \psi^{1+2} \det \begin{pmatrix} \tilde{f}_1 & \tilde{f}_2 \\ \tilde{f}_1' & \tilde{f}_2' \end{pmatrix}. \end{aligned}$$

Consequently it implies that $W(z)(dz)^N$ defines a global section of $H^0(X, K^{\otimes N})$.

Moreover, from the expression of $W(z)$ we have

$$\begin{aligned}
\mathrm{ord}_p(W(z)) &= \Big(\sum_{i=1}^{g} a_i\Big) - (0+1+\cdots+(g-1)) \\
&= \Big(\sum_{i=1}^{g}(a_i+1)\Big) - (1+2+\cdots+g) \\
&= \Big(\sum_{n\in G_p} n\Big) - g(g+1)/2 \\
&= w(p).
\end{aligned}$$

Since $\deg(K^{\otimes N}) = (g-1)g(g+1)$, the desired formula follows right away. □

**Corollary 4.5.** *Let $X$ be a genus $g$ curve for $g \geq 2$. Then it has at least $2g+2$ distinct Weierstrass points. Moreover, the lower bound is attained if and only if $X$ is a hyperelliptic curve.*

*Proof.* By definition, a Weierstrass point has strictly positive weight, so $X$ has finitely many Weierstrass points. Since $w(p) \leq (g-1)g/2$, we conclude that $X$ has at least $2g+2$ distinct Weierstrass points. This lower bound is attained if and only if each Weierstrass point $p$ satisfies $2 \in H_p$, i.e. $X$ admits a double cover of $\mathbb{P}^1$ with $p$ as a ramification point, hence if and only if $X$ is hyperelliptic. □

**Exercise 4.6.** Let $X$ be a hyperelliptic curve of genus $\geq 2$ with a double cover $\pi : X \to \mathbb{P}^1$ and $p \in X$ a point. Prove directly that

(1) $p$ is a ramification point of $\pi$ if and only if $G_p = \{1, 3, \ldots, 2g-1\}$;

(2) $p$ is not a ramification point of $\pi$ if and only if $G_p = \{1, 2, \ldots, g\}$.

**Exercise 4.7.** Determine all possible Weierstrass gap sequences for a point on a genus 3 curve.

4.3. **Automorphisms of a curve.** As an application of Weierstrass points, we have the following result.

**Proposition 4.8.** *The automorphism group of a genus $g \geq 2$ curve is finite.*

*Proof.* Let $X$ be a genus $g$ curve. Since $g \geq 2$, $X$ has finitely many Weierstrass points. In particular, an automorphism $\tau$ of $X$ sends a Weierstrass point to a Weierstrass point.

If $X$ is hyperelliptic, there are $2g+2$ Weierstrass points, corresponding to the $2g+2$ ramification points of the associated double cover $\pi : X \to \mathbb{P}^1$. Modulo the hyperelliptic involution, $\tau$ is induced by an automorphism of $\mathbb{P}^1$ that sends the $2g+2$ branch points to themselves. In other words, $\mathrm{Aut}(X)$ is a $(\mathbb{Z}/2)$-extension of the automorphism group of a $(2g+2)$-pointed $\mathbb{P}^1$. Since $2g+2 > 3$, such automorphisms of $\mathbb{P}^1$ are of finitely many, because any automorphism of $\mathbb{P}^1$ fixing three points must be the x.

Suppose $X$ is non-hyperelliptic. It is sufficient to show that if an automorphism $\tau$ fixes all Weierstrass points of $X$, then $\tau$ is the identity. Suppose $\tau$ is not identity and fixes all Weierstrass points of $X$. Take general points $p_1, \ldots, p_{g+1}$ on $X$ such that any $g$ of them are not contained in a hyperplane in the canonical embedding of $X$ and moreover none of them is fixed by $\tau$. By Geometric Riemann-Roch, there exists $f \in \mathscr{M}(X)$ such that

$$(f)_\infty = p_1 + \cdots + p_{g+1}.$$

Consider $h = f - \tau^* f \in \mathscr{M}(X)$. Since $p_i$'s are general, $h$ is not identically zero. Since $f$ has $g+1$ poles, then $h$ can have at most $2g+2$ poles, and hence at most $2g+2$ zeros. If $p$ is a fixed point of $\tau$, then $h(p) = f(p) - f(\tau(p)) = 0$, hence $p$ is a zero of $h$. So the number of fixed points of $\tau$ is bounded from above by $2g+2$. But $X$ has more than $2g+2$ Weierstrass points, since it is not hyperelliptic, leading to a contradiction. □

**Remark 4.9.** Indeed, Hurwitz's automorphisms theorem says that $|\operatorname{Aut}(X)| \leq 84(g-1)$ for a curve $X$ of genus $g \geq 2$ in the case of characteristic zero.

Let us give another proof of Proposition 4.8 for the case of Riemann surfaces. Let $f : X \to X$ be a continuous map of a compact triangulable space $X$. Define the *Lefschetz number*

$$\Lambda_f = \sum_{k \geq 0} (-1)^k \operatorname{Tr}(f^* : H^k(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})).$$

If $f$ has finitely many fixed points, then the Lefschetz Fixed Point Theorem says that

$$\Lambda_f = \sum_{x = f(x)} i(f, x),$$

where $i(f, x)$ is the index of a fixed point $x$.

In our situation, since $X$ is a complex manifold, all the indices are positive, coming from the intersection multiplicity of the graph of $\phi$ and the diagonal $\Delta$ in $X \times X$ for $\phi \in \operatorname{Aut}(X)$.

*Proof of Proposition 4.8.* It suffices to show that if $\phi \in \operatorname{Aut}(X)$ fixes all the Weierstrass points for a non-hyperelliptic Riemann surface $X$, then $\phi$ is the identity. Suppose it is not. Then $\phi$ has only finitely many fixed points, since $X$ is compact. Moreover, it is easy to see that $\phi$ is of finite order, hence its eigenvalues are unit roots. By the Lefschetz Fixed Point Theorem, we have

$$\Lambda_\phi \leq \sum_k \dim H^k(X, \mathbb{Q}) = 2g + 2,$$

i.e. $\phi$ has at most $2g+2$ fixed points, contradicting that $X$ is not hyperelliptic. □

4.4. **Weierstrass points on genus four curves.** Let $X$ be a non-hyperelliptic curve of genus four. Use $g^r_d$ to denote a $r$-dimensional linear system associated to a divisor of degree $d$. Recall that the canonical embedding of $X$ is contained in a quadric surface $Q$ in $\mathbb{P}^3$. If $Q$ is smooth, $X$ has two $g^1_3$'s, residual to each other with respect to $K_X$. If $Q$ is singular, $X$ has a unique $g^1_3$ induced by a half-canonical divisor.

**Lemma 4.10.** *In the above setting, if* $3p$ *induces a* $g^1_3$ *and if* $6p$ *is a canonical divisor, then* $X$ *has a unique* $g^1_3$.

*Proof.* Otherwise, we have $3p$ and $p+q+r$ as the two $g^1_3$'s of $X$, where $q, r \neq p$. Then $6p \sim 4p+q+r$ and $2p \sim q+r$, contradicting that $X$ is non-hyperelliptic. □

For a Weierstrass point on $X$, all the possible gap sequences $G$ are $\{1,2,4,7\}$, $\{1,2,4,5\}$, $\{1,2,3,7\}$, $\{1,2,3,6\}$ and $\{1,2,3,5\}$ of weights 4, 2, 3, 2 and 1, respectively. Let $n_1$, $n_2$ and $n_3$ be the numbers of Weierstrass points whose gap sequences are $\{1,2,4,7\}$, $\{1,2,4,5\}$ and $\{1,2,3,5\}$, respectively.

**Theorem 4.11.** *(i) $n_1 \leq 6$. (ii) If $n_1 > 0$, then $n_2 = 0$ and $2n_1 + n_3 \geq 12$. (iii) $n_2 \leq 12$.*

*Proof.* (i) Suppose $p$ has gap sequence $\{1, 2, 4, 7\}$. Then we have $6p \sim K_X$ and $3p$ induces a $g^1_3$. By the lemma, $X$ has a unique $g^1_3$. If $q$ also has gap sequence $\{1, 2, 4, 7\}$, then $3q$ induces the same $g^1_3$. Then the unique triple cover $X \to \mathbb{P}^1$ has total ramification order 12 by Riemann-Hurwitz. Each triple ramification point contributes 2, hence we get $n_1 \leq 6$.

(ii) If $n_1 > 0$, let $p$ be a point with gap sequence $\{1, 2, 4, 7\}$. By the lemma, $X$ has a unique $g^1_3$ and $3p$ induces the unique triple cover $X \to \mathbb{P}^1$. Suppose $q$ has gap sequence $\{1, 2, 4, 5\}$. Then $3 \in H_q$ implies that $3q$ also induces the unique $g^1_3$, hence $6q$ is a canonical divisor. Then we have $7 \in G_q$, leading to a contradiction.

Next, suppose $q$ is a ramification point of order one of the triple cover, i.e. $2q + r \sim 3p$ for some $r \neq q$. Then $4q + 2r$ is a canonical divisor, hence $5 \in G_q$. Moreover, $3 \notin H_q$ for otherwise $3q$ would also induce the same $g^1_3$, contradicting that $q \neq r$. Therefore, we conclude that $G_q = \{1, 2, 3, 5\}$. By Riemann-Hurwitz, we have $2n_1 + n_3 \geq 12$.

(iii) Let $p_i$ be the points with gap sequence $\{1, 2, 4, 5\}$ for $i = 1, \ldots, n_2$. Since $3 \in H_{p_i}$, it implies that $3p_i$ induces a $g^1_3$. Then at least half of the $3p_i$'s belong to the same $g^1_3$ of $X$ for $1 \leq i \leq n_2$. Using Riemann-Hurwitz associated to that triple cover, we conclude that $2([(n_2 + 1)/2]) \leq 12$, hence $n_2 \leq 12$. $\square$

**Corollary 4.12.** *Every non-hyperelliptic curve of genus four admits a 4-sheeted covering of $\mathbb{P}^1$ such that $\infty \in \mathbb{P}^1$ has a unique inverse image.*

*Proof.* It is equivalent to show that there exists $p \in X$ such that $4 \in H_p$. Note that the total Weierstrass weight equals 60. By Theorem 4.11, we have $4n_1 + 2n_2 \leq 24 < 60$. The claim follows right away. Indeed, there are at least 12 Weierstrass points on $X$ whose first non-gap equals 4. $\square$

This discussion motivates a question: which numerical semigroups occur as Weierstrass semigroups? A necessary condition can be easily imposed as follows.

**Proposition 4.13.** *For a semigroup $H$, let $G$ be its gap set of cardinality $g$, and $nG$ the set of sums of arbitrary $n$ elements in $G$. If for some $n > 1$ we have $|nG| > (2n - 1)(g - 1)$, then $H$ cannot be a Weierstrass semigroup.*

*Proof.* Suppose $G = \{a_1, \ldots, a_g\}$ is the Weierstrass gap sequence at $p \in X$. Then there exist a basis $\omega_1, \ldots, \omega_g$ of $H^0(X, K)$ such that $\omega_i$ has vanishing order $a_i - 1$ at $p$. Note that $h^0(X, K^{\otimes n}) = (2n - 1)(g - 1)$ for $n > 1$. Therefore, $nG - n$ has cardinality at most $(2n - 1)(g - 1)$, so does $|nG|$. $\square$

**Remark 4.14.** The above criterion is due to Buchweitz. He also found the first example of a numerical semigroup $H$ in $g = 16$ that violates this criterion.

## 5. Hilbert scheme

In this section we want to find a parameter space for subschemes of $\mathbb{P}^n$. Since we haven't introduced schemes, roughly speaking, one can think of the corresponding defining (saturated) ideals instead. Moreover, in contrast to varieties, schemes can be non-reduced or with embedded components.

5.1. **Parameterize geometric objects in $\mathbb{P}^n$.** Define the *degree* of a curve $C$ in $\mathbb{P}^n$ by the intersection number $C \cdot H$, counting with multiplicity, where $H$ is a general hyperplane. We would like to construct a parameter space for curves of the same degree and the same (arithmetic) genus, or in general, for subschemes with the same numerical invariants (to be made clear later). In addition, we want this parameter space itself to be a projective scheme with a universal object (explained later), so that we can study its geometry. Before we formally introduce the idea, let us consider some examples first.

**Example 5.1.** Consider degree 2 curves in $\mathbb{P}^2$. Such a curve is uniquely determined by a degree 2 homogenous polynomial in three variables, mod scalar. Hence the desired parameter space oughts to be $\mathbb{P}^5$. Note that even if this curve becomes non-reduced, i.e. a double line defined by $X^2 = 0$, it is still uniquely determined by the underlying reduced line $X = 0$. Hence no information is lost if we forget the double structure.

**Example 5.2.** Consider how to parameterize two points (unordered) in $\mathbb{P}^2$. A natural choice for such a parameter space is $\mathrm{Sym}^2 \mathbb{P}^2$. Away from the diagonal $\Delta = \{(p, p)\}$, there is no problem. Nevertheless, what happens if the two points coincide? Fix $p_1$ with ideal $I_1 = (x, y)$ in affine coordinate. Let $p_2$ approach $p_1$ along a line of slope $a$, i.e. it has ideal $I_2(t) = (x - t, y - at)$. For $t \neq 0$, the union of $p_1, p_2$ as a subscheme of $\mathbb{P}^2$ has ideal

$$I(t) = (x(x - t), x(y - at), y(x - t), y(y - at))$$

which contains $ax - y$ since $t \neq 0$. As $t$ approaches 0, we have the limiting ideal

$$I(0) = (x^2, xy, y^2, y - ax) = (y', x^2),$$

where $y' = y - ax$. In other words, the limit as a subscheme is a *double point* supported at $p$ contained in the line $y - ax = 0$, which remembers the approaching direction of $p_2$ towards $p_1$! In summary, there is a $\mathbb{P}^1$-family of double point structures at a given point in $\mathbb{P}^2$. Indeed, the corresponding parameter space in this sense is the blowup of $\mathrm{Sym}^2 \mathbb{P}^2$ along $\Delta$.

**Example 5.3.** Consider two skew lines $L_1$ and $L_2(t)$ in $\mathbb{P}^3$. Suppose their defining ideas are $I_1 = (X, Y)$ and $I_2(t) = (X - tZ, Y - tW)$ for $t \neq 0$. Their union as a subscheme of $\mathbb{P}^3$ has ideal

$$I(t) = (X(X - tZ), X(Y - tW), Y(X - tZ), Y(Y - tW)),$$

which contains $XW - YZ$ since $t \neq 0$. As $t$ tends to 0, $L_2(t)$ approaches $L_1$ and the limiting ideal is

$$I(0) = (X^2, XY, Y^2, XW - YZ),$$

which defines a double line $N$. Note that $N$ is not planar, since there is no linear polynomial contained in its ideal. Alternatively, we can see this because $L_1 \cup L_2$ as a curve has arithmetic genus $p_a = -1$, whereas a plane double line has $p_a = 0$, assuming that $p_a$ is invariant in the degeneration process.

The above examples illustrate two parameter spaces of different type. One parameterizes only the underlying reduced support of a subscheme, but loses information of possibly non-reduced structures or embedded components. This leads to the *Chow variety*. The other, called *Hilbert scheme*, keeps track of every piece of information, by parameterizing the defining ideal of a subscheme, no matter it gives rise to non-reduced structures or embedded components. Of course the Hilbert scheme carries finer information than the Chow variety. Indeed, with a suitable scheme structure on the Chow variety, there is a forgetful morphism from Hilbert to Chow, sending a subscheme to its reduced support with multiplicity. Only in some special cases these two spaces are the same, e.g. conics in $\mathbb{P}^2$, or in general, hypersurfaces in $\mathbb{P}^n$. From now on we will mainly focus on the Hilbert scheme.

One minor issue is that different ideals may define the same scheme. For instance, $(X)$ and $(X^2, XY)$ both define the same reduced, one-point subscheme in $\mathbb{P}^1$. To get rid of this issue, we introduce the notion of saturation. For an ideal $J \subset S = k[x_0, \ldots, x_n]$, define its *saturation* to be

$$I = \{F \in S : F \cdot S_n \subset J \text{ for some } n\}.$$

We say that $J$ is *saturated* if $J$ equals its saturation. Then there is a bijective correspondence between subschemes of $\mathbb{P}^n$ and saturated ideals.

5.2. **Hilbert function and Hilbert polynomial.** Let $Z$ be a subscheme of $\mathbb{P}^n$ defined by a homogenous saturated ideal $I$ in the homogeneous (graded) coordinate ring $S = k[X_0, \ldots, X_n]$. Define its *Hilbert function* as

$$h(m) = \dim(S/I)_m.$$

In other words, $\dim S_m - h(m)$ is the dimension of the space of degree $m$ homogeneous polynomials vanishing on $Z$. A key fact is that for $m \gg 0$, the Hilbert function turns out to be a polynomial $p(m)$, called the *Hilbert polynomial*. Moreover, the leading term of $p(m)$ is

$$\frac{d}{k!}m^k,$$

where $d = \deg Z$ and $k = \dim Z$. The condition $m \gg 0$ is important, because in general the Hilbert function $h$ and the Hilbert polynomial $p$ are different. The reason why $h$ fails to be a polynomial in low degree is due to special geometric properties of $Z$.

**Example 5.4.** For a 0-dimensional subscheme $Z$, define its *length* as $h^0(Z, \mathscr{O})$. Consider a length-3 subscheme $Z$ in $\mathbb{P}^2$. If the three points are in general position, i.e. if they are not collinear, then one checks that $h(m) = 3$ for every $m > 0$. Nevertheless, if they are collinear, then $h(1) = 3 - 1 = 2$, and $h(m) = 3$ for every $m > 1$.

In general, if $Z$ is a 0-dimensional subscheme, by the exact sequence

$$0 \to I_Z(m) \to \mathscr{O}_{\mathbb{P}^n}(m) \to \mathscr{O}_Z(m) \to 0,$$

we have

$$h^0(Z, \mathscr{O}_Z) = h^0(Z, \mathscr{O}_Z(m)) = h(m) = p(m)$$

for $m \gg 0$, because $\mathscr{O}(1)$ is ample, hence $h^1(\mathbb{P}^n, I_Z(m)) = 0$ for $m \gg 0$.

**Corollary 5.5.** *The Hilbert polynomial of a* 0*-dimensional subscheme is a constant, equal to the length of the subscheme.*

**Exercise 5.6.** Verify that for any length-2 subscheme of $\mathbb{P}^2$, i.e. the union of two distinct points or a double point, the Hilbert function equals the Hilbert polynomial.

**Proposition 5.7.** *Suppose $C$ is a smooth, connected and complete curve of degree $d$ and genus $g$ in $\mathbb{P}^n$. Then its Hilbert polynomial is*

$$p(m) = dm + 1 - g.$$

*Proof.* By the exact sequence

$$0 \to I_C(m) \to \mathscr{O}_{\mathbb{P}^n}(m) \to \mathscr{O}_C(m) \to 0,$$

for $m \gg 0$ we have

$$h(m) = h^0(C, \mathscr{O}(m)) = 1 - g + dm$$

by the Riemann-Roch formula. Here we again use the fact that $h^1(\mathbb{P}^n, I_C(m)) = 0$ for $m \gg 0$. □

**Exercise 5.8.** Find all possible ideals with Hilbert polynomial $p(m) = m + 2$ for subschemes of $\mathbb{P}^2$ up to projective equivalence.

5.3. **Construction of the Hilbert scheme and its universal property.** The Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^n)$ parameterizes subschemes of $\mathbb{P}^n$ with a given Hilbert polynomial $p$. Grothendieck constructed it and established its universal property. In order to construct $\mathrm{Hilb}^p(\mathbb{P}^n)$, the following result plays a key role.

**Lemma 5.9.** *Given a Hilbert polynomial $p$, there exists an integer $m$ such that for all subschemes $X$ in $\mathbb{P}^n$ whose Hilbert polynomial is $p$, $X$ is uniquely determined by $I_X(m)$, $h^i(X, I_X(m)) = 0$ for all $i > 0$, $h^0(X, \mathscr{O}(m)) = p(m)$ and $h^0(X, I_X(m)) = \binom{m+n}{m} - p(m)$.*

The consequence is that the $m$-th piece of $I_X$ can be regarded as a subspace $\Lambda_X$ of codimension $p(m)$ in $H^0(\mathbb{P}^n, \mathscr{O}(m))$, which determines the subscheme $X$. Set $N = \binom{m+n}{m}$. Then $\Lambda_X$ corresponds to a point in the Grassmannian $G(N-p(m), N)$. In this way one can realize $\mathrm{Hilb}^p(\mathbb{P}^n)$ as a subscheme of $G(N - p(m), N)$. Note that for an arbitrary subspace $\Lambda$ of codimension $p(m)$ in $H^0(\mathbb{P}^n, \mathscr{O}(m))$, it may not arise from a subscheme $X$, hence in general $\mathrm{Hilb}^p(\mathbb{P}^n)$ only forms a subscheme of $G(N - p(m), N)$.

**Example 5.10.** Consider $\mathrm{Hilb}^2(\mathbb{P}^2)$, the Hilbert scheme of length-2 subschemes in $\mathbb{P}^2$. Let $Z$ be a length-2 subscheme with ideal $I$. Then $Z$ is either the union of two distinct points or a double point. In both cases one checks that $\dim(S/I)_2 = 2$ and $Z$ is uniquely determined by $I_2$. Hence we obtain an embedding $\mathrm{Hilb}^2(\mathbb{P}^2) \hookrightarrow G(4, 6)$, where $G(4, 6)$ parameterizes 4-dimensional subspaces in the 6-dimensional space of degree 2 homogeneous polynomials in three variables. Since $\dim \mathrm{Hilb}^2(\mathbb{P}^2) = 4$ and $\dim G(2, 6) = 8$, it implies that *not* every 4-dimensional subspace of vanishing conics gives rise to a point in $\mathrm{Hilb}^2(\mathbb{P}^2)$. For instance, consider $I = (X^2, Y^2, Z^2, XY)$. Obviously it does not define any (non-empty) subscheme.

**Exercise 5.11.** Carry out the above construction explicitly for $\mathrm{Hilb}^3(\mathbb{P}^2)$, the Hilbert scheme of length-3 subschemes in $\mathbb{P}^2$.

Alternatively, there is a functorial description of the Hilbert scheme. It is the scheme that represents the *functor*

$$\mathrm{Hilb}^p(\mathbb{P}^n) : \{\text{schemes}\} \to \{\text{sets}\}$$

which associates to a scheme $S$ the set of families $\mathcal{X} \subset S \times \mathbb{P}^n$ of subschemes in $\mathbb{P}^n$ with a given Hilbert polynomial $p$. Equivalently speaking, there exists a unique *universal object* $\mathcal{U} \subset \mathrm{Hilb}^p(\mathbb{P}^n) \times \mathbb{P}^n$ such that for any family $\mathcal{X} \to S$ of subschemes with Hilbert polynomial $p$, there is a unique morphism $f : S \to \mathrm{Hilb}^p(\mathbb{P}^n)$ satisfying $f^*\mathcal{U} = \mathcal{X}$. In general, a parameter space with this universal property is called a *fine moduli space*.

**Example 5.12.** The Grassmannian $G(k, n)$ is a special case of the Hilbert scheme. Its universal object is the tautological bundle of rank $k$ whose fiber over a point $[V]$ represents the subspace $V$.

**Remark 5.13.** There is another terminology called *flat family*. In the setting of the Hilbert scheme, a family $\mathcal{X} \subset S \times \mathbb{P}^n$ is flat if and only if every fiber of $\mathcal{X} \to S$ has the same Hilbert polynomial. In particular, a family of curves in $\mathbb{P}^n$ is flat if and only if they have the same degree and arithmetic genus.

**Remark 5.14.** Not every moduli space is a fine moduli space. Roughly speaking, a moduli space may fail to be fine at a special point when the corresponding object has more automorphisms. For instance, the $j$-line parameterizing elliptic curves does not possess a universal elliptic curve satisfying the above property, due to the extra automorphisms for the elliptic curves with $j = 0$ and $j = 1728$. However, it satisfies a weaker property that every family of elliptic curves admits a unique morphism to it. In this case, we call it a *course moduli space*.

5.4. **Tangent space to the Hilbert scheme.** Suppose $H = \mathrm{Hilb}^p(\mathbb{P}^n)$ sits inside certain Grassmannian $G$ in the above construction. A point $[X] \in H$ defined by the ideal $I$ is given by a special codimension-$p(m)$ subspace $[I_m] \in G$, where $I_m$ is the degree $m$-piece of the ideal of $X$. Any tangent vector of $T_G|_{[X]}$ is induced by a linear map

$$\varphi : I_m \to (S/I)_m.$$

In other words, it corresponds to a collection

$$I_{\mathcal{X}} = \{f + \epsilon\varphi(f)\}_{f \in I_m}$$

defining a family $\mathcal{X} \subset [\,] \times \mathbb{P}^n$, where $\epsilon$ is the dual number and $[\,] = \operatorname{Spec} k[\epsilon]/\epsilon^2$.

When does this tangent vector belong to the tangent space of $H$ at $[X]$? The answer is if and only if $\mathcal{X}$ is *flat* over $[\,]$ by the universal property of the Hilbert scheme. This is further equivalent to that $\varphi$ extends to an $S$-module homomorphism

$$\varphi : I_{l \geq m} \to (S/I)_{l \geq m}.$$

Indeed, if this condition does *not* hold, there exist $\alpha_i \in S$ and $f_i \in I$ such that $\sum \alpha_i f_i = 0$ on the left but $\sum \alpha_i \varphi(f_i) \neq 0$ on the right. Then we have $\sum \alpha_i(f_i + \epsilon\varphi(f_i)) \neq 0$ in $I_{\mathcal{X}} \otimes_{k[\epsilon]/\epsilon^2} k$ but it will go to 0 in $S$. It implies that the exact sequence of $S \otimes k[\epsilon]/\epsilon^2$-modules

$$0 \to I_{\mathcal{X}} \to S \otimes k[\epsilon]/\epsilon^2 \to A_{\mathcal{X}} \to 0$$

is *not* flat after we tensor the $k[\epsilon]/\epsilon^2$-module $k$.

The map $\varphi : I_{l \geq m} \to (S/I)_{l \geq m}$ of $S$-modules determines a map of sheaves

$$\mathscr{I} \to \mathscr{O}_{\mathbb{P}^n}/\mathscr{I} \cong \mathscr{O}_X,$$

where $\mathscr{I}$ is the ideal sheaf of $X$ in $\mathbb{P}^n$. By $S$-linearity, the kernel contains $\mathscr{I}^2$. Define $N_{X/\mathbb{P}^n} = \hom(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X)$ to be the *normal sheaf* of $X$ in $\mathbb{P}^n$. In general, $N_{X/\mathbb{P}^n}$ may not be locally free. However, when $X$ is smooth it is the normal bundle defined as follows:

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to N_{X/\mathbb{P}^n} \to 0.$$

Putting everything together, we obtain the following fundamental result.

**Proposition 5.15.** *The tangent space of $H$ at $[X]$ is $H^0(X, N_{X/\mathbb{P}^n})$. In particular, $\dim H|_{[X]} \leq h^0(X, N_{X/\mathbb{P}^n})$ and $H$ is smooth at $[X]$ if and only if the equality holds.*

**Remark 5.16.** Indeed, one can show that the *obstruction group* of $X$ in $\mathbb{P}^n$ is contained in $H^1(X, N_{X/\mathbb{P}^n})$. Hence if $H^1(X, N_{X/\mathbb{P}^n})$ is zero, we conclude that $H$ is smooth at $[X]$, but this is only a sufficient condition.

**Example 5.17.** Consider the Hilbert scheme of degree $d$ hypersurfaces in $\mathbb{P}^n$. We know it is isomorphic to $\mathbb{P}^N$ for $N = \binom{n+d}{d} - 1$. Let us check the dimension of its tangent space. Suppose $X$ is a hypersurface of degree $d$. Since $\mathscr{I}_X = \mathscr{O}_{\mathbb{P}^n}(-d)$, we have

$$N_{X/\mathbb{P}^n} \cong \hom(\mathscr{I}_X/\mathscr{I}_X^2, \mathscr{O}_X) \cong \mathscr{O}_X(d).$$

Therefore, by the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(d) \to \mathscr{O}_X(d) \to 0,$$

we read off

$$h^0(X, N_{X/\mathbb{P}^n}) = h^0(\mathbb{P}^n, \mathscr{O}(d)) - 1 = N.$$

Here we used the fact that $h^1(\mathbb{P}^n, \mathscr{O}) = 0$.

In order to calculate $h^0(X, N)$, it would be useful to know some information about $T_{\mathbb{P}^n}$. In fact, we have

$$0 \to \mathscr{O}(-1) \to \mathscr{O}^{\oplus(n+1)} \to Q \to 0,$$

where $\mathscr{O}(-1)$ is the tautological line bundle and $Q$ is the quotient bundle of rank $n$. Tensor the sequence with $O(1)$ and we obtain the following *Euler sequence*:

$$0 \to \mathscr{O} \to \mathscr{O}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0.$$

Another useful fact is the *adjunction formula*.

**Proposition 5.18.** *Let $D \subset X$ be a smooth divisor in a smooth ambient space. Then $N_{D/X} \cong \mathscr{O}_X(D)$ and $K_D \sim (K_X + D)|_D$.*

*Proof.* The first part is a direct consequence of $N_D = \hom(\mathscr{I}/\mathscr{I}^2, \mathscr{O})$. For the other part, consider the exact sequence

$$0 \to T_D \to T_X|_D \to N_D \to 0.$$

Take the first Chern class of each term and the desired formula follows. $\square$

**Example 5.19.** Suppose $D$ is a hypersurface of degree $d$ in $\mathbb{P}^n$. Using the Euler sequence we read off $K_{\mathbb{P}^n} \cong \mathscr{O}_{\mathbb{P}^n}(-n-1)$, hence $K_D = \mathscr{O}_D(d-n-1)$.

5.5. **Hilbert scheme of twisted cubics.** A twist cubic curve $C$ has Hilbert polynomial $3m+1$. So here we consider the Hilbert scheme $H = \mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$.

Restricting the Euler sequence to $C$, we have

$$0 \to \mathscr{O}_C \to \mathscr{O}(1)^{\oplus 4}|_C \to T_{\mathbb{P}^3}|_C \to 0.$$

Then we get $\deg(T_{\mathbb{P}^3}|_C) = 12$. Using the exact sequence

$$0 \to T_C \to T_{\mathbb{P}^3}|_C \to N_{C/\mathbb{P}^3} \to 0,$$

we obtain that $\deg(N_{C/\mathbb{P}^3}) = 10$. By a result of Grothendieck, any vector bundle on $\mathbb{P}^1$ splits into a direct sum of line bundles. Suppose that $N_{C/\mathbb{P}^3} \cong \mathscr{O}(a) \oplus \mathscr{O}(b)$ for $a_1 \leq a_2$ and $a_1 + a_2 = 10$. Consider the exact sequence

$$0 \to N_{C/Q} \to N_{C/\mathbb{P}^3} \to N_{Q/\mathbb{P}^3}|_C \to 0,$$

where $Q$ is a quadric surface in $\mathbb{P}^3$ containing $C$. Since $N_{Q/\mathbb{P}^3}|_C \cong \mathscr{O}_{\mathbb{P}^1}(6)$, the sequence reduces to

$$0 \to \mathscr{O}(4) \to \mathscr{O}(a_1) \oplus \mathscr{O}(a_2) \to \mathscr{O}(6) \to 0$$

on $\mathbb{P}^1$. By the remark below, if $a_1 < 4$, then the map restricted to $\mathscr{O}(a_1)$ is zero, so the cokernel has torsion, leading to a contradiction. Therefore, we obtain that $a_1, a_2$ are either $5, 5$ or $4, 6$. In both cases, we have

$$h^0(C, N_{C/\mathbb{P}^3}) = (a+1) + (b+1) = 12.$$

Indeed, a twisted cubic is given by

$$[X, Y] \to [F_0, F_1, F_2, F_3],$$

where $F_i$'s are general homogeneous polynomials of degree 3 in $X, Y$. Modulo the automorphism group of $\mathbb{P}^1$, the dimension of the Hilbert scheme (component) parameterizing twisted cubics is (at least) equal to $4 \times 4 - 1 - 3 = 12$. In particular, it implies that $H$ is smooth at $[C]$. We can also see this by checking that $H^1(C, N_{C/\mathbb{P}^3}) = 0$.

**Remark 5.20.** Suppose $\mathscr{O}_{\mathbb{P}^1}(m) \to \mathscr{O}_{\mathbb{P}^1}(n)$ is a non-zero map. Twist $m$ and $n$ so we can assume they have vanishing $H^1$. Then its cokernel is supported at isolated points. It implies that $h^0(\mathscr{O}_{\mathbb{P}^1}(m)) \leq h^0(\mathscr{O}_{\mathbb{P}^1}(n))$, hence $m \leq n$. In the above example, the splitting type of $N_{C/\mathbb{P}^3}$ is indeed *balanced*, i.e. $\mathscr{O}(5) \oplus \mathscr{O}(5)$. This holds in general for the normal bundle of any rational normal curve.

**Exercise 5.21.** Calculate the dimension of the tangent space of the Hilbert scheme $H$ at a point parameterizing a rational normal curve $C$ in $\mathbb{P}^n$. Prove that $H$ is smooth at $C$.

The above analysis implies that the closure of the locus of twisted cubics forms an irreducible component $H_1$ of $H$, which has dimension 12. What about the other components of $H$? Consider a plane cubic $E$ union a point $p$, denoted by $Z$. It has the same Hilbert polynomial $3m + 1$. We have

$$\begin{aligned} h^0(N_{Z/\mathbb{P}^3}) &= h^0(N_{E/\mathbb{P}^3}) + 3 \\ &= h^0(N_{E/\mathbb{P}^2}) + 3 + 3 \\ &= 15 \end{aligned}$$

Moreover, the locus of such $Z$ has dimension equal to 15, since choosing $p$ has 3-dimensional freedom, choosing a plane contributes 3, and choosing $E$ in the plane contributes 9. Therefore, we conclude that the closure of this locus forms another irreducible component $H_2$ of $H$.

How do $H_1$ and $H_2$ intersect? Apparently along their common locus a twisted cubic has to degenerate to a plane singular curve, and a general such curve is a plane nodal cubic. Moreover, the isolated point in $Z$ has to approach the plane curve, and hence gives rise to an embedded point. In general, a scheme $X \subset \mathbb{A}^n$ has an *embedded component* if for some open subset $U$ of $\mathbb{A}^n$ meeting $X$ in a dense subset of $X$ the closure $X \cap U$ does not equal to $X$. Equivalently, the primary decomposition of $I_X$ contains an embedded prime. If this embedded prime is maximal, namely, if $U$ may be taken to be the complement of a point, then $X$ has an *embedded point*.

For instance, consider $I = (XZ, YZ, Z^2, Y^2W - X^2(X+W))$. The primary decomposition of $I$ is

$$I = (Z, Y^2W - X^2(X+W)) \cap (X, Y, Z)^2.$$

The corresponding scheme $X$ has reduced support equal to the plane nodal cubic $E$ defined by $(Z, Y^2W - X^2(X+W))$. Nevertheless, $X$ possesses an embedded point at the node $[0,0,0,1]$ of $E$. One checks that $X$ has Hilbert polynomial equal to $3m+1$. Moreover, $X$ can be realized as a degeneration of both twist cubics and plane cubics union an isolated point. The closure of the locus parameterizing such $X$ is indeed $H_1 \cap H_2$, which has dimension equal to $3+8=11$. In other words, it forms a divisor in $H_1$.

We summarize these remarkable results due to Piene and Schlessinger as follows.

**Theorem 5.22.** *The Hilbert scheme $H = \mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$ consists of two irreducible components $H_1$ and $H_2$, of dimension* 12 *and* 15*, respectively. Both of $H_1$ and $H_2$ are smooth and rational, they intersect transversally, and their intersection is non-singular, rational, of dimension* 11*.*

*Any subscheme $C$ parameterized in $H_1 \cap H_2$ consists of a plane cubic with a spatial embedded point, emerging from the plane, at a singular point of the cubic. Up to projective equivalence, it has ideal of the form $(XZ, YZ, Z^2, Q(X,Y,W))$, where $Q$ is a cubic form singular at $[X,Y,W] = [0,0,1]$. Moreover,* $\dim T_{[C]}H = 16$.

5.6. **Dimension of the Hilbert scheme of curves.** Let $C$ be a smooth curve of degree $d$ and genus $g$ in $\mathbb{P}^r$. By the exact sequences

$$0 \to T_C \to T_{\mathbb{P}^r}|_C \to N_C \to 0$$

and

$$0 \to \mathscr{O}_C \to \mathscr{O}_C(1)^{\oplus(r+1)} \to T_{\mathbb{P}^r}|_C \to 0,$$

we have

$$\chi(N_C) = (r+1)\chi(\mathscr{O}_C(1)) - (1-g) - (3-3g) = (r+1)d - (r-3)(g-1).$$

Call this number the *Hilbert number* and denote it by $h_{d,g,r}$. Combining the preceding section, we obtain the following result.

**Proposition 5.23.** *Let $H$ be the Hilbert scheme (component) that contains $[C]$. Then*

$$h^0(C, N_C) \geq \dim(H) \geq h_{d,g,r} = (r+1)d - (r-3)(g-1).$$

*Moreover, if $\mathscr{O}_C(1)$ is non-special, i.e. if $h^1(C, \mathscr{O}(1)) = 0$, then* $\dim(H) = h_{d,g,r}$ *and $H$ is smooth $[C]$.*

*Proof.* The only part we need to prove is if $h^1(C, \mathscr{O}(1)) = 0$, then $H$ is smooth at $[C]$. But in this case, we have $h^1(C, T_{\mathbb{P}^r}|_C) = 0$ and consequently $h^1(C, N_C) = 0$. □

**Remark 5.24.** The Hilbert number $h_{d,g,r}$ only provides an *expected* dimension. For many values of $d, g, r$, it can be even negative, which gives no information about the actual dimension of the Hilbert scheme.

**Example 5.25.** In the case $r = 3$, the Hilbert number reduces to $4d$. Hence we conclude that any Hilbert scheme component whose general point parameterizes smooth curves of degree $d$, genus $g$ in $\mathbb{P}^3$ has dimension $\geq 4d$. Note that this lower bound is attained for twisted cubics. However, in general a Hilbert scheme component parameterizing space curves may have dimension strictly bigger than $4d$.

The following is still an open problem regarding the dimension of the Hilbert scheme of curves .

**Question 5.26.** *For fixed $d, g$ and $r$, what is the lower bound for the dimension of all Hilbert scheme components whose general point parameterizes smooth curves of degree $d$, genus $g$ in $\mathbb{P}^r$?*

5.7. **Pathologies of Hilbert schemes.** The geometry of Hilbert schemes can be quite nasty, e.g. they may have many components, with arbitrarily bad singularities or even non-reduced structures. This phenomenon is formulated by Harris and Morrison as follows:

*Murphy's law for Hilbert schemes*: There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme.

Let us use the following example, due to Mumford, to explain the meaning of this law. First recall some basic facts about a smooth cubic surface $S$ in $\mathbb{P}^3$. A smooth cubic surface can be realized as the blow-up of $\mathbb{P}^2$ at 6 general points as follows. Let $\pi : S \to \mathbb{P}^2$ be the blow-up at 6 general points $p_1, \ldots, p_6$. Let $l$ be a line class and $e_i$ be the class of an exceptional curve for $1 \leq i \leq 6$. Consider the 3-dimensional linear series of plane cubics passing through $p_1, \ldots, p_6$ on $\mathbb{P}^2$. The corresponding linear system on $S$ is $\sigma = |\pi^* 3l - e_1 - \cdots - e_6|$, which is 3-dimensional and very ample. It gives rise to an embedding of $S$ into $\mathbb{P}^3$. The degree of $S$ is $D^2 = 9 - 6 = 3$ for any $D \in \sigma$, hence $S$ is a cubic surface. Moreover, $e_i \cdot D = 1$, hence these exceptional curves are lines under the embedding.

Now switch our notation. Let $H$ be a plane section of $S$ and $L$ the class of a line (an exceptional curve) contained in $S \subset \mathbb{P}^3$. We have

$$H^2 = 3, \quad L^2 = -1,$$

$$H \cdot L = 1, \quad K_S = -H.$$

The last one follows from the adjunction formula.

Consider curves $C$ in $S$ with class $4H + 2L$. The degree $d$ of $C$ is

$$d = H \cdot (4H + 2L) = 14.$$

By the adjunction formula, the genus $g$ of $C$ is

$$g = 1 + \frac{1}{2}(4H + 2L) \cdot (3H + 2L) = 24.$$

Hence we will study the Hilbert scheme parameterizing one-dimensional subschemes of degree 14 and genus 24 in $\mathbb{P}^3$.

First, using the exact sequences

$$0 \to \mathscr{O}_S(4H) \to \mathscr{O}_S(4H + L) \to \mathscr{O}_L(4H + L) \to 0,$$

$$0 \to \mathscr{O}_S(4H + L) \to \mathscr{O}_S(4H + 2L) \to \mathscr{O}_L(4H + 2L) \to 0,$$

we can deduce that $\mathscr{O}_S(4H)$ is globally generated and has vanishing $H^1$, and so are $\mathscr{O}_S(4H + L)$ as well as $\mathscr{O}_S(4H + 2L)$. In other words, the linear system $|4H + 2L|$ is base point free. Then by an argument of Bertini type, its general element represents a smooth and irreducible curve $C$ with class $4H + 2L$. By the exact sequence

$$0 \to \mathscr{O}_S \to \mathscr{O}_S(C) \to \mathscr{O}_C(C) \to 0,$$

we obtain that

$$\dim |4H + 2L| = h^0(S, \mathscr{O}(C)) - 1 = 1 - g + (4H + 2L)^2 = 37.$$

Since the family of cubic surfaces has dimension 19, we thus obtain a sublocus $J_3$ of $R$ of dimension $37 + 19 = 56$, parameterizing such curves $C$. Apparently $J_3$ is irreducible by its construction. The key question is: is $J_3$ open and dense in a component of the Hilbert scheme?

Now consider an arbitrary curve $C$ of degree 14 and genus 24 in $\mathbb{P}^3$. Since

$$\begin{aligned} h^0(C, \mathscr{O}_C(3)) &= 1 - g + 3d + h^1(C, \mathscr{O}_C(-3)) \\ &= 19 + h^0(C, K_C(-3)), \end{aligned}$$

where $K_C(-3)$ has degree $2g - 2 - 3d = 4$, if $h^0(C, K_C(-3))$ is positive, then $H^0(\mathbb{P}^3, \mathscr{O}(3)) \to H^0(C, \mathscr{O}(3))$ may or may not have a non-zero kernel. In other words, $C$ may or may not lie on a cubic surface.

Suppose $C$ does not lie on any cubic surface. We have

$$h^0(\mathbb{P}^3, \mathscr{O}(4)) = 35,$$

$$h^0(C, \mathscr{O}(4)) = 1 - g + 4d = 33,$$

hence $C$ must lie on a pencil of quartic surfaces. Since the degree of the intersection curve of two quartics is 16, $C$ is residual to a degree 2 curve $D$, i.e. a plane conic. Now reverse our engine and start from a conic $D$, which has 8-dimensional freedom. The space $\Lambda$ of quartics containing $D$ has dimension $h^0(\mathbb{P}^3, \mathscr{O}(4)) - h^0(D, \mathscr{O}(4)) - 1 = 25$. A pencil of such quartics corresponds to a point in $\mathbb{G}(1, 25)$, which has dimension 48. Altogether, we obtain a locus $J_4$ of all such $C$ and the dimension of $J_4$ is $48 + 8 = 56$. Since $J_3$ and $J_4$ have the same dimension, we obtain that a general curve of class $4H + 2L$ on a smooth cubic surface is *not* the specialization of a curve not lying on a cubic. This implies that the closure of $J_3$ is an irreducible component of the Hilbert scheme.

Finally, let us return to $C$ contained in a cubic surface $S$. By the exact sequence

$$0 \to N_{C/S} \to N_{C/\mathbb{P}^3} \to N_{S/\mathbb{P}^3}|_C \to 0,$$

we obtain that

$$0 \to \mathscr{O}_C(4H + 2L) \to N_{C/\mathbb{P}^3} \to \mathscr{O}_C(3) \to 0.$$

Since $K_C = \mathscr{O}_C(3H + 2L)$, we conclude that $N_{C/S} = \mathscr{O}_C(4H + 2L)$ is non-special. Hence

$$\begin{aligned} h^0(C, N_C) &= h^0(C, \mathscr{O}(4H + 2L)) + h^0(C, \mathscr{O}(3)) \\ &= 1 - g + (4H + 2L)^2 + 19 + h^0(C, \mathscr{O}(2L)) \\ &= 57. \end{aligned}$$

Thus $J_3$ is singular at $[C]$. But $C$ represents a general point in $J_3$, hence $J_3$ is non-reduced! In summary, any actual deformation of $C$ lies on a cubic surface, but there exists a first-order infinitesimal deformation of $C$ which does not lie on a cubic surface, and it accounts for the extra dimension in the tangent space to the Hilbert scheme.

5.8. **Hilbert scheme of complete intersections.** Besides hypersurfaces, the next simple example of subschemes in $\mathbb{P}^n$ arises from taking complete intersection of hypersurfaces. For instance, let $F$ and $G$ are two general surfaces of degree $m$ and $n$ in $\mathbb{P}^3$ for $m \leq n$. Let $C$ be the intersection of $F$ and $G$. In other words, $C$ had ideal $I_C = (F, G)$. The degree of $C$ is $d = mn$. In order to calculate its genus, we use the adjunction formula for $C \subset F$:

$$2g - 2 = (K_F + C) \cdot C = (m-4)mn + mnn = mn(m+n-4),$$

hence we conclude that

$$g = 1 + \frac{1}{2}mn(m+n-4).$$

We can get more precise information regarding the normal bundle of $C$. Consider the exact sequence

$$0 \to \mathscr{O}(-m-n) \to \mathscr{O}(-m) \oplus \mathscr{O}(-n) \to I_C \to 0.$$

The left map is given by $A \mapsto (AG, -AF)$ and the right map is given by $(B, C) \mapsto BF + CG$. Dualizing it by $\hom_{\mathscr{O}_C}(\cdot, \mathscr{O}_C)$. Since both $F$ and $G$ are vanishing on $C$, the transpose of the first map vanishes. We thus obtain

$$0 \to N_{C,\mathbb{P}^3} \to \mathscr{O}_C(m) \oplus \mathscr{O}_C(n) \to 0.$$

Therefore, we conclude that

$$N_{C,\mathbb{P}^3} \cong \mathscr{O}_C(m) \oplus \mathscr{O}_C(n).$$

Note that the line bundle $\mathscr{O}_C(m)$ has degree $m^2n$, which could be smaller than $2g-2$, hence the normal bundle of $C$ may have non-zero $H^1$.

Let $H$ be the locus in the corresponding Hilbert scheme parameterizing such complete intersections $C$. Apparently $H$ is irreducible. A naive dimension count says that

$$\dim H = \binom{m+3}{3} - 1 + \binom{n-m+3}{3} - 1.$$

Although $H^1(C, N_{C/\mathbb{P}^3})$ could be non-zero, using the map

$$\mathbb{P}^{\binom{m+3}{3}-1} \times \mathbb{P}^{\binom{n+3}{3}-1} \dashrightarrow H,$$

we can still conclude the smoothness of $H$. In other words, a complete intersection is always *unobstructed*.

**Exercise 5.27.** Find a complete intersection $C$ in $\mathbb{P}^3$ such that $H^1(C, N_{C/\mathbb{P}^3}) \neq 0$.

5.9. **Hilbert scheme of determinantal varieties.** We first briefly introduce determinantal varieties. Let $M = M(m,n)$ be the space of $m \times n$ matrices. As a variety, $M$ is isomorphic to $\mathbb{A}^{mn}$. For $0 \leq k \leq \min(m,n)$, denote by $M_k = M_k(m,n)$ the locus of matrices of rank at most $k$, namely, $M_k$ is cut out by all the $(k+1) \times (k+1)$ minors. We say that $M_k$ is the *kth generic determinantal variety.*

**Proposition 5.28.** *$M_k$ is an irreducible subvariety of codimension $(m-k)(n-k)$ in $M$.*

*Proof.* Regard an $m \times n$ matrix $A$ as a linear map $\mathbb{C}^n \to \mathbb{C}^m$. Then $\operatorname{rank}(A) \leq k$ if and only if $\dim \ker(A) \geq n - k$. Define the incidence correspondence

$$\widetilde{M}_k = \{(A, W) \mid A \cdot W = 0\} \subset M \times G(n-k, n).$$

Then $\widetilde{M_k}$ admits two projections $p_1$ and $p_2$ to $M$ and $G(n-k,n)$, respectively. The map $p_1$ is onto $M_k$ and generically one to one. In other words, $p_1$ has positive dimensional fiber over $A$ if and only if the rank of $A$ is at most $k-1$. On the other hand, for a fixed $W \in G(n-k,n)$, the fiber of $p_2$ over $W$ is isomorphic to $\mathbb{A}^{mk}$. It implies that $\widetilde{M_k}$ is a vector bundle on $G(n-k,n)$, hence is irreducible. So is $M_k$. Moreover, we conclude that

$$\begin{aligned}\dim M_k &= \dim \widetilde{M_k} \\ &= \dim G(n-k,n) + mk \\ &= (m+n)k - k^2.\end{aligned}$$

Therefore, the codimension of $M_k$ in $M$ is equal to

$$mn - (m+n)k + k^2 = (m-k)(n-k).$$

□

**Remark 5.29.** As a vector bundle, $\widetilde{M_k}$ is smooth, but the contraction $M_k$ could be singular. With a more detailed study of the tangent space, one can show that the singular locus of $M_k$ is exactly $M_{k-1}$.

Now consider an $n$ by $(n+1)$ matrix $M$ whose entries are general linear forms in $\mathbb{P}^3$. Let $C$ be the subscheme cut out by the maximal minors of $M$. By Proposition 5.28, the codimension of $C$ in $\mathbb{P}^3$ is 2, hence $C$ is a curve.

**Example 5.30.** A twisted cubic $C$ in $\mathbb{P}^3$ is a determinantal curve. The ideal of $C$ can be generated by the 2 by 2 minors of

$$\begin{pmatrix} X & Y & Z \\ Y & Z & W \end{pmatrix}.$$

We have the following resolution of $I_C$:

$$0 \to \mathscr{O}_{\mathbb{P}^3}^{\oplus 2}(-3) \to \mathscr{O}_{\mathbb{P}^3}^{\oplus 3}(-2) \to I_C \to 0$$

where the left map is multiplication by

$$\begin{pmatrix} X & Y & Z \\ Y & Z & W \end{pmatrix}$$

and the right one is multiplication by

$$\begin{pmatrix} YW - Z^2 \\ -XW + YZ \\ XZ - Y^2 \end{pmatrix}.$$

In this case, we know $C$ has degree 3 and genus 0.

In general, let us calculate the degree and genus of $C$. Note that we have the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^3}^{\oplus n}(-n-1) \to \mathscr{O}_{\mathbb{P}^3}^{\oplus n+1}(-n) \to I_C \to 0.$$

Twist it by $m \gg 0$. Then we have

$$h^0(C, I_C(m)) = (n+1)\binom{m-n+3}{3} - n\binom{m-n+2}{3}.$$

Consequently the Hilbert polynomial of $C$ equals

$$h^0(\mathbb{P}^3, \mathscr{O}(m)) - h^0(C, I_C(m)) = \frac{1}{2}(n^2+n)m - \frac{1}{6}(2n^3 - 3n^2 - 5n).$$

We thus obtain the degree and genus of $C$ as follows:

$$d = \frac{1}{2}(n^2 + n),$$

$$g = 1 + \frac{1}{6}(2n^3 - 3n^2 - 5n).$$

Indeed, one can prove that the locus of such curves is open in the Hilbert scheme. Let $U$ be the corresponding Hilbert component. The dimension of $U$ is

$$4n(n+1) - 1 - \dim \mathrm{PGL}(n) - \dim \mathrm{PGL}(n+1) = 4d.$$

In other words, $U$ has the expected dimension as predicted by the Hilbert number $h_{d,g,3} = 4d$.

**Exercise 5.31.** Define similarly a determinantal surface in $\mathbb{P}^4$ and calculate its Hilbert polynomial.

## 6. Moduli space of curves

Let $\mathcal{M}_g$ denote the moduli space of smooth, connected and complete genus $g$ curves, and $\overline{\mathcal{M}}_g$ the Deligne-Mumford compactification of $\mathcal{M}_g$ by adding stable nodal curves in the boundary. In this section, we will give an elementary introduction to the construction and basic properties of $\overline{\mathcal{M}}_g$.

6.1. **Geometric Invariant Theory.** One approach to construct the moduli space of curves is by GIT (the Geometric Invariant Theory). The idea is to embed curves of genus $g$ into a projective space $\mathbb{P}^n$ in an uniform way. Then consider the corresponding Hilbert scheme and take its quotient by modulo $\mathrm{PGL}(n+1)$.

Consider a motivating example first. Let $\mathbb{P}^9$ be the space of plane cubics. Then $\mathrm{GL}(3)$ acts on the coordinates of $\mathbb{P}^2$, which induces an action on $\mathbb{P}^g$. We naturally expect the quotient space to parameterize genus one curves as a compactification of the $j$-line. However, we expect more, e.g. the quotient better has an algebraic structure, say, a scheme. In addition, plane cubics can be very singular or even non-reduced, so we do not want all of the singular cubics to show up in the boundary of the compactification. Finally, we want the quotient to be a moduli space, i.e. with certain universal property as discussed before.

The major technique in getting an algebraic quotient is the GIT (Geometric Invariant Theory) approach, see Harris-Morrison's book, Chapter 4. In general, suppose a group $G$ acts on a variety $K$. We want to construct a reasonable quotient $q : K \dashrightarrow K/G$ such that $K/G$ is a scheme. Let $X = \mathrm{Spec}(R)$ is a $G$-invariant affine subscheme of $K$ and $R^G$ is the subring of $G$-invariant elements of $R$. Define $X/G = \mathrm{Spec}(R^G)$. The restriction of $q$ to $X$ should be induced by $R^G \hookrightarrow R$. Then we patch all these local characterizations together. In other words, if $\mathbb{C}[K]$ is the homogeneous coordinate ring of $K$ and $\mathbb{C}[K]^G$ is the subring of $G$-invariants, then we would like to take $K/G = \mathrm{Proj}(\mathbb{C}[K]^G)$.

There are several technical issues in this naive definition. Among them one is to make sure the subring $\mathbb{C}[K]^G$ is finitely generated, otherwise taking Proj is not well-defined. Luckily this is not a problem for a reductive group $G$. A *reductive* group is an algebraic group $G$ over an algebraically closed field such that the unipotent radical of $G$ is trivial (i.e. the group of unipotent elements of the radical of $G$). Recall that an element $a$ in a group is called *unipotent* if for some $m$ we have $(a-1)^m = 0$. The *radical* of an algebraic group is the identity component of its maximal normal solvable subgroup. In our applications we will only use $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, which are reductive.

**Theorem 6.1** (Hilbert-Weyl-Haboush). *If a reductive group $G$ acts algebraically on a noetherian ring $R$, then $R^G$ is finitely generated.*

A group $G$ is called *geometrically reductive* if for any representation of $G$, there is a non-constant invariant homogeneous polynomial not identically 0 on the invariant subspace. A reductive group is geometrically reductive, proved in general by Haboush.

**Theorem 6.2** (Hilbert-Nagata). *If $G$ is geometrically reductive and $W$ is a representation of $G$, then values of homogeneous invariant polynomials separate disjoint closed $G$-invariant subsets of $\mathbb{P}W$.*

**Corollary 6.3.** *The quotient map $q : \mathbb{P}W \dashrightarrow \mathbb{P}W/G$ has base locus exactly the set of points $[x] \in \mathbb{P}W$ such that for any non-zero lifting $x$ of $[x]$ to $W$, the origin $0$ of $W$ lies in the closure $\overline{G \cdot x}$ of the $G$-orbit of $x$.*

In this sense, we call the base locus of $q$ the *non-semistable locus* (or *unstable locus*) and denote it by $\mathbb{P}W_n$. Call its complement the *semistable locus* and denote it by $\mathbb{P}W_{ss}$. Moreover, a point $[x]$ is called *stable* if the orbit $G \cdot x$ of any lifting is closed (i.e. $[x] \in \mathbb{P}W_s$) and if the stabilizer $\mathrm{stab}_G(x)$ is of minimal dimension among all stabilizers of points in $\mathbb{P}W$. The locus of stable points is called the *stable locus* and denoted by $\mathbb{P}W_s$. For a subvariety $K \subset \mathbb{P}W$, we define analogous loci in $K$ by restricting to $K$ from $\mathbb{P}W$. In all the examples we will consider, the minimal dimension of a stabilizer will be finite, hence for simplicity from now on we assume that $K$ always contains points with finite stabilizers.

**Corollary 6.4.** *(1) Two points $[x]$ and $[y]$ in $K_s$ lie in the same $G$-orbit if and only if $q([x]) = q([y])$.*

*(2) If $[x]$ and $[y]$ are in $K_{ss}$, then $q([x]) = q([y])$ if and only if*

$$\overline{G \cdot x} \cap \overline{G \cdot y} \cap K_{ss} \neq \emptyset.$$

The above corollaries show that we can only hope to have a quotient whose closed points correspond to orbits in $K$ beneath the semistable locus of $K$. Therefore, the fundamental problem in the GIT approach is then to determine the stable and strictly semistable loci. Luckily there is a numerical criterion due to Mumford.

To see how it works, consider a simple case $G = \mathbb{C}^*$. Then we have

$$\mathbb{C}[\mathbb{C}^*] = \mathbb{C}[t, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}t^i.$$

Since $\mathbb{C}^*$ is abelian, all its irreducible representations are one-dimensional, and hence its characters correspond bijectively with the integers. As a result, we can decompose any finite-dimensional representation $\lambda : \mathbb{C}^* \to \mathrm{GL}(W)$ of $\mathbb{C}^*$ as

$$W = \bigoplus_{i \in S_\lambda(W)} W_i$$

where $W_i$ is the set of vectors $w \in W$ on which $\mathbb{C}^*$ acts by $\lambda(t) \cdot w = t^i \cdot w$ and $S_\lambda(W)$ is the finite set of integers for which $W_i$ is non-zero.

We call the element $i$ of $S_\lambda(W)$ the *$\lambda$-weights* of $W$ and the space $W_i$ the *ith $\lambda$-weight space* of $W$. For $x \in W$, define $S_\lambda(x)$ to be the set of $i$ in $S_\lambda(W)$ for which the component $x_i$ of $x$ in the decomposition subspace $W_i$ is non-zero. Call the elements of $S_\lambda(x)$ the $\lambda$-weights of $x$ and define

$$\mu_\lambda(x) = \min(S_\lambda(x)).$$

We also define these invariants for $[x] \in \mathbb{P}W$ by using their common value on any lifting $x$ of $[x]$.

More concretely, the above amounts to choosing a basis $B = \{b_1, \ldots, b_N\}$ of $W$ such that for $t \in \mathbb{C}^*$, $\lambda(t)$ acts on $W$ by the diagonal matrix

$$\operatorname{diag}(t^{w_1}, \ldots, t^{w_N}).$$

The subspace $W_i$ is just the span of those $b_j$ for which $w_j = i$, so $S_\lambda(W)$ is the set of distinct $w_j$'s and $S_\lambda(x)$ is the set of distinct $w_j$'s for which the $j$th coordinate of $x$ in terms of the basis $B$ is non-zero.

Using this interpretation, we can write

$$\lambda(t) \cdot x = \sum_{i \in S_\lambda(x)} t^i \cdot x_i.$$

Then we conclude the following equivalences:

$$\mu_\lambda(x) \geq 0 \Leftrightarrow \lim_{t \to 0} \lambda(t) \cdot x \text{ exists} \Leftrightarrow x \text{ is not stable;}$$
$$\mu_\lambda(x) > 0 \Leftrightarrow \lim_{t \to 0} \lambda(t) \cdot x = 0 \Leftrightarrow x \text{ is not semistable.}$$

The two left equivalences are obvious. The limit in the middle on the first line is either $x$ (in which case $x = x_0$ hence all of $\lambda(\mathbb{C}^*)$ lies in $\operatorname{stab}(x)$, which therefore has positive dimension) or a point of the closure of the $\lambda$-orbit of $x$ not lying in that orbit (in which case the orbit is not closed). These possibilities correspond exactly to the two ways in which $x$ can fail to be stable. The right equivalence of the second line follows from the definition of semistability.

Mumford's idea was that this easy case actually takes care of the general case. We call an algebraic group homomorphism $\lambda : \mathbb{C}^* \to G$ a *one-parameter subgroup* of $G$ and $\lambda$ is called *non-trivial* if its image is. We thus obtain the following criterion.

**Theorem 6.5** (Hilbert-Mumford Criterion)**.** *(1) $[x]$ is non-semistable if and only if for some one-parameter subgroup $\lambda$ of $G$, $\mu_\lambda([x]) > 0$.*

*(2) $[x]$ is semistable if and only if for every one-parameter subgroup $\lambda$ of $G$, $\mu_\lambda([x]) \leq 0$.*

*(3) $[x]$ is non-stable if and only if for some non-trivial one-parameter subgroup $\lambda$ of $G$, $\mu_\lambda([x]) \geq 0$.*

*(4) $[x]$ is stable if and only if for every non-trivial one-parameter subgroup $\lambda$ of $G$, $\mu_\lambda([x]) < 0$.*

We often rephrase the criterion as that $[x]$ is stable (resp. semistable) if and only if every non-trivial one-parameter subgroup $\lambda$ of $G$ acts on it with a negative (resp. non-positive) weight. Since the weights of the subgroup $\lambda^{-1}$ are minus those of $\lambda$, one can formulate the criterion more symmetrically as that $[x]$ is stable (resp. semistable) if and only if every non-trivial one-parameter subgroup of $G$ acts on it with both positive and negative (resp. non-negative and non-positive) weights.

In summary, if the orbit $G \cdot x$ is not closed, then it is possible to find a disc $\Delta$ in $\mathbb{P}W$ such that $\Delta^* \subset G \cdot x$ and 0 lies in $\overline{G \cdot x}$ but not in $G \cdot x$. The criterion says that we can actually take this disc to be the image of the unit circle in a one-parameter subgroup of $G$ under the map $g \mapsto g \cdot x$.

6.2. **Stability of plane curves.** Let us consider plane curves of degree $d$. The space $K = \mathbb{P}\operatorname{Sym}^d(\mathbb{C}^3)^\vee$ parameterizes all plane curves of degree $d$. The group $G$ is thus $\mathrm{SL}(3)$, and the representation $W$ is $\operatorname{Sym}^d(\mathbb{C}^3)^\vee$, i.e., the degree $d$ piece of the homogeneous coordinate ring of $\mathbb{P}^2$.

Fix a one-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{SL}(3)$. Then there are homogeneous coordinates $X, Y, Z$ on $\mathbb{C}^3$ such that $\lambda(t)$ acts like

$$\lambda(t) = \operatorname{diag}(t^a, t^b, t^c).$$

Moreover, since $\det(\lambda(t)) = 1$, we have $a + b + c = 0$. Acting on a monomial of degree $d$, we have

$$\lambda(t) \cdot X^i Y^j Z^k = t^{ai+bj+ck} X^i Y^j Z^k.$$

If $C$ is a degree $d$ plane curve with defining equation

$$F(X, Y, Z) = \sum_{i+j+k=d} c_{ijk} X^i Y^j Z^k = 0,$$

then using the monomial basis $S_\lambda(F)$ is just the set of monomials whose coefficients in $F$ are non-zero. Define $L(i, j, k) = ai + bj + ck$ with respect to the action of $\lambda$. By Theorem 6.5, $C$ is $\lambda$-stable (resp. semistable) if and only if for some monomial $X^i Y^j Z^k$ in $S_\lambda(F)$ we have $L(i, j, k) < 0$ (resp. $\leq 0$).

Now we return to the case $d = 3$, i.e. plane cubics. Take $(a, b, c) = (-5, 1, 4)$. In order to get $-5i + j + 4k \leq 0$, the possible $(i, j, k)$-triples are $(3, 0, 0)$, $(2, 1, 0)$, $(2, 0, 1)$, $(1, 2, 0)$ and $(1, 1, 1)$. Without loss of generality, suppose that $C$ contains the point $P = [1, 0, 0]$, i.e. $c_{300} = 0$. Below we use an $*$ to denote a general non-zero form and $\langle\rangle$ to denote a general element of an ideal. Starting from $(3, 0, 0)$, we gradually let the coefficients of these monomials to be zero one by one.

$c_{300} = 0$: $F = \langle Y, Z\rangle$, so $P$ is contained in $C$.

$c_{210} = 0$: $F = *X^2 Z + \langle Y, Z\rangle^2$, so $Z = 0$ is tangent to $C$ at $P$.

$c_{201} = 0$: $F = \langle Y, Z\rangle^2$, so $P$ is a double point of $C$.

$c_{120} = 0$: $F = *XYZ + *XZ^2 + \langle Y, Z\rangle^3$, so $Z = 0$ is tangent to a branch of $C$ at the node $P$.

$c_{111} = 0$: $F = *XZ^2 + \langle Y, Z\rangle^3$, so $P$ is a cusp of $C$ with tangent cone defined by $Z^2 = 0$.

$c_{102} = 0$: $F = \langle Y, Z\rangle^3$, so $P$ is a triple point of $C$.

Combining with the above analysis, it implies that such $C$ is $\lambda$-stable (resp. $\lambda$-semistable) if and only if $C$ has at worst a node at $P$ such that $Z = 0$ is not tangent to a branch at $P$ (resp. $C$ has at worst a node at $P$). Now rotating the line $L$ around the central point $(1, 1, 1)$ and changing coordinates, we conclude that $C$ is stable (resp. semistable) if and only if $C$ is smooth (resp. has worst nodes).

This example illustrates a very interesting connection between curve stability and curve singularities. Roughly speaking, more singular curves are less stable. Nevertheless, in general it is quite non-trivial to classify all stable/semistable objects for a GIT problem.

**Exercise 6.6.** Prove that a quartic plane curve with at worst cusps are semistable under the action $\mathrm{SL}(3)$ on the space of plane quartics.

**Exercise 6.7.** Let $W = \mathbb{P}\operatorname{Sym}^d(\mathbb{C}^2)^\vee$ be the space of degree $d$ homogeneous polynomials $F$ in two variables $X$ and $Y$ up to rescaling. In other words, $W$ is the linear system $|\mathscr{O}_{\mathbb{P}^1}(d)| \cong \mathbb{P}^d$, where $[X, Y]$ is the homogeneous coordinate of $\mathbb{P}^1$. Consider $\mathrm{SL}(2)$ acting on $\mathbb{C}^2 = \{(X, Y)\}$ and the induced action on $W$.

(1) Classify the stable, strictly semistable and non-semistable elements $F$ under this action.

(2) Note that the roots of $F$ uniquely determine $F$ up to rescaling, hence this GIT problem is relevant to the construction of a moduli space parameterizing the isomorphism classes of $n$ unordered points (not necessarily distinct) on $\mathbb{P}^1$. In this sense, interpret your results in terms of the roots of $F$.

6.3. **nodal curves.** As we have seen, the more singular a curve is, the less stable it can possibly be. The simplest curve singularity is nodal singularity. A *node* is where two smooth branches of a curve intersect transversally, i.e. locally it is defined by $xy = 0$. In this section we introduce some basic properties of nodal curves.

Let $C$ be a connected nodal curve with $n$ nodes $p_1, ...p_n$ and $k$ irreducible components $C_1, \ldots, C_k$, where $C_i$ is possible nodal and has geometric genus $g_i$. Let $\widetilde{C}$ and $\widetilde{C}_i$ be the normalization of $C$ and $C_i$, respectively. Let $\pi : \widetilde{C} \to C$ the normalization map. We have the exact sequence

$$0 \to \mathscr{O}_C \to \pi_* \mathscr{O}_{\widetilde{C}} \to \sum_{j=1}^{n} \mathbb{C}_{p_j} \to 0.$$

Moreover, $H^i(\widetilde{C}, \mathscr{O}) \cong H^i(C, \pi_* \mathscr{O}_{\widetilde{C}})$ for all $i$, which is due to the fact that $R^i \pi_* \mathscr{O}_{\widetilde{C}} = 0$ for all $i > 0$. We thus conclude the following.

**Proposition 6.8.** *In the above setting, the arithmetic genus of $C$ is*

$$g = \sum_{i=1}^{k} g_i + n - k + 1.$$

*Proof.* Taking the cohomology of the exact sequence, we have

$$\begin{aligned} 1 - g &= \chi(\mathscr{O}_{\widetilde{C}}) - n \\ &= \sum_{i=1}^{k} (1 - g_i) - n \\ &= k - \sum_{i=1}^{k} g_i - n. \end{aligned}$$

The desired formula follows immediately. $\square$

We say that a node is *separating* if its removal disconnects the whole curve. Otherwise we say that it is *non-separating*. The above Proposition can be restated as that adding a separating node does not change the arithmetic genus while adding a non-separating node increases the genus by one.

For a smooth and connected curve, the canonical line bundle plays a central role in determining the geometry of $C$. There is an analogue of canonical line bundle, called the dualizing sheaf $\omega_C$ for a nodal curve $C$. Let $q_i, r_i$ be the pre-images of a node $p_i$ in $\widetilde{C}$. Then $\omega_C$ can be identified as the pushforward of the *sheaf of rational differentials* on $\widetilde{C}$. In other words, its sections on an open subset $U \subset C$ are rational one-forms $\eta$ on $\pi^{-1}(U) \subset \widetilde{C}$ with at worst simple poles at $q_i$ and $r_i$ lying over each $p_i \in U$ and such that $\mathrm{Res}_{q_i}(\eta) + \mathrm{Res}_{r_i}(\eta) = 0$.

**Proposition 6.9.** *Let $C$ be a connected nodal curve of arithmetic genus $g$. Then $\omega_C$ is a line bundle of degree $2g-2$ and $h^0(C,\omega)=g$.*

*Proof.* Let us treat the case when $C$ has only one node $p$. The general case follows by induction. At the node $p$, suppose the two branches have local coordinates $x$ and $y$. Differentiating the relation $xy=0$, we have

$$\frac{dx}{x} = -\frac{dy}{y}.$$

Then locally $\omega_C$ at $p$ is a free $\mathscr{O}$-module with one generator, say $\frac{dx}{x}$. Therefore, $\omega_C$ is an invertible sheaf.

For its degree and global sections, first suppose $p$ is a separating node. Then $C$ consists of two irreducible, smooth components $C_1$ and $C_2$ of genus $g_1$ and $g_2$, respectively, intersecting transversally at $p$, such that $g_1+g_2=g$. By definition, $\omega_C|_{C_i} \cong K_{C_i}(p)$ is the canonical line bundle of $C_i$ tensor with $\mathscr{O}_{C_i}(p)$. Therefore, we have

$$\deg(\omega_C) = \deg(\omega_1) + \deg(\omega_2) + 2 = g.$$

Moreover, it is easy to see that $p$ is a base point of $K_{C_i}(p)$. Hence

$$H^0(C,\omega) \cong H^0(C_1,\omega) \oplus H^0(C_2,\omega) \cong \mathbb{C}^{g_1} \oplus \mathbb{C}^{g_2} \cong \mathbb{C}^g.$$

Next, consider the case when $p$ is a non-separating node. Let $\pi : \widetilde{C} \to C$ be the normalization and $\pi^{-1}(p) = \{q,r\}$. Then the genus of $\widetilde{C}$ is $g-1$. A section of $\omega_C$ can be identified with a section of $K_{\widetilde{C}}(q+r)$ that satisfies the residue condition at $q$ and $r$. It implies that

$$\deg(\omega_C) = \deg(K_{\widetilde{C}}) + 2 = 2g-4+2 = 2g-2.$$

Moreover, we have

$$H^0(C,\omega) \cong H^0(C, \pi_* K_{\widetilde{C}}(q+r)) \cong H^0(\widetilde{C}, K_{\widetilde{C}}(q+r)) \cong \mathbb{C}^g.$$

□

In general, the dualizing sheaf satisfies the Serre duality. As a consequence, for a connected nodal curve $C$, the *Riemann-Roch formula* holds as follows:

$$h^0(C,L) - h^0(C, \omega_C \otimes L^{-1}) = 1 - g + \deg(L).$$

In order to use nodal curves to compatify the moduli space of smooth curves, we have to take care of the boundedness issue. For instance, one can keep adding $\mathbb{P}^1$ tails to a curve while keeping the arithmetic genus unchanged, but we may not want to parameterize curves with unbounded number of components. Note that for a smooth connected curve of genus $\geq 2$, its automorphism group is finite. For an elliptic curve, fixing one point on it makes the automorphism group finite. Finally, any automorphism fixing three points on $\mathbb{P}^1$ has to be the identity, but fixing only two points on $\mathbb{P}^1$ does not give rise to a finite automorphism group. This simple analysis motivates the following definition.

We say a point in the normalization of a nodal curve is *special* if its image is a node. A connected nodal curve is called (moduli) *stable* if the normalization of any of its genus zero (resp. one) components contains at least one (resp. three) special points. Note that here the stability condition is not (yet) defined under a GIT setting.

**Exercise 6.10.** Verify that a stable nodal curve has finite automorphism group.

There is another equivalent definition of stable nodal curves in terms of the ampleness of the dualizing sheaf, which has advantage of generalizing to other cases.

**Proposition 6.11.** *A connected nodal curve $C$ is stable if and only if $\omega_C$ is ample.*

*Proof.* It suffices to show that $C$ is stable if and only if $\omega_C$ has positive degree restricted to every irreducible component $Z$ of $C$. Suppose $Z$ intersects $\overline{C\backslash Z}$ at $k$ nodes. Then $\deg(\omega_C|_Z) = \deg(\omega_Z)+k = 2g_Z-2+k$. If $g_Z \geq 2$, then $\deg(\omega_C|_Z) > 0$. If $g_Z = 1$, then $\deg(\omega_C|_Z) > 0$ if and only if $k \geq 1$. If $g_Z = 0$, then $\deg(\omega_C|_Z) > 0$ if and only if $k \geq 3$. $\square$

**Exercise 6.12.** Find an example of a nodal curve whose dualizing sheaf is not very ample.

With a little more work, one can show that $\omega_C^{\otimes n}$ is very ample for any stable nodal curve $C$ and every $n \geq 3$.

Moreover, if we have a family of curves $\pi : \mathcal{X} \to B$, one can define the *relative dualizing sheaf* $\omega_{\mathcal{X}/B}$ as an invertible sheaf on $\mathcal{X}$, whose dual satisfies the following exact sequence:

$$0 \to \pi^* T_B \to T_{\mathcal{X}} \to \omega^{\vee}_{\mathcal{X}/B} \to 0,$$

where $T_B$ and $T_{\mathcal{X}}$ denote the tangent bundles of $B$ and $\mathcal{X}$, respectively. Note that for each fiber $C$ in the family, $\omega_{\mathcal{X}/B}|_C \cong \omega_C$.

6.4. **Construction of $\overline{\mathcal{M}}_g$.** The idea is to use the Hilbert scheme of $n$-canonical curves and take the corresponding GIT quotient. Let us consider a general case first. Suppose a Hilbert scheme component $\mathcal{H}$ parameterizes curves of degree $d$ and genus $g$ in $\mathbb{P}^r$. Take $m \gg 0$ in the construction of the Hilbert scheme such that using the degree $m$ piece of the ideals we have an embedding $\mathcal{H}$ into a Grassmannian $\mathcal{G} = G(P(m), N)$, where $P(m) = 1-g+dm$ and $N = \binom{r+m}{m}$. Further embed $\mathcal{G}$ into a projective space $\mathbb{P}W$ by using the Plücker basis. Then the $\mathrm{SL}(r+1)$ action on the monomials of degree $m$ induces an action on the representation $W$. The question boils down to study stable and semistable loci of the action restricted to the image of $\mathcal{H}$ in $\mathbb{P}W$. Under this setting, we call a curve $C$ in the stable (resp. semistable) locus as *$m$th Hilbert stable* (resp. semistable). If $C$ is $m$th Hilbert stable for all $m \gg 0$, we say that $C$ is *Hilbert stable*. In what follows we present a fundamental result due to Gieseker.

**Theorem 6.13** (Stability of smooth curves of high degree)**.** *Suppose that $C$ is a smooth curve of genus $g \geq 2$ embedded in $\mathbb{P}^r$ by a complete linear system $|L|$ of degree $d \geq 2g$. Then $C$ is Hilbert stable. Moreover, there exists a uniform bound $M$ for all such curves $C$ such that for all $m \geq M$, $C$ is $m$th Hilbert stable.*

Now fix $g \geq 2$ and $n \geq 5$. Consider the $n$-canonical embedding of $C$ to $\mathbb{P}^r$, where $r = (g-1)(2n-1)-1$. Let $\mathrm{SL}(r+1)$ act on the corresponding Hilbert scheme component of $n$-canonical genus $g$ curves.

**Theorem 6.14.** *The Hilbert semistable $n$-canonical curves are all Hilbert stable, consisting of every (moduli) stable nodal curves of genus $g$.*

Define $\overline{\mathcal{M}}_g$ to be the GIT quotient of the Hilbert scheme component of the $n$-canonical curves for $n \geq 4$. We remark that $\overline{\mathcal{M}}_g$ is the *course moduli scheme* of the

Deligne-Mumford compactification of $\mathcal{M}_g$. Moreover, $\overline{\mathcal{M}}_g$ is projective, irreducible, and has dimension $3g-3$.

**Remark 6.15.** For $n=3$, Schubert showed that the GIT quotient of the Hilbert scheme of tri-canonical curves gives rise to a moduli space different from $\overline{\mathcal{M}}_g$, parameterizing curves with at worst cusps but without elliptic tails.

6.5. **Picard group of $\overline{\mathcal{M}}_g$.** Let $\Delta = \overline{\mathcal{M}}_g - \mathcal{M}_g$ denote the boundary of $\overline{\mathcal{M}}_g$ parameterizing stable curves of genus $g$ with at least one node. It is not hard to see that $\Delta$ has codimension one in $\overline{\mathcal{M}}_g$ and it breaks into a number of components $\Delta_0, \ldots, \Delta_{[g/2]}$, where $\Delta_i$ is the closure of the locus parameterizing two curves of genus $i$ and $g-i$ attached at a node for $1 \leq i \leq [g/2]$ and $\Delta_0$ is the closure of the locus parameterizing an irreducible one-nodal curve of geometric genus $g-1$.

**Exercise 6.16.** Assume you know that $\overline{\mathcal{M}}_g$ is irreducible and has dimension equal to $3g-3$ for every $g \geq 2$. Prove that $\Delta_i$ is an irreducible divisor in $\overline{\mathcal{M}}_g$ for $0 \leq i \leq [g/2]$.

Alternatively, a stable nodal curve is parameterized in $\Delta_0$ if and only if it possesses a non-separating node. A stable nodal curve is parameterized in $\Delta_i$ for $i > 0$ if and only if it possesses a separating node such that the two branches of the curve separated by the node has arithmetic genus $i$ and $g-i$, respectively. Note that the boundary divisors can intersect each other or self intersect.

**Exercise 6.17.** Draw a nodal curve that lies in the intersection of $\Delta_0$, $\Delta_i$ and $\Delta_j$ for $i+j \leq g-1$.

Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_g$ be the universal curve. Define the *Hodge bundle* $\mathbb{E}$ as $\pi_*\omega$, where $\omega$ is the relative dualizing sheaf of $\pi$. Since $h^0(C, \omega_C) = g$ and $h^1(C, \omega_C) = 1$ for every stable genus $g$ curve $C$, $\mathbb{E}$ is a vector bundle of rank $g$. Define the *Hodge class* as

$$\lambda = c_1(\mathbb{E}).$$

We remark that since $\overline{\mathcal{M}}_g$ in general is not a fine moduli space, this universal curve might exist only up to a finite base change. Nevertheless, in what follows we will only consider the Picard group of $\overline{\mathcal{M}}_g$ over $\mathbb{Q}$, hence we may pretend that $\mathcal{C}$ exists. Alternatively, one can define the divisor class $\lambda$ associated to any family of stable nodal curves where the universal curve exists. We also use $\delta_i$ to denote the divisor class of the boundary divisor $\Delta_i$ for $0 \leq i \leq [g/2]$.

**Theorem 6.18** (The rational Picard group of $\overline{\mathcal{M}}_g$)**.** *For $g \geq 3$, the divisor classes $\lambda, \delta_0, \ldots, \delta_{[g/2]}$ freely generate* $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$.

**Remark 6.19.** For $g=2$, $\lambda$, $\delta_0$ and $\delta_1$ still generate the rational Picard group of $\overline{\mathcal{M}}_2$. However, they satisfy the following relation:

$$\lambda = \frac{1}{10}\delta_0 + \delta_1.$$

This special relation is essentially due to the fact that every genus two curve is hyperelliptic.

**Example 6.20.** Consider the case $g=1$. Let $B$ be a general pencil of plane cubics. In other words, take two general degree three homogeneous polynomials $F$ and $G$ in three variables. Consider the cubic $C_{[s,t]}$ as the vanishing locus of $sF + tG$ for

$[s,t] \in B \cong \mathbb{P}^1$. Blow up the 9 base points of the pencil. We obtain a universal cubic $S \to B$.

Let $n$ be the number of rational one-nodal curves in $B$. Note that for a smooth cubic $C$, its topological Euler characteristic is $\chi_{\text{top}}(C) = 0$, while for a rational nodal cubic $C'$ we have $\chi_{\text{top}}(C') = 1$. Moreover, $\chi_{\text{top}}(S) = \chi_{\text{top}}(\mathbb{P}^2) + 9 = 12$. Therefore, by the Riemann-Hurwitz formula

$$\chi_{\text{top}}(S) = \chi_{\text{top}}(C) \cdot \chi_{\text{top}}(B) + n \cdot (\chi_{\text{top}}(C') - \chi_{\text{top}}(C)),$$

we read off that $n = 12$. Hence a general pencil of plane cubics contains 12 nodal cubics.

Later on we will see that $\lambda|_B$ has degree 1. Therefore, we obtain the relation $12\lambda = \delta_0$ for the moduli space of elliptic curves $\overline{\mathcal{M}}_{1,1}$. We need a marked point just in order to make the automorphism group of an elliptic curve finite.

6.6. **Grothendieck-Riemann-Roch.** Let $E$ be a vector bundle of rank $r$ on a variety $X$. Let $\alpha_1, \ldots, \alpha_r$ be its formal *Chern roots*, i.e. the total Chern class of $E$ satisfies a relation in the Chow ring of $X$:

$$c(E) = \sum_i c_i(E) = \prod_i (1 + \alpha_i).$$

In other words, we have

$$\begin{aligned} c_1(E) &= \sum_i \alpha_i, \\ c_2(E) &= \sum_{i<j} \alpha_i \alpha_j \end{aligned}$$

and etc. Define the *Chern character* of $E$ as

$$\begin{aligned} \operatorname{ch}(E) &= \sum_i e^{\alpha_i} \\ &= \sum_i \sum_j \frac{\alpha_i^j}{j!} \\ &= \operatorname{rank}(E) + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2} + \cdots . \end{aligned}$$

Furthermore, we define the *Todd class* of $E$ as

$$\operatorname{td}(E) = \prod_i \frac{\alpha_i}{1 - e^{-\alpha_i}}.$$

Using the expansion

$$\frac{\alpha}{1 - e^{-\alpha}} = 1 + \frac{\alpha}{2} + \frac{\alpha^2}{12} + \cdots ,$$

we obtain that

$$\operatorname{td}(E) = 1 + \frac{c_1(E)}{2} + \frac{c_1^2(E) + c_2(E)}{12} + \cdots .$$

We also write $\operatorname{td}(X)$ for the Todd class of the tangent bundle of $X$.

**Exercise 6.21.** Let $E$, $F$ and $G$ be vector bundles satisfying the exact sequence:

$$0 \to E \to F \to G \to 0.$$

Verify that

$$\operatorname{ch}(F) = \operatorname{ch}(E) + \operatorname{ch}(G),$$

$$\mathrm{ch}(E \otimes G) = \mathrm{ch}(E) \cdot \mathrm{ch}(G),$$
$$\mathrm{td}(F) = \mathrm{td}(E) \cdot \mathrm{td}(G).$$

The formal properties of ch ensures that it can be defined for any coherent sheaf using its resolution by locally free sheaves. In other words, ch induces a homomorphism from the Grothendieck group to the Chow ring.

Finally, for a morphism $\pi : X \to B$, define the *shriek* of $E$ by $\pi$ as

$$\pi_!(E) = \sum_i (-1)^i R^i \pi_*(E),$$

where $R^i\pi_*(E)$ denotes the $i$th direct image sheaf of $E$. We are mainly interested in the case when $R^0\pi_*(E) = \pi_*(E)$ is a vector bundle, especially when $h^i(X_b, E|_{X_b})$ is the same for all fibers $X_b$.

**Theorem 6.22** (Grothendieck-Riemann-Roch)**.** *Let $E$ be a vector bundle on $X$, and $\pi : X \to B$ is a morphism with smooth base $B$. Then*

$$\mathrm{ch}(\pi_!(E)) \cdot \mathrm{td}(B) = \pi_*(\mathrm{ch}(E) \cdot \mathrm{td}(X)).$$

*Alternatively, using the pullback-pushdown formula, the formula can be rewritten as*

$$\mathrm{ch}(\pi_!(E)) = \pi_*(\mathrm{ch}(E) \cdot (\mathrm{td}(X)/\pi^* \mathrm{td}(B))).$$

Note that the $\pi_*$ on the right hand side is the Chow ring morphism $\pi_* : A_*(X) \to A_*(B)$ by pushing down cycle classes from $X$ to $B$. In particular, if $D \subset X$ has image of smaller dimension in $B$, then $\pi_*(D) = 0$.

**Example 6.23.** Let $B$ be a point $b$, $X$ a smooth curve of genus $g$ and $E$ a line bundle of degree $d$ on $X$. Then we have $\mathrm{td}(b) = 1$, $\pi_*(E) = H^0(E)$ and $R^1\pi_*(E) = H^1(E)$ as vector bundles over $b$. Then we obtain that $\mathrm{ch}(\pi_!(E)) = \mathrm{ch}(H^0(E)) - \mathrm{ch}(H^1(E)) = h^0(X, E) - h^1(X, E)$. On the other hand, $\mathrm{ch}(E) = 1 + E$ and $\mathrm{td}(X) = 1 - \frac{K}{2}$, where we identify $E$ with its divisor class and $K$ is the cotangent line bundle class of $X$. Therefore,

$$\pi_*(\mathrm{ch}(E) \cdot \mathrm{td}(X)) = \pi_*(1 - K/2 + E) = 1 - g + d$$

in $A_*(b) \cong \mathbb{Z}$. We thus recover the ordinary Riemann-Roch formula for curves.

**Example 6.24.** Let $X \to Y$ be a branched cover of degree $n$ between two smooth curves of genus $g$ and $h$, respectively. Let $E$ be a line bundle of degree $d$ on $X$. Then we have

$$\mathrm{td}(X) = 1 - \frac{1}{2}K_X,$$
$$\mathrm{td}(Y) = 1 - \frac{1}{2}K_Y,$$
$$\mathrm{ch}(E) = 1 + E.$$

Moreover, $\pi_*(E)$ is a vector bundle of degree $n$ on $Y$, because $\pi^{-1}(y)$ is a zero-dimensional scheme of length $n$ for any $y \in Y$. By the same token, $R^i\pi_*(E) = 0$ for $i > 0$. Therefore, we have

$$(n + c_1(\pi_*(E))) \cdot (1 - K_Y/2) = \pi_*((1 + E) \cdot (1 - K_X/2)),$$
$$c_1(\pi_*(E)) - \frac{n}{2}K_Y = d - g + 1,$$
$$c_1(\pi_*(E)) = 1 - g + d + n(h - 1).$$

This tells us the degree of the vector bundle $\pi_*(E)$.

**Exercise 6.25.** Simplify the Grothendieck-Riemann-Roch formula for the case when $B$ is a point.

Consider $\pi : \mathcal{C} \to \mathcal{M}_g$ the morphism from the universal curve to the moduli space of genus $g$ Riemann surfaces. Let $\omega$ be the relative dualizing sheaf of $\pi$. We have seen that $\mathbb{E} = \pi_*(\omega)$ is the Hodge bundle of rank $g$. Since $\omega$ is a line bundle on $\mathcal{C}$, let $\gamma$ be its first Chern class. Then via the exact sequence

$$0 \to \pi^* T_{\mathcal{M}_g} \to T_{\mathcal{C}} \to \omega^\vee \to 0,$$

we conclude that

$$\frac{\operatorname{td}(\mathcal{C})}{\pi^* \operatorname{td}(\mathcal{M}_g)} = \operatorname{td}(\omega^\vee) = 1 - \frac{\gamma}{2} + \frac{\gamma^2}{12} - \cdots .$$

By Grothendieck-Riemann-Roch, we have

$$\begin{aligned} \operatorname{ch}(\pi_!(\omega)) &= \pi_* \left( \left(1 - \frac{\gamma}{2} + \frac{\gamma^2}{12} + \cdots \right) \cdot \left(1 + \gamma + \frac{\gamma^2}{2} + \frac{\gamma^3}{6} + \cdots \right) \right) \\ &= \pi_* \left(1 + \frac{\gamma}{2} + \frac{\gamma^2}{12} + \cdots \right). \end{aligned}$$

Note that $\pi_*\omega = \mathbb{E}$, $R^1\pi_*\omega \cong \mathscr{O}_{\mathcal{M}_g}$ and $R^i\pi_*\omega = 0$ for $i > 1$. We thus conclude that

$$\operatorname{ch}(\mathbb{E}) - 1 = \pi_* \left(1 + \frac{\gamma}{2} + \frac{\gamma^2}{12} + \cdots \right).$$

Using the definition of ch, we read off

$$\operatorname{rank}(\mathbb{E}) - 1 = \frac{1}{2}(\pi_*\gamma) = g - 1,$$

since $\gamma$ has degree $2g - 2$ along each fiber of $\pi$. Moreover, we have

$$\lambda = c_1(\mathbb{E}) = \frac{1}{12}\pi_*\gamma^2$$

and etc.

Write $\lambda_i = c_i(\mathbb{E})$ and $\kappa_i = \pi_*(\gamma^{i+1})$. We also set $\lambda = \lambda_1$ and $\kappa = \kappa_1$ as divisor classes. Then we have the relation

$$\lambda = \frac{1}{12}\kappa.$$

Furthermore, from the above calculation it is clear that the $\lambda$-classes can be represented by polynomials of $\kappa$-classes.

Since the universal curve and the Hodge bundle extend to $\overline{\mathcal{M}}_g$, one can take the boundary into account and redo the calculation. Nevertheless, the relative cotangent bundle $\Omega$ of $\pi$ differs from the relative dualizing sheaf $\omega$ along the locus $Z$ of nodes in $\mathcal{C}$. More precisely, $\Omega \cong \mathcal{I}_Z \otimes \omega$, where $\mathcal{I}_Z$ is the ideal sheaf of $Z$ in $\mathcal{C}$. Applying Grothendieck-Riemann-Roch to $i : Z \to \overline{\mathcal{M}}_g$, one can work out $\operatorname{ch}(\mathcal{I}_Z)$. Then apply Grothendieck-Riemann-Roch to $\pi : \mathcal{C} \to \overline{\mathcal{M}}_g$, we obtain the following *Mumford relation* (also called the *Noether formula*):

$$\lambda = \frac{\kappa + \delta}{12},$$

where $\delta = \sum_{i=0}^{[g/2]} \delta_i$ denotes the total boundary class.

**Example 6.26.** Let us finish the calculation of the degree of $\lambda$ on a general pencil $B \cong \mathbb{P}^1$ of plane cubics. Blow up the 9 base points. We obtain a universal cubic $\pi : S \to B$. One can regard $S \subset \mathbb{P} = \mathbb{P}^2 \otimes \mathbb{P}^1$ as a surface of divisor class $(3, 1)$. By the adjunction formula, we have

$$K_S = (K_{\mathbb{P}} + S)|_S = [(-3, -2) + (3, 1)]|_S = (0, -1)|_S.$$

Since $\pi^* K_{\mathbb{P}^1}$ has class $(0, -2)$, the relative dualizing line bundle $\omega$ of $\pi$ has divisor class

$$(0, -1) - (0, -2) = (0, 1)$$

restricted to $S$. Let $h$ and $l$ denote the pullback of the hyperplane class from $\mathbb{P}^2$ and $\mathbb{P}^2$ to $\mathbb{P}$, respectively. We have

$$h^2 \cdot l = 1$$

and all the other top intersection numbers are zero. Putting everything together, we obtain that

$$\omega^2 = l^2(3h + l) = 0.$$

In other words, we have $\kappa \cdot B = 0$. Using Mumford's relation, we obtain that

$$\lambda \cdot B = \frac{\delta \cdot B + \kappa \cdot B}{12} = 1.$$

This calculation implies that the relation $12\lambda = \delta$ holds in the rational Picard group of $\overline{\mathcal{M}}_{1,1}$.

**Exercise 6.27.** Calculate the degree of $\delta$ and $\lambda$ on a general pencil of plane quartic curves.

6.7. **Porteous' formula.** Porteous' formula expresses the class of the locus where the rank of a map between two vector bundles is less than or equal to a given bound. In what follows we briefly review this formula.

For a vector bundle $E$, define its *Chern polynomial* as

$$c_t(E) = \sum_i c_i(E) t^i,$$

where $c_i(E)$ is the $i$th Chern class of $E$ and $t$ is a formal variable. For any integer $p$ and any positive integer $q$, define a $q \times q$ matrix $M_{p,q}(c_t)$ whose $(i, j)$th entry is $c_{p+j-i}$ for a formal power series $c_t = \sum_i c_i t^i$. Define its determinant as

$$\Delta_{p,q}(c_t) = \det(M_{p,q}(c_t)).$$

**Theorem 6.28** (Porteous' Formula)**.** *Let $\phi : E \to F$ be a homomorphism between vector bundles of ranks $m$ and $n$, respectively, on a variety $X$. Consider the degenerate locus*

$$D_k = \{x \in X \mid \operatorname{rank}(\phi_x) \le k\}$$

*and let $[D_k]$ be its cycle class in the Chow ring of $X$. If $D_k$ is either empty or of the expected codimension $(m-k)(n-k)$, then*

$$[D_k] = \Delta_{n-k,m-k}((c_t(F)/c_t(E)).$$

Even if $D_k$ has dimension higher than expected, we can still compute the *virtual class* of $D_k$ using Porteous' formula. If $D_k$ is of the expected dimension, the virtual class thus equals its actual class.

6.8. **The hyperelliptic locus in $\mathcal{M}_3$.** In $g=3$, the locus of hyperelliptic curves forms a divisor $H$ in $\mathcal{M}_3$. Since $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_3)$ is generated by $\lambda$, we would like to calculate the class of $H$ in terms of $\lambda$ and the main tool of the calculation is Porteous' formula.

Let us treat this question in a more general setting. Suppose $\pi : \mathcal{X} \to B$ is a general family of genus $g$ curves. Let $b \in B$ be a point, $C_b$ the fiber curve over $b$ and $p \in C_b$ a point. We want to construct two vector bundles $E$ and $F$ on $\mathcal{X}$, such that $E|_{(b,p)} \cong H^0(C_b, K)$ and $F|_{(b,p)} \cong H^0(C_b, K/K(-2p))$. In other words, $E$ is just the Hodge bundle of rank $g$ and the fiber of the rank 2 bundle $F$ parameterizes differentials in a neighborhood of $p$ in $C_b$ modulo those vanishing to order 2 at $p$. Consider the natural map $\phi : E \to F$ induced by the exact sequence

$$0 \to K(-2p) \to K \to K/K(-2p) \cong \mathscr{O}_{2p}(K) \to 0.$$

Then the locus in $\mathcal{X}$ where $\mathrm{rank}(\phi) \leq 1$ consists of points $(b,p)$ where

$$h^0(C_b, K(-2p)) \geq g-1,$$

i.e. $h^0(C_b, 2p) \geq 2$, or equivalently, $C$ is a hyperelliptic curve and $p$ is a Weierstrass point.

In order to construct $E$ and $F$, let $\mathcal{X}_2 = \mathcal{X} \times_B \mathcal{X}$ be the fiber product parameterizing $(b,p,q)$, where $p,q$ are two points in $C_b$ not necessarily distinct. Let $\pi_1$ and $\pi_2$ be the projections from $\mathcal{X}_2$ to $\mathcal{X}$. Then $E$ is thus the pullback of the Hodge bundle, i.e.

$$E = (\pi_1)_*(\pi_2^*\omega_{\mathcal{X}/B}).$$

Let $\Delta \subset \mathcal{X}_2$ be the diagonal parameterizing points $(b,p,p)$. Then $F$ can be defined as

$$F = (\pi_1)_*\left((\pi_2^*\omega_{\mathcal{X}/B}) \otimes (\mathscr{O}_{\mathcal{X}_2}/I_\Delta^2)\right).$$

Then the map $\phi : E \to F$ is just the push-forward under $\pi_1$ of the restriction map

$$\pi_2^*\omega_{\mathcal{X}/B} \to (\pi_2^*\omega_{\mathcal{X}/B}) \otimes (\mathscr{O}_{\mathcal{X}_2}/I_\Delta^2).$$

Now we specialize to the case $g=3$. Since we only care about a divisorial locus, we can skip Chern classes of codimension-2 or higher in the base $B$. In this sense, we have

$$c(E) = 1 + \pi^*\lambda.$$

To calculate the Chern class of $F$, we use a two-term filtration

$$0 \to F_2 \to F \to F_1 \to 0,$$

where the fibers of $F_1$ and $F_2$ are $H^0(C_b, K/K(-p))$ and $H^0(C_b, K(-p)/K(-2p))$ at $(b,p)$, respectively. Note that $H^0(C_b, K/K(-p))$ is thus $K|_{(b,p)}$, hence $F_1$ is nothing but the relative dualizing sheaf $\omega_{\mathcal{X}/B}$. Moreover, let $\mathfrak{m}_p$ be the maximal ideal of $p$ in $C_b$. Then the canonical bundle $K$ of $C$ restricted to $p$ is $\mathfrak{m}_p/\mathfrak{m}_p^2$, which implies that $F_2$ is isomorphic to $\omega_{\mathcal{X}/B}^{\otimes 2}$. Therefore, we read off

$$c(F) = c(F_1)\cdot c(F_2) = (1+\gamma)(1+2\gamma) = 1 + 3\gamma + 2\gamma^2,$$

where $\gamma = c_1(\omega_{\mathcal{X}/B})$.

By Porteous' formula, the locus $W = \{(b,p)\}$ in $\mathcal{X}$ where $C_b$ is hyperelliptic of genus 3 with $p$ a Weierstrass point has class

$$[W] = \Delta_{1,2}((c_t(F)/c_t(E)) = c_1^2 - c_2,$$

where $c_1 = 3\gamma - \pi^*\lambda$ and $c_2 = 2\gamma^2 - 3\gamma(\pi^*\lambda)$. Therefore, we obtain that

$$[W] = 7\gamma^2 - 3\gamma(\pi^*\lambda) + (\pi^*\lambda)^2.$$

Note that $\pi_*(\gamma^2) = \kappa = 12\lambda$ by the Mumford relation. Moreover, $\gamma$ has degree $2g - 2 = 4$ along each fiber, hence we have

$$\pi_*(\gamma(\pi^*\lambda)) = 4\lambda.$$

Finally, $\pi_*((\pi^*\lambda)^2) = 0$, because $\pi^*\lambda$ is a multiple fiber class and the image of $(\pi^*\lambda)^2$ under $\pi$ drops dimension. Altogether we obtain that

$$\pi_*[W] = 72\lambda$$

in $B$. Since a hyperelliptic curve of genus 3 has 8 Weierstrass points, the class $[H]$ of hyperelliptic curves in $B$ equals $\pi_*[W]/8$. We thus conclude that

$$[H] = 9\lambda$$

in $\mathcal{M}_3$.

**Exercise 6.29.** Suppose $[H] = 9\lambda - b_0\delta_0 - b_1\delta_1$ in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_3)$. Use a general pencil of plane quartics to calculate the coefficient $b_0$.