MT845: RIEMANN SURFACES – FROM ANALYTIC AND ALGEBRAIC VIEWPOINTS

DAWEI CHEN

Contents

1. Preliminaries

Definition 1.1 (Riemann surface)**.** A *Riemann surface* X is a one-dimensional complex manifold, i.e., X is a real surface with a complex atlas of charts

$$\{\phi_i : U_i \to V_i \subset \mathbb{C}\},$$

where $\{U_i\}$ is an open covering of X, ϕ_i is a homeomorphism, and

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is biholomorphic. We say that such a complex atlas of charts is a *complex structure* on the underlying real surface.

Example 1.2. (a) Suppose X is a Riemann surface. Let $Y \subset X$ be a (connected) open subset. Then Y is a Riemann surface, whose complex structure is given by taking all $U \subset Y$ from charts of X.

(b) Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, homeomorphic to the real sphere. Take

$$U_1 = \mathbb{P}^1 \backslash \{\infty\} = \mathbb{C},$$

$$U_2 = \mathbb{P}^1 \backslash \{0\} = \mathbb{C}^* \cup \{\infty\}.$$

Define $\phi_1(z) = z$, $\phi_2(z) = 1/z$ for $z \neq \infty$ and $\phi_2(\infty) = 0$. Then $\phi_2 \circ \phi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^*$ is given by $z \mapsto 1/z$, which is biholomorphic. Therefore, $\mathbb{P}^1$ is a Riemann surface, called the *Riemann sphere* or the *projective line* (over the field of complex numbers).

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(c) Suppose $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. Define

$$\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\},$$

called the lattice spanned by ω_1 and ω_2. Define an equivalence relation $z_1 \sim z_2$ on $\mathbb{C}$ if $z_1 - z_2 \in \Gamma$. Denote by E the set of equivalence classes, i.e., $E = \mathbb{C}/\Gamma$. Equip E with the quotient topology, i.e., $U \subset E$ is open if and only if $\pi^{-1}(U)$ is open for the projection $\pi : \mathbb{C} \to E$. Then E is homeomorphic to a torus.

One can enable E a complex structure as follows. Let $V \subset \mathbb{C}$ be an open subset such that no two points in V are equivalent with respect to Γ. Then $U = \pi(V)$ is open in E, and let $\phi : U \to V$ be the inverse of π, which forms a complex chart on E.

Exercise 1.3. Prove that the above complex charts form a complex structure on E.

Definition 1.4 (Holomorphic function)**.** Let X be a Riemann surface. A function $f : X \to \mathbb{C}$ is called *holomorphic*, if for every chart $\phi : U \to V \subset \mathbb{C}$ on X the function

$$f \circ \phi^{-1} : V \to \mathbb{C}$$

is holomorphic. The set of all holomorphic functions on X is denoted by $\mathscr{O}(X)$.

Definition 1.5 (Holomorphic map)**.** Let X and Y be Riemann surfaces. A continuous map $f : X \to Y$ is a called *holomorphic*, if for every chart $\phi : U_1 \to V_1 \subset \mathbb{C}$ on X and $\psi : U_2 \to V_2 \subset \mathbb{C}$ on Y with $f(U_1) \subset U_2$ the function

$$\psi \circ f \circ \phi^{-1} : V_1 \to V_2$$

is holomorphic.

If f is bijective and both f and f^{-1} are holomorphic, we say that f is *biholomorphic*, and X and Y are called *isomorphic*.

Definition 1.6 (Meromorphic function)**.** Let X be a Riemann surface and $\Sigma = \{p_1, \dots, p_n\} \subset X$ be a finite set of points. Suppose $f : X \backslash \Sigma \to \mathbb{C}$ is a holomorphic function such that

$$\lim_{x \to p_i} |f(x)| = \infty$$

for every $p_i \in \Sigma$, then f is called *meromorphic*, and the points p_i are *poles* of f. The set of all meromorphic functions on X is denoted by $\mathscr{M}(X)$.

Remark 1.7. In a neighborhood of a pole p, let z be a suitable coordinate with $z(p) = 0$. Then f can be expanded in a Laurent series

$$f = \sum_{i=-k}^{\infty} a_i z^i$$

for some $k \in \mathbb{Z}^+$ with $a_{-k} \neq 0$. We say that k is the *pole order* of f at p.

Example 1.8. Let $f = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial on $\mathbb{C} \subset \mathbb{P}^1$. Then $\lim_{z \to \infty} |f(z)| = \infty$, hence $f \in \mathscr{M}(\mathbb{P}^1)$.

Exercise 1.9. Prove that in the above example the pole order of f at ∞ is n.

Theorem 1.10. *Suppose X is a Riemann surface and $f \in \mathscr{M}(X)$. For each pole p of f, define $f(p) = \infty$. Then $f : X \to \mathbb{P}^1$ is a holomorphic map.*

Proof. Since the set of poles of f is a finite set, the result follows from the Riemann's theorem on removable singularities. Alternatively, expand

$$f = \sum_{i=-k}^{\infty} a_i z^i = z^{-k} g$$

locally at a pole p with a suitable coordinate $z(p) = 0$, where k is the pole order of p and $g(p) \neq 0$. Then $1/f = z^k(1/g)$ is holomorphic at p.

□

Exercise 1.11. Prove that every holomorphic function on a compact Riemann surface is constant. *Hint: a non-constant holomorphic function is open.*

Exercise 1.12. Prove that every meromorphic functions on $\mathbb{P}^1$ is *rational*, i.e., can be expressed as the quotient of two polynomials. *Hint: subtract the principal parts of the Laurent expansions at the poles.*

Definition 1.13 (Covering, branch and ramification)**.** Suppose X and Y are Riemann surfaces and $\pi : Y \to X$ is a non-constant holomorphic map. Then π is called a *covering*.

A point $y \in Y$ is called a *ramification point* of π, if there is no neighborhood V of y such that $\pi|_V$ is injective. A point $x \in X$ is called a *branch point* of π, if $\pi^{-1}(x)$ contains a ramification point. The map π is called *unramified* if it has no branch point (i.e., no ramification point).

For $p \in Y$ and $q = \pi(p) \in X$, by properties of non-constant holomorphic functions, there exist neighborhoods U of p and V of q as well as suitable coordinates x and y such that $x(p) = y(q) = 0$ and $f|_U(x) = x^k$ for some $k \in \mathbb{Z}^+$. If $k = 1$, then p is not a ramification point of π. If $k > 1$, then p is a ramification point. We say that the *multiplicity* of π at p is k, denoted by $\mathrm{mult}_p(\pi)$, and the *ramification order* of π at p is $k-1$, denoted by $\mathrm{ord}_p(\pi)$. In particular, $\mathrm{mult}_p(\pi) - 1 = \mathrm{ord}_p(\pi)$.

Example 1.14. (a) Let $\pi : \mathbb{C} \to \mathbb{C}$ be $\pi(z) = z^k$ for an integer $k \geq 2$. Then 0 is the only ramification point.

(b) Consider the *exponential map* $\exp : \mathbb{C} \to \mathbb{C}^*$ by $z \mapsto e^z$. Then exp is unramified. In particular, if $V \subset \mathbb{C}$ does not contain two points that differ by an integer multiple of $2\pi i$, then $\exp|_V$ is injective.

(c) Let Γ be a lattice in $\mathbb{C}$, and $E = \mathbb{C}/\Gamma$ the associated torus. Then the projection $\mathbb{C} \to E$ is unramified.

Theorem 1.15 (Degree of covering maps)**.** *Let $\pi : Y \to X$ be a covering map of two compact Riemann surfaces. Then for $q \in X$, the fiber cardinality over q, counting multiplicities, is independent of q. Namely,*

$$d = \sum_{p \in \pi^{-1}(q)} \mathrm{mult}_p(\pi)$$

is constant for all $q \in X$.

We say that d is the *degree* of π.

Proof. Since π is non-constant, π is open, $\pi(Y)$ is both open and closed in X, hence π is onto. For $q \in X$, suppose $\pi^{-1}(q) = \{p_1, \ldots, p_n\}$. At each preimage p_i, suppose π is of type $z \mapsto z^{k_i}$. Then $d = \sum_{i=1}^n k_i$ for all q' in a neighborhood of q, and hence d is locally constant. Since X is connected, d is globally constant. □

Exercise 1.16. Suppose X is a compact Riemann surface and f is a non-constant meromorphic function on X. Then f has as many zeros as poles, counting with multiplicities.

2. Sheaves and cohomology

2.1. **Sheaves.** Let X be a topological space. A *sheaf* $\mathscr{F}$ on X associates to each open set U an abelian group $\mathscr{F}(U)$, called the *sections* of $\mathscr{F}$ over U, along with a *restriction map* $r_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ for any open sets $U \subset V$ (for a section $\sigma \in \mathscr{F}(V)$, we often write $\sigma|_U$ to denote $r_{V,U}(\sigma)$), satisfying the following conditions:

(1) For any open sets $U \subset V \subset W$, $r_{V,U} \circ r_{W,V} = r_{W,U}$;

(2) For a collection of open sets $\{U_i\}_{i\in I}$ and sections $\alpha_i \in \mathscr{F}(U_i)$, if $\alpha_i|_{U_i\cap U_j} = \alpha_j|_{U_i\cap U_j}$ for any $i, j \in I$, then there exists a unique $\alpha \in \mathscr{F}(\cup_i U_i)$ such that $\alpha|_{U_i} = \alpha_i$ for any i.

Remark 2.1. If $\mathscr{F}$ satisfies (1) only, we call it a *presheaf*. One can perform *sheafification* for a presheaf to make it become a sheaf. For many sheaves we consider, the restriction maps are the obvious ones by restricting functions from bigger subsets to smaller ones.

Exercise 2.2. Show that $\mathscr{F}(\emptyset)$ consists of exactly one element.

Example 2.3. Let G be an abelian group. We have the *sheaf of locally constant functions* $\mathbb{G}$ on a topological space X, where $\mathbb{G}(U)$ is the group of locally constant maps $f : U \to G$ on a non-empty open set $U \subset X$ and $\mathbb{G}(\emptyset) = 0$.

Exercise 2.4. Show that for the sheaf $\mathbb{G}$ of locally constant functions, we have $\mathbb{G}(U) = G$ for any non-empty connected open set U.

Exercise 2.5. Suppose we define $\mathbb{G}(U) = G$ as the set of constant functions on a non-empty open set U with the natural restriction maps. If G contains at least two elements and if X has two disjoint non-empty open subsets, show that $\mathbb{G}$ is a sheaf.

Example 2.6. Let X be a complex manifold and $U \subset X$ an open set.

(1) *Sheaf $\mathscr{O}$ of holomorphic functions*:

$$\mathscr{O}(U) = \{\text{holomorphic functions on } U\}.$$

The group law is given by addition.

(2) *Sheaf $\mathscr{O}^*$ of everywhere nonzero holomorphic functions*:

$$\mathscr{O}^*(U) = \{\text{holomorphic functions } f \text{ on } U : f(p) \neq 0 \text{ for any } p \in U\}.$$

The group law is given by multiplication.

(3) *Sheaf $\mathscr{M}$ of meromorphic functions*: strictly speaking, a meromorphic function is not a function, even we take ∞ into account. If X is compact, we cannot take a meromorphic function as a quotient of two holomorphic functions, since any globally defined holomorphic function is a constant on X. Instead, we define $f \in \mathscr{M}(U)$ as local quotients of holomorphic functions compatible with each other. Namely, there exists an open covering $\{U_i\}$ of U such that on each U_i, f is given by g_i/h_i for some $g_i, h_i \in \mathscr{O}(U_i)$ satisfying $g_i/h_i = g_j/h_j$, i.e. $g_ih_j = g_jh_i \in \mathscr{O}(U_i \cap U_j)$, hence these local quotients can be glued together over U.

(4) *Sheaf $\mathscr{M}^*$ of meromorphic functions not identically zero*: this is defined similarly and the group law is given by multiplication.

2.2. **Maps between sheaves.** Let $\mathscr{E}$ and $\mathscr{F}$ be two sheaves on a topological space X. A *map* $f : \mathscr{E} \to \mathscr{F}$ is a collection of group homomorphisms

$$\{f_U : \mathscr{E}(U) \to \mathscr{F}(U)\}$$

such that they commute with the restriction maps, i.e. for open sets $U \subset V$ and $\sigma \in \mathscr{E}(V)$ we have

$$f_V(\sigma)|_U = f_U(\sigma|_U).$$

Define the *sheaf of kernel* $\ker(f)$ as

$$\ker(f)(U) = \{\ker(f_U : \mathscr{E}(U) \to \mathscr{F}(U))\}.$$

Exercise 2.7. Prove that in the above definition $\ker(f)$ is a sheaf.

Example 2.8. Let X be a complex manifold. Define the *exponential map*

$$\exp : \mathscr{O} \to \mathscr{O}^*$$

by $\exp(h) = e^{2\pi\sqrt{-1}h}$ for any open set $U \subset X$ and section $h \in \mathscr{O}(U)$. It is easy to see that $\ker(\exp)$ is the locally constant sheaf $\mathbb{Z}$.

The *sheaf of cokernel* is harder to define. Naively, one would like to define $\operatorname{coker}(f)(U) = \operatorname{coker}(f_U : \mathscr{E}(U) \to \mathscr{F}(U))$, but this is problematic. For instance, consider the exponential map $\exp : \mathscr{O} \to \mathscr{O}^*$ on the punctured plane $\mathbb{C}\backslash\{0\}$. The section $z \in \mathscr{O}^*(\mathbb{C}\backslash\{0\})$ is not in the image of f, hence it would define a section in the cokernel. Nevertheless, restricted to any contractible open set $U \subset \mathbb{C}\backslash\{0\}$, z lies in the image of f. Now cover $\mathbb{C}\backslash\{0\}$ by contractible open sets. By the gluing property of sheaves, z would be zero everywhere, leading to a contradiction.

Instead, we define a section of $\operatorname{coker}(f)(U)$ to be a collection of sections $\sigma_\alpha \in \mathscr{F}(U_\alpha)$ for an open covering $\{U_\alpha\}$ of U such that for all α, β we have

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in f_{U_\alpha \cap U_\beta}(\mathscr{E}(U_\alpha \cap U_\beta)).$$

Here the definition still depends on the choice of an open covering. To get rid of this, we pass to the direct limit. More precisely, we further identify two collections $\{(U_\alpha, \sigma_\alpha)\}$ and $\{(V_\beta, \sigma_\beta\}$ if for all $p \in U_\alpha \cap V_\beta$, there exists an open set W satisfying $p \in W \subset U_\alpha \cap V_\beta$ such that

$$\sigma_\alpha|_W - \sigma_\beta|_W \in f_W(\mathscr{E}(W)).$$

This identification yields an equivalence relation and correspondingly we define $\operatorname{coker}(f)(U)$ as the group of equivalence classes of the above sections.

Exercise 2.9. Prove that in the above definition $\operatorname{coker}(f)$ is a sheaf.

If $\ker(f)$ (resp. $\operatorname{coker}(f)$) is the zero sheaf, we say that f is *injective* (resp. *surjective*).

Consider the following sequence of maps between sheaves:

$$0 \longrightarrow \mathscr{E} \xrightarrow{\ \alpha\ } \mathscr{F} \xrightarrow{\ \beta\ } \mathscr{G} \longrightarrow 0.$$

We say that it is a *short exact sequence* if $\mathscr{E} = \ker(\beta)$ and $\mathscr{G} = \operatorname{coker}(\alpha)$. In this case we also say that $\mathscr{E}$ is a *subsheaf* of $\mathscr{F}$ and $\mathscr{G}$ is the *quotient sheaf* $\mathscr{F}/\mathscr{E}$.

Example 2.10. Let X be a complex manifold. We have the exact *exponential sequence*:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } \mathscr{O} \xrightarrow{\ \exp\ } \mathscr{O}^* \longrightarrow 0,$$

where i is the natural inclusion and $\exp(f) = e^{2\pi\sqrt{-1}f}$ for $f \in \mathscr{O}(U)$.

Exercise 2.11. Prove that the exponential sequence is exact.

Example 2.12. Let X be a complex manifold and $Y \subset X$ a submanifold. Define the *ideal sheaf* $\mathscr{I}_{Y/X}$ of Y in X (or simply $\mathscr{I}_Y$ if there is no confusion) by

$$\mathscr{I}_Y(U) = \{\text{holomorphic functions in } U \text{ vanishing on } Y \cap U\}.$$

We have the exact sequence:

$$0 \longrightarrow \mathscr{I}_Y \xrightarrow{\ i\ } \mathscr{O}_X \xrightarrow{\ r\ } i_*\mathscr{O}_Y \longrightarrow 0,$$

where i is the natural inclusion and r is defined by the natural restriction map. Here $i_*\mathscr{O}_Y$ is the extension of $\mathscr{O}_Y$ by zero outside Y, as a sheaf defined on X.

Exercise 2.13. Prove that the above sequence is exact.

2.3. **Stalks and germs.** Let $\mathscr{F}$ be a sheaf on a topological space X and $p \in X$ a point. Suppose U and V are two open subsets, both containing p, with two sections $\alpha \in \mathscr{F}(U)$ and $\beta \in \mathscr{F}(V)$. Define an equivalence relation $\alpha \sim \beta$, if there exists an open subset W satisfying $p \in W \subset U \cap V$ such that $\alpha|_W = \beta|_W$. Define the *stalk* $\mathscr{F}_p$ as the union of all sections in open neighborhoods of p modulo this equivalence relation. Namely, $\mathscr{F}_p$ is the direct limit

$$\mathscr{F}_p := \varinjlim_{U \ni p} \mathscr{F}(U) = \Big(\bigsqcup_{U \ni p} \mathscr{F}(U) \Big) / \sim .$$

Note that $\mathscr{F}_p$ is also a group, by adding representatives of two equivalence classes. There is a group homomorphism $r_U : \mathscr{F}(U) \to \mathscr{F}_p$ mapping a section $\alpha \in \mathscr{F}(U)$ to its equivalence class. The image is called the *germ* of α.

Example 2.14 (Skyscraper sheaf)**.** Let $p \in X$ be a point on a topological space X. Define the *skyscraper sheaf* $\mathscr{F}$ at p by $\mathscr{F}(U) = \{0\}$ for $p \notin U$ and $\mathscr{F}(U) = A$ for $p \in U$, where A is an abelian group. The restriction maps are either the identity map $A \to A$ or the zero map. For $q \neq p$, the stalk $\mathscr{F}_q = \{0\}$. At p, we have $\mathscr{F}_p = A$. Note that $\mathscr{F}$ can also be obtained by extending the constant sheaf A at p by zero to $X \backslash \{p\}$.

Exercise 2.15. Let X be a Riemann surface and $p \in X$ a point. Let $\mathscr{I}_p$ be the ideal sheaf of p in X parameterizing holomorphic functions vanishing at p. We have the exact sequence

$$0 \longrightarrow \mathscr{I}_p \xrightarrow{\ i\ } \mathscr{O}_X \xrightarrow{\ r\ } \mathscr{O}_p \longrightarrow 0.$$

Show that the quotient sheaf $\mathscr{O}_p$ is isomorphic to the skyscraper sheaf with stalk $\mathbb{C}$ at p.

It is more convenient to verify injections and surjections for maps of sheaves by the language of stalks.

Proposition 2.16. *Let $\phi : \mathscr{E} \to \mathscr{F}$ be a map for sheaves $\mathscr{E}$ and $\mathscr{F}$ on a topological space X.*

(1) ϕ is injective if and only if the induced map $\phi_p : \mathscr{E}_p \to \mathscr{F}_p$ is injective for the stalks at every point p.

(2) ϕ is surjective if and only if the induced map $\phi_p : \mathscr{E}_p \to \mathscr{F}_p$ is surjective for the stalks at every point p.

(3) ϕ is an isomorphism if and only if the induced map $\phi_p : \mathscr{E}_p \to \mathscr{F}_p$ is an isomorphism for the stalks at every point p.

Proof. The claim (3) follows from (1) and (2). Let us prove (1) only, and one can easily find the proof of (2) in many books, e.g. Hartshorne.

Suppose ϕ is injective. Take a section $\sigma \in \mathscr{E}(U)$ on an open subset U. If $\phi([\sigma]) = 0 \in \mathscr{F}_p$, there exists a smaller open subset $V \subset U$ such that $\phi_V(\sigma) = 0 \in \mathscr{F}(V)$, hence $\sigma|_V = 0 \in \mathscr{E}(V)$. Consequently the equivalence class $[\sigma] = 0 \in \mathscr{E}_p$ and we conclude that ϕ_p is injective.

Conversely, suppose ϕ_p is injective for every point p. Take a section $\sigma \in \mathscr{E}(U)$. If $\phi(\sigma) = 0 \in \mathscr{F}(U)$, then for every point $p \in U$, $[\phi(\sigma)] = 0 \in \mathscr{F}_p$. Since ϕ_p is injective, it implies that $[\sigma] = 0 \in \mathscr{E}_p$ i.e. there exists an open subset $U_p \ni p$ such that $\sigma|_{U_p} = 0 \in \mathscr{E}(U_p)$. Applying the gluing property to the open covering $\{U_p\}$ of U, we conclude that $\sigma = 0 \in \mathscr{E}(U)$. □

Remark 2.17. The image of ϕ does not automatically form a sheaf. In general, it is only a *presheaf.* If the sheafification of $\mathrm{Im}(\phi)$ equals $\mathscr{F}$, we say that ϕ is surjective. In particular, it does *not* mean $\mathscr{E}(U) \to \mathscr{F}(U)$ is surjective for every open set U. Sometimes one has to pass to a smaller open set in order to obtain a surjection between sections.

Example 2.18. Consider the exponential map $\exp : \mathscr{O} \to \mathscr{O}^*$ on the punctured plane $\mathbb{C}\backslash\{0\}$. As a sheaf map it is surjective, but the section z over $\mathbb{C}\backslash\{0\}$ does not have an inverse. It does have an inverse over any contractible open subset.

2.4. **Cohomology of sheaves.** Let $\mathscr{F}$ be a sheaf on a topological space X. Take an open covering $\underline{U} = \{U_\alpha\}$ of X. Define the k-th *cochain group*

$$C^k(\underline{U}, \mathscr{F}) := \prod_{\alpha_0,\dots,\alpha_k} \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}).$$

An element σ of $C^k(\underline{U}, \mathscr{F})$ consists of a section $\sigma_{\alpha_0,\dots,\alpha_k} \in \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$ for every $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$.

Define a *coboundary map* $\delta : C^k(\underline{U}, \mathscr{F}) \to C^{k+1}(\underline{U}, \mathscr{F})$ by

$$(\delta\sigma)_{\alpha_0,\dots,\alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \sigma_{\alpha_0,\dots,\hat{\alpha}_j,\dots,\alpha_{k+1}}|_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_{k+1}}}.$$

Example 2.19. Consider $\underline{U} = \{U_1, U_2, U_3\}$ as an open covering of X. Take a cochain element $\sigma \in C^0(\underline{U}, \mathscr{F})$, i.e. σ is a collection of a section $\sigma_i \in \mathscr{F}(U_i)$ for every i. Then we have

$$(\delta\sigma)_{ij} = (\sigma_j - \sigma_i)|_{U_i \cap U_j} \in \mathscr{F}(U_i \cap U_j).$$

Now take $\tau \in C^1(\underline{U}, \mathscr{F})$, i.e. τ is a collection of a section $\tau_{ij} \in \mathscr{F}(U_i \cap U_j)$ for every pair i, j. Then we have

$$(\delta\tau)_{123} = (\tau_{23} - \tau_{13} + \tau_{12})|_{U_1\cap U_2 \cap U_3} \in \mathscr{F}(U_1 \cap U_2 \cap U_3).$$

A cochain $\sigma \in C^k(\underline{U}, \mathscr{F})$ is called a *cocyle* if $\delta\sigma = 0$. We say that σ is a *coboundary* if there exists $\tau \in C^{k-1}(\underline{U}, \mathscr{F})$ such that $\delta\tau = \sigma$.

Lemma 2.20. *A coboundary is a cocycle, i.e.* $\delta \circ \delta = 0$.

Proof. Let us prove it for the above example. The same idea applies in general with messier notation. Under the above setting, we have

$$\begin{aligned}((\delta\circ\delta)\sigma)_{123} &= (\delta\sigma)_{23}-(\delta\sigma)_{13}+(\delta\sigma)_{12}\\ &= (\sigma_3-\sigma_2)-(\sigma_3-\sigma_1)+(\sigma_2-\sigma_1)\\ &= 0\in\mathscr{F}(U_1\cap U_2\cap U_3).\end{aligned}$$

Here we omit the restriction notation, since it is obvious. □

Exercise 2.21. Prove in full generality that $\delta\circ\delta=0$.

For the coboundary map $\delta_k : C^k(\underline{U},\mathscr{F})\to C^{k+1}(\underline{U},\mathscr{F})$, define the *k-th cohomology group* (respect to $\underline{U}$) by

$$H^k(\underline{U},\mathscr{F}) := \frac{\ker(\delta_k)}{\mathrm{Im}(\delta_{k-1})}.$$

This is well-defined due to the above lemma.

Example 2.22. For $k=0$, we have $H^0(\underline{U},\mathscr{F})=\ker(\delta_0)$. Take an element $\{\sigma_i\in\mathscr{F}(U_i)\}$ in this group. Because it is a cocycle, it satisfies

$$\sigma_i|_{U_i\cap U_j}=\sigma_j|_{U_i\cap U_j}\in\mathscr{F}(U_i\cap U_j).$$

By the gluing property of sheaves, there exists a global section $\sigma\in\mathscr{F}(X)$ such that $\sigma|_{U_i}=\sigma_i$. Conversely, if σ is a global section, then define $\sigma_i=\sigma|_{U_i}\in\mathscr{F}(U_i)$. In this way we obtain a cocycle in $C^1(\underline{U},\mathscr{F})$. From the discussion we see that $H^0(\underline{U},\mathscr{F})=\mathscr{F}(X)$, which is independent of the choice of an open covering. Hence $H^0(\underline{U},\mathscr{F})$ is called the *group of global sections* of $\mathscr{F}$ and we often denote it by $H^0(X,\mathscr{F})$ or simply $H^0(\mathscr{F})$.

In general, we would like to define cohomology independent of open coverings. Take two open coverings $\underline{U}=\{U_\alpha\}_{\alpha\in I}$ and $\underline{V}=\{V_\beta\}_{\beta\in J}$. We say that $\underline{U}$ is a refinement of $\underline{V}$ if for every U_α there exists a V_β such that $U_\alpha\subset V_\beta$ and we write it as $\underline{U}<\underline{V}$. Then we also have an index map $\phi: I\to J$ sending α to β. It induces a map

$$\rho_\phi : C^k(\underline{V},\mathscr{F})\to C^k(\underline{U},\mathscr{F})$$

given by

$$\rho_\phi(\sigma)_{\alpha_0,\dots,\alpha_k}=\sigma_{\phi(\alpha_0),\dots,\phi(\alpha_p)}|_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_k}}.$$

One checks that it commutes with the coboundary map δ, i.e. $\delta\circ\rho_\phi=\rho_\phi\circ\delta$. Moreover, it induces a map

$$\rho: H^k(\underline{V},\mathscr{F})\to H^k(\underline{U},\mathscr{F}),$$

which is independent of the choice of ϕ. Finally, we define the k-th (Čech) cohomology group by passing to the direct limit:

$$H^k(X,\mathscr{F}) := \varinjlim H^k(\underline{U},\mathscr{F}).$$

The definition involves direct limit, which is inconvenient to use in practice. Nevertheless, we can simplify the situation once the open covering $\underline{U}$ is fine enough. We say that $\underline{U}=\{U_i\}_{i\in I}$ is *acyclic* respect to $\mathscr{F}$, if for any $k>0$ and $i_1,\dots,i_l\in I$ we have

$$H^k(U_{i_1}\cap\cdots\cap U_{i_l},\mathscr{F})=0.$$

Theorem 2.23 (Leray's Theorem)**.** *If the open covering $\underline{U}$ is acyclic respect to $\mathscr{F}$, then $H^*(\underline{U},\mathscr{F})\cong H^*(X,\mathscr{F})$.*

Remark 2.24. In the context of complex manifolds, if U_i's are contractible, then $\underline{U}$ is acyclic respect to the sheaves we will consider. While for varieties, if U_i's are affine, then $\underline{U}$ is acyclic.

Example 2.25. Let us compute the cohomology of the structure sheaf $\mathscr{O}$ on $\mathbb{P}^1_{\mathbb{C}}$. It is clear that $H^0(\mathbb{P}^1, \mathscr{O}) = \mathbb{C}$, since any global holomorphic function on a compact complex manifold is constant. For higher cohomology, use $[X, Y]$ to denote the coordinates of $\mathbb{P}^1$. Take the standard open covering $U = \{[X, Y] : X \neq 0\}$ and $V = \{[X, Y] : Y \neq 0\}$. It is acyclic respect to the structure sheaf $\mathscr{O}$ (morally because $U, V \cong \mathbb{C}$ is contractible). Let $s = Y/X$ and $t = X/Y$ as affine coordinates of U and V, respectively. Suppose h is an element in $C^1(\{U, V\}, \mathscr{O})$, i.e. $h \in \mathscr{O}(U \cap V)$. We can write

$$h = \sum_{i=-\infty}^{\infty} a_i s^i.$$

Now take

$$f = -\sum_{i=0}^{\infty} a_i s^i \in \mathscr{O}(U),$$

$$g = \sum_{i=-\infty}^{-1} a_i s^i = \sum_{i=-\infty}^{-1} a_i t^{-i} \in \mathscr{O}(V).$$

Then we have $(f, g) \in C^0(\{U, V\}, \mathscr{O})$ and $\delta((f, g)) = g - f = h$. It implies that $H^1(\mathbb{P}^1, \mathscr{O}) = 0$. All the other $H^k(\mathbb{P}^1, \mathscr{O}) = 0$ for $k > 1$, since there are only two open subsets in the covering.

Example 2.26. Let Ω denote the *sheaf of holomorphic one-forms* on a Riemann surface, i.e. locally a section of Ω can be expressed as $f(z)dz$, where z is local coordinate and $f(z)$ a holomorphic function. Let us compute the cohomology of Ω on $\mathbb{P}^1$. Take the above open covering. Suppose ω is a global holomorphic one-form. Then on the open chart U, it can be written as

$$\Big(\sum_{i=0}^{\infty} a_i s^i\Big) ds.$$

Using the relation $s = 1/t$ and $ds = -dt/t^2$, on V it can be expressed as

$$-\Big(\sum_{i=0}^{\infty} a_i t^{-i-2}\Big) dt,$$

which is holomorphic if and only if $a_i = 0$ for all i. Hence w is the zero one-form and $H^0(\mathbb{P}^1, \Omega) = 0$. Now take $\omega \in C^1(\{U, V\}, \Omega)$, i.e. $\omega \in \Omega(U \cap V) = \Omega(\mathbb{C}^*)$, we express it as

$$\omega = \Big(\sum_{i=-\infty}^{\infty} a_i t^i\Big) dt.$$

Note that any $\alpha \in \Omega(U)$ and $\beta \in \Omega(V)$ can be written as

$$\alpha = \Big(\sum_{i=0}^{\infty} b_i s^i\Big) ds,$$

$$\beta = \Big(\sum_{i=0}^{\infty} c_i t^i\Big) dt.$$

Hence on $U \cap V$ we have

$$\delta((\alpha,\beta)) = \beta - \alpha = -\Big(\sum_{i=0}^{\infty} b_i t^{-i-2}\Big)dt + \Big(\sum_{i=0}^{\infty} c_i t^i\Big)dt.$$

Note that only the term t^{-1} is missing from the expression. We conclude that $H^1(\mathbb{P}^1, \Omega) = \{a_{-1}t^{-1}dt\} \cong \mathbb{C}$.

Remark 2.27. If $\mathbb{P}^1$ is defined over an algebraically closed field K, replacing holomorphic functions by regular functions, we have $H^0(\mathbb{P}^1, \mathscr{O}) \cong K$ and $H^1(\mathbb{P}^1, \mathscr{O}) = 0$. Indeed, we have seen that the coordinate ring $A(U \cap V)$ is given by $K[s, 1/s]$, hence the above argument goes word by word. Similarly replacing holomorphic one-forms by *regular differentials*, i.e. in the expression $f(z)dz$, $f(z)$ is regular and dz satisfies the usual differentiation rules, we have $H^0(\mathbb{P}^1, \Omega) = 0$ and $H^1(\mathbb{P}^1, \Omega) \cong K$. In general, the rank of $H^1(X, \mathscr{O}) \cong H^0(X, \Omega)$ (by Serre Duality) is called the *genus* of a Riemann surface (or an algebraic curve) X.

Exercise 2.28. Let $D = p_1 + \cdots + p_n$ be a collection of n points in $\mathbb{P}^1$. We say that D is an *effective divisor of degree n*. Define the sheaf $\mathscr{O}(D)$ on $\mathbb{P}^1$ by

$$\mathscr{O}(D)(U) = \{f \in \mathscr{M}(U) : f \in \mathscr{O}(U \backslash \{p_1, \ldots, p_n\}) \text{ with at worst a simple pole at each } p_i\}.$$

Assume that the standard covering of $\mathbb{P}^1$ is acyclic respect to $\mathscr{O}(D)$. Use it to calculate the cohomology groups $H^*(\mathbb{P}^1, \mathscr{O}(D))$.

As many other homology/cohomology theories, one can associate a *long exact sequence of cohomology* to a short exact sequence. Suppose we have a short exact sequence of sheaves

$$0 \longrightarrow \mathscr{E} \xrightarrow{\ \alpha\ } \mathscr{F} \xrightarrow{\ \beta\ } \mathscr{G} \longrightarrow 0.$$

Then α and β induce maps

$$\alpha : C^k(\underline{U}, \mathscr{E}) \to C^k(\underline{U}, \mathscr{F}), \quad \beta : C^k(\underline{U}, \mathscr{F}) \to C^k(\underline{U}, \mathscr{G}).$$

Since the coboundary map δ is given by alternating sums of restrictions, α and β commute with δ, hence they send a cocycle to cocycle and a coboundary to coboundary. Consequently they induce maps for colomology

$$\alpha_* : H^k(X, \mathscr{E}) \to H^k(X, \mathscr{F}), \quad \beta_* : H^k(X, \mathscr{F}) \to H^k(X, \mathscr{G}).$$

Next we define the coboundary map

$$\delta_* : H^k(X, \mathscr{G}) \to H^{k+1}(X, \mathscr{E}).$$

For $\sigma \in C^k(\underline{U}, \mathscr{G})$ satisfying $\delta\sigma = 0$, after refining $\underline{U}$ (still denoted by $\underline{U}$) such that there exists $\tau \in C^k(\underline{U}, \mathscr{F})$ satisfying $\beta(\tau) = \sigma$, because β is surjective. Then $\beta(\delta\tau) = \delta(\beta(\tau)) = \delta\sigma = 0$, hence after refining further there exists $\mu \in C^{k+1}(\underline{U}, \mathscr{E})$ satisfying $\alpha(\mu) = \delta\tau$. Note that μ is a cocycle. It is because $\alpha(\delta\mu) = \delta(\alpha(\mu)) = \delta\delta(\tau) = 0$ and α is injective, hence $\delta\mu = 0$ and $\mu \in \ker(\delta)$. We thus take $\delta_*\sigma := [\mu] \in H^{k+1}(X, \mathscr{E})$. One checks that this is independent of the choice of τ and μ.

We say that a sequence of maps

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\ \alpha_{n-1}\ } A_n \xrightarrow{\ \alpha_n\ } A_{n+1} \longrightarrow \cdots$$

is *exact* if $\operatorname{Im}(\alpha_{n-1}) = \ker(\alpha_n)$.

Proposition 2.29. *The long sequence of cohomology associated to a short exact sequence of sheaves is exact.*

Proof. We prove it under an extra assumption that there exists an acyclic open covering $\underline{U}$ such that for any $U = U_{i_1} \cap \cdots \cap U_{i_k}$ we have the short exact sequence:

$$0 \to \mathscr{E}(U) \to \mathscr{F}(U) \to \mathscr{G}(U) \to 0.$$

At least for sheaves considered in this course, this assumption always holds. It further implies the following sequence is exact:

$$0 \to C^k(\underline{U}, \mathscr{E}) \to C^k(\underline{U}, \mathscr{F}) \to C^k(\underline{U}, \mathscr{G}) \to 0.$$

Let us prove that

$$H^k(\underline{U}, \mathscr{F}) \xrightarrow{\beta_*} H^k(\underline{U}, \mathscr{G}) \xrightarrow{\delta_*} H^{k+1}(\underline{U}, \mathscr{E})$$

is exact. The other cases are easier.

Consider $\tau \in Z^k(\underline{U}, \mathscr{F})$. In the definition of δ_*, take $\sigma = \beta(\tau)$. Then there exists $\mu \in C^k(\underline{U}, \mathscr{E})$ such that $\alpha(\mu) = \delta\tau = 0$. Then we have $\mu = 0$ since α is injective. Consequently $\delta_*\beta_*(\tau) = \delta_*(\sigma) = \mu = 0$, hence $\delta_*\beta_* = 0$ and $\mathrm{Im}(\beta_*) \subset \ker(\delta_*)$.

Conversely, suppose $\delta_*\sigma = 0$ for $\sigma \in Z^k(\underline{U}, \mathscr{G})$. In the definition of δ_*, it implies that $\mu = 0 \in H^{k+1}(\underline{U}, \mathscr{E})$, hence there exists $\gamma \in C^k(\underline{U}, \mathscr{E})$ such that $\delta\gamma = \mu$. Since $\alpha(\mu) = \delta\tau$, we have $\delta\tau = \delta\alpha(\gamma)$ and $\tau - \alpha(\gamma) \in Z^k(\underline{U}, \mathscr{F})$ is a cocycle. Moreover, $\beta(\tau - \alpha(\gamma)) = \beta(\tau) = \sigma$, hence $\beta_*(\tau - \alpha(\gamma)) = \sigma$. We conclude that $\ker(\delta_*) \subset \mathrm{Im}(\beta_*)$. □

Exercise 2.30. Prove in general the cohomology sequence is exact.

Example 2.31. Consider the short exact sequence

$$0 \longrightarrow \mathscr{I}_p \xrightarrow{i} \mathscr{O}_{\mathbb{P}^1} \xrightarrow{r} \mathscr{O}_p \longrightarrow 0.$$

Its long exact sequence of cohomology is as follows:

$$0 \to H^0(\mathscr{I}_p) \to H^0(\mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathscr{O}_p) \to H^1(\mathscr{I}_p) \to H^1(\mathscr{O}_{\mathbb{P}^1}) \to 0.$$

The last term is zero because p is a point so it does not have higher cohomology. We have $H^0(\mathscr{O}_{\mathbb{P}^1}) = K$ because any global regular function on $\mathbb{P}^1$ is constant. Note that $H^0(\mathscr{I}_p) = 0$, because vanishing at p forces such a constant function to be zero. Moreover we have seen that $H^1(\mathscr{O}_{\mathbb{P}^1}) = 0$. Altogether it implies $H^1(\mathscr{I}_p) = 0$, because $H^0(\mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathscr{O}_p)$ is an isomorphism by evaluating at p.

Exercise 2.32. Let D be an effective divisor of degree n on $\mathbb{P}^1$. We have the short exact sequence

$$0 \longrightarrow \mathscr{I}_D \xrightarrow{i} \mathscr{O}_{\mathbb{P}^1} \xrightarrow{r} \mathscr{O}_D \longrightarrow 0.$$

Use the associated long exact sequence to calculate the cohomology $H^*(\mathbb{P}^1, \mathscr{I}(D))$.

3. Vector bundles, line bundles and divisors

3.1. **Holomorphic vector bundles.** Let k be a positive integer. Consider $\pi : E \to X$ a holomorphic map between complex manifolds, such that for every $x \in X$ the fiber $E_x = \pi^{-1}(x)$ is isomorphic to $\mathbb{C}^k$ and there exists an open neighborhood U of x along with an isomorphism

$$\phi_U : E_U = \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^k$$

mapping E_x to $\{x\} \times \mathbb{C}^k$ which is a linear isomorphism between vector spaces. Then E is called a *holomorphic vector bundle of rank* k on X and has a *trivialization* $\{(U, \phi_U)\}$. If E is of rank one, we say that E is a *line bundle.*

We give another characterization of vector bundles based on transition functions. Suppose $\underline{U} = \{U_\alpha\}$ is an open covering of X. Given holomorphic functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}^k),$$

we can construct a vector bundle E by gluing $U_\alpha \times \mathbb{C}^k$ together. More precisely,

$$E = \sqcup (U_\alpha \times \mathbb{C}^k)/\sim$$

as a complex manifold is defined by identifying (x, v) with $(x, g_{\alpha\beta}(v))$ for $x \in U_\alpha \cap U_\beta$ and $v \in \mathbb{C}^k$ and $E \to X$ is given by projection to the bases U_α. Call $\{g_{\alpha\beta}\}$ the transition functions of E. They have to satisfy the following compatibility conditions:

$$g_{\alpha\beta}(x) \cdot g_{\beta\alpha}(x) = I, \quad \text{for all } x \in U_\alpha \cap U_\beta,$$
$$g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = I, \quad \text{for all } x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Exercise 3.1. Let E and F be two vector bundles on X of rank k and l, respectively. Define the *direct sum* $E \oplus F$, the *tensor product* $E \otimes F$, the *dual* E^*, and the *wedge product* $\wedge^r E$ for $r \leq k$. Calculate the ranks of these bundles and represent their transition functions in terms of the transition functions of E and F.

A *map* between two vector bundles E and F on X is given by a holomorphic map $f : E \to F$ such that $f(E_x) \subset F_x$ and $f_x = f|_{E_x} : E_x \to F_x$ is linear. Note that if $f(E_x)$ has the same rank for every x, then $\ker(f)$ and $\mathrm{Im}(f)$ are naturally subbundles of E and F, respectively. We say that E and F are *isomorphic* if f_x is a linear isomorphism for every x. A vector bundle is called *trivial* if it is isomorphic to $X \times \mathbb{C}^k$.

Exercise 3.2. Give an example of a map between vector bundles $f : E \to F$ on X such that the the image of f is not a vector bundle.

Exercise 3.3. Let L be a line bundle on X. Prove that $L \otimes L^*$ is a trivial line bundle.

Define a *section* σ as a holomorphic map $\sigma : X \to E$ such that $\sigma(x) \in E_x$ for every $x \in X$, i.e. $\pi \circ \sigma$ is identity. If $\sigma(x) = 0 \in E_x$, we say that σ is vanishing on x.

Exercise 3.4. Let L be a line bundle on X. Prove that L is trivial if and only if it possesses a nowhere vanishing section.

Example 3.5 (Holomorphic tangent bundles)**.** Let X be an n-dimensional complex manifold. Suppose $\phi_U : U \to \mathbb{C}^n$ are coordinate charts of X. Define the (holomorphic) *tangent bundle* T_X by setting $T_X = \sqcup T_x$ with

$$T_x = \mathbb{C}\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\} \cong \mathbb{C}^n$$

as well as transition functions $g_{UV} = J(\phi_V \phi_U^{-1})$, where J denotes the Jacobian matrix $(\frac{\partial w_i}{\partial z_j})$ for $1 \leq i, j \leq n$. The dual bundle T_X^* is called the *cotangent bundle* of X. The determinant $\det(T_X^*)$ is called the *canonical line bundle* of X.

Remark 3.6. Alternatively, one can define vector bundles on a topological space, a differential manifold and an algebraic variety. The above definitions and properties go through word by word once replacing “holomorphic map” by “homomorphism”, “smooth map” or “regular map”.

3.2. **Vector bundles and locally free sheaves.** There is a *one-to-one correspondence* between isomorphism classes of vector bundles of rank n and isomorphism classes of locally free sheaves of rank n on a variety X. Here we briefly explain the idea. The reader can refer to Hartshorne II 5, especially Ex. 5.18 for more details.

Let $\mathscr{O}_X$ be the structure sheaf of a variety X. Note that $\mathscr{O}_X(U)$ has a *ring* structure (not only a group) for any open set U. A *sheaf of $\mathscr{O}_X$-modules* is a sheaf $\mathscr{F}$ on X such that for each open set U, the group $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module. An $\mathscr{O}_X$-module $\mathscr{F}$ is called *free* if it is isomorphic to a direct sum of $\mathscr{O}_X$. It is called *locally free* if there is an open covering $\underline{U} = \{U_\alpha\}$ such that for each open subset U_α, $\mathscr{F}|_{U_\alpha}$ is a free $\mathscr{O}_X|_U$-module. The *rank* of $\mathscr{F}$ on U is the number of copies of $\mathscr{O}$ in the summation. In this course we only consider the rank being finite (such sheaves are called *coherent sheaves*). If X is connected, the rank of $\mathscr{F}$ does not vary with the open subsets. In particular, a locally free sheaf of rank 1 is also called an *invertible sheaf.*

Roughly speaking, if $\mathscr{F}$ is locally free of rank n, we can choose a set of n generators $x_1, \ldots, x_n$ for the $\mathscr{O}_X(U)$-module $\mathscr{F}(U)$. They span an n-dimensional affine space $A[x_1, \ldots, x_n]$ over U, where A is the coordinate ring of U. By changing to a different set of generators over another open subset, one can write down the transition functions, hence it associates to $\mathscr{F}$ a vector bundle structure. Conversely if F is a vector bundle on X, locally we have $F|_U \cong U \times \mathbb{A}^n$ with $x_1, \ldots, x_n$ a basis (i.e. n linearly independent sections) of $\mathbb{A}^n$ over U. Then we can associate to $F|_U$ an $\mathscr{O}_X(U)$-module of rank n using $x_1, \ldots, x_n$ as generators.

Example 3.7. Let $X \subset \mathbb{P}^n$ be a (smooth) variety and $Y \subset X$ a hypersurface, i.e. Y is cut out (transversely) by a hypersurface F in $\mathbb{P}^n$ with X. We have the short exact sequence

$$0 \to \mathscr{I}_{Y/X} \to \mathscr{O}_X \to \mathscr{O}_Y \to 0.$$

The ideal sheaf $\mathscr{I}_{Y/X}$ is an invertible sheaf. Indeed, for an open subset $U \subset X$, $\mathscr{I}_{Y/X}(U)$ can be expressed as $(F|_U) \cdot \mathscr{O}_X(U)$, hence is locally free of rank 1. The sheaf $\mathscr{O}_Y$ (extended to X by zero) is *not* locally free. For $U \cap Y = \emptyset$, $\mathscr{O}_Y(U) = 0$ and for $U \cap Y \neq \emptyset$, $\mathscr{O}_Y(U)$ is non-zero. Later we will see how to construct a line bundle corresponding to $\mathscr{I}_{Y/X}$.

3.3. **Divisors.** Let X be a variety. Suppose $Y \subset X$ is an irreducible subvariety of codimension one. We say that Y is an *irreducible divisor* of X. More precisely, for every $p \in Y$ there exists an open neighborhood $U \subset X$ of p such that $U \cap Y$ is cut out by a (holomorphic or regular) function f. We call f a local defining equation for Y near p. A *divisor* D on X is a formal linear combination of irreducible divisors:

$$D = \sum_{i=i}^{n} a_i Y_i,$$

where $a_i \in \mathbb{Z}$ (or $\mathbb{Q}$, $\mathbb{R}$ depending on the context). If $a_i \geq 0$ for all i, we say that D is *effective* and denote it by $D \geq 0$. The divisors on X form an additive group $\mathrm{Div}(X)$.

Suppose f is a local defining equation of an irreducible divisor $Y \subset X$ on an open subset $U \subset X$. For another function g on X, locally we can write

$$g = f^a \cdot h$$

such that the regular function h is coprime with f in $\mathscr{O}_X(U)$. We say that a is the *vanishing order* of g along $Y \cap U$. Note that the vanishing order is locally a constant, hence is independent of U. We use

$$\operatorname{ord}_Y(g) = a$$

to denote the vanishing order of g along Y.

For two regular functions g, h on X, we have

$$\operatorname{ord}_Y(gh) = \operatorname{ord}_Y(g) + \operatorname{ord}_Y(h).$$

For a function $f = g/h$, we define

$$\operatorname{ord}_Y(f) = \operatorname{ord}_Y(g) - \operatorname{ord}_Y(h).$$

If $\operatorname{ord}_Y(f) > 0$, we say that f has a *zero* along Y. If $\operatorname{ord}_Y(f) < 0$, we say that f has a *pole* along Y. We also define the *divisor associated to* f by

$$(f) = \sum_Y \operatorname{ord}_Y(f),$$

as well as the *divisor of zeros*

$$(f)_0 = \sum_Y \operatorname{ord}_Y(g)$$

and the *divisor of poles*

$$(f)_\infty = \sum_Y \operatorname{ord}_Y(h).$$

They satisfy

$$(f) = (f)_0 - (f)_\infty.$$

If $D = (f)$ is the associated divisor of a global meromorphic function f, D is called a *principal divisor*.

Let $\mathscr{M}^*$ be the multiplicative sheaf of (not identically zero) meromorphic functions and $\mathscr{O}^*$ the multiplicative sheaf of nowhere vanishing regular functions, which is a subsheaf of $\mathscr{M}^*$.

Proposition 3.8. *We have a correspondence* $\operatorname{Div}(X) \cong H^0(X, \mathscr{M}^*/\mathscr{O}^*)$.

Proof. Suppose $\{f_\alpha\}$ represents a global section of $\mathscr{M}^*/\mathscr{O}^*$ with respect to an open covering $\underline{U} = \{U_\alpha\}$. Associate to it a divisor $D_\alpha = (f_\alpha)$ in U_α. We claim that $D_\alpha = D_\beta$ in $U_\alpha \cap U_\beta$. This is due to

$$\frac{f_\alpha}{f_\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta),$$

hence f_α and f_β define the same divisor. Consequently $\{D_\alpha\}$ defines a global divisor. Moreover, if $\{f_\alpha\}$ and $\{g_\alpha\}$ define the same divisor, then $f_\alpha/g_\alpha \in \mathscr{O}^*(U_\alpha)$, hence $\{f_\alpha\}$ and $\{g_\alpha\}$ represent the same section of $\mathscr{M}^*/\mathscr{O}^*$. This shows an injection

$$H^0(X, \mathscr{M}^*/\mathscr{O}^*) \hookrightarrow \operatorname{Div}(X).$$

Conversely, suppose $D = \sum a_i Y_i$ is a divisor on X with $a_i \in \mathbb{Z}$ and Y_i effective. We can choose an open covering $\underline{U} = \{U_\alpha\}$ such that Y_i is locally defined by $g_{i\alpha} \in \mathscr{O}(U_\alpha)$. Consider

$$f_\alpha = \prod_i (g_{i\alpha})^{a_i} \in \mathscr{M}^*(U_\alpha).$$

Then we have

$$\frac{f_\alpha}{f_\beta} = \prod_i \Big(\frac{g_{i\alpha}}{g_{i\beta}}\Big)^{a_i}.$$

Both $g_{i\alpha}$ and $g_{i\beta}$ cut out the same divisor $Y_i|_{U_\alpha\cap U_\beta}$ in $U_\alpha \cap U_\beta$, hence we conclude that

$$\frac{g_{i\alpha}}{g_{i\beta}} \in \mathscr{O}^*(U_\alpha \cap U_\beta), \quad \frac{f_\alpha}{f_\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta).$$

Then $\{f_\alpha\}$ defines a global section of $\mathscr{M}^*/\mathscr{O}^*$. Finally if D determines the zero section of $\mathscr{M}^*/\mathscr{O}^*$ (which is 1 since the group structure is multiplicative), it means locally $f_\alpha \in \mathscr{O}^*(U_\alpha)$ (after refining the open covering). Then it does not have zeros or poles, hence $D|_{U_\alpha} = 0$ for each U_α and D is globally zero. This shows the other injection

$$\mathrm{Div}(X) \hookrightarrow H^0(X, \mathscr{M}^*/\mathscr{O}^*).$$

□

3.4. **Line bundles.** Recall that a line bundle L on X is a vector bundle of rank 1. Equivalently, it is a locally free sheaf of rank 1. Define the *Picard group* $\mathrm{Pic}(X)$ parameterizing isomorphism classes of line bundles on X. The group law is given by tensor product. We can interpret $\mathrm{Pic}(X)$ as a cohomology group.

Proposition 3.9. *There is a one-to-one correspondence between the isomorphism classes of line bundles on X and $H^1(X, \mathscr{O}^*)$, i.e.*

$$\mathrm{Pic}(X) \cong H^1(X, \mathscr{O}^*).$$

Proof. Take an open covering $\underline{U} = \{U_\alpha\}$ of X with respect to the trivialization of a line bundle L. The transition function

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$$

can be regarded as a section of $\mathscr{O}^*(U_\alpha \cap U_\beta)$, satisfying

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = 1,$$

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1.$$

Therefore, $\{g_{\alpha\beta}\}$ is a cocycle in $C^1(\underline{U}, \mathscr{O}^*)$, hence represents a cohomology class in $H^1(X, \mathscr{O}^*)$.

Suppose M is another line bundle with transition functions $\{h_{\alpha\beta}\}$. If M and L are isomorphic, then $L \otimes M^*$ is trivial, i.e. $\{g_{\alpha\beta}/h_{\alpha\beta}\}$ are transition functions of $L \otimes M^*$, which has a nowhere vanishing section σ. Suppose on U_α we have $\sigma_\alpha : U_\alpha \to \mathbb{C}^*$ as the restriction of σ. Then on $U_\alpha \cap U_\beta$ we have

$$\frac{g_{\alpha\beta}}{h_{\alpha\beta}} \cdot \sigma_\alpha = \sigma_\beta.$$

Therefore we conclude that

$$\frac{g_{\alpha\beta}}{h_{\alpha\beta}} = \frac{\sigma_\beta}{\sigma_\alpha} \in \delta C^0(\underline{U}, \mathscr{O}^*).$$

□

Now we describe another important correspondence between line bundles and divisors. Suppose D is a divisor on X with local defining equations $\{f_\alpha\}$ such that $f_\alpha \in \mathscr{M}^*(U_\alpha)$. Define

$$g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

Then we have $g_{\alpha\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta)$. Moreover, $\{g_{\alpha\beta}\}$ satisfy the assumptions imposed to transition functions, hence they define a line bundle, denoted by $L = [D]$ or $L = \mathscr{O}_X(D)$. We have a group homomorphism

$$\mathrm{Div}(X) \to \mathrm{Pic}(X)$$

induced by

$$D + D' \mapsto [D] \otimes [D'].$$

We say that D and D' are *linearly equivalent*, if $[D]$ and $[D']$ are isomorphic line bundles. We denote linear equivalence by

$$D \sim D'.$$

The following result says that the kernel of the above map consists of principal divisors. In other words, two divisors $D \sim D'$ if and only if $D - D'$ is a principal divisor.

Proposition 3.10. *The associated line bundle $[D]$ is trivial if and only if D is a principal divisor, i.e. $D = (f)$ for some $f \in \mathscr{M}^*(X)$.*

Proof. Suppose $D = (f)$ is the associated divisor of a meromorphic function f on X. Then D has local defining equations $\{f_\alpha = f|_{U_\alpha}\}$. The transition functions associated to $[D]$ are all equal to 1, hence $[D]$ is a trivial line bundle. Conversely, suppose $[D]$ is trivial. Then it has a nowhere vanishing section σ whose restriction to U_α is denoted by σ_α. The transition functions $g_{\alpha\beta} = f_\beta / f_\alpha$ defined above satisfy

$$g_{\alpha\beta} \cdot \sigma_\alpha = \sigma_\beta,$$

hence we have

$$\frac{f_\alpha}{\sigma_\alpha} = \frac{f_\beta}{\sigma_\beta} \in \mathscr{M}^*(U_\alpha \cap U_\beta).$$

We can glue $\{f_\alpha / \sigma_\alpha\}$ to form a global function $f \in \mathscr{M}^*(X)$. Since σ is nowhere vanishing, we obtain that $(f) = D$. □

Let us summarize, using the short exact sequence

$$0 \to \mathscr{O}^* \to \mathscr{M}^* \to \mathscr{M}^*/\mathscr{O}^* \to 0.$$

Recall that

$$\mathrm{Div}(X) \cong H^0(X, \mathscr{M}^*/\mathscr{O}^*), \quad \mathrm{Pic}(X) \cong H^1(X, \mathscr{O}^*).$$

Then we have the long exact sequence

$$0 \to H^0(X, \mathscr{O}^*) \to H^0(X, \mathscr{M}^*) \xrightarrow{(\cdot)} \mathrm{Div}(X) \xrightarrow{[\cdot]} \mathrm{Pic}(X) \to \cdots$$

which encodes all the information in the above discussions.

3.5. **Sections of a line bundle.** Let L be a line bundle on X with transition functions $\{g_{\alpha\beta}\}$. A *holomorphic section* s of L has restriction $s_\alpha \in \mathscr{O}(U_\alpha)$, satisfying

$$g_{\alpha\beta} s_\alpha = s_\beta.$$

Conversely, a collection $\{s_\alpha \in \mathscr{O}(U_\alpha)\}$ such that $s_\beta / s_\alpha = g_{\alpha\beta}$ determines a section of L.

Similarly, we define a *meromorphic section* s to be a collection

$$\{s_\alpha \in \mathscr{M}(U_\alpha)\}$$

such that $g_{\alpha\beta}s_\alpha = s_\beta$. Suppose $t \not\equiv 0$ is another meromorphic section with collection $\{t_\alpha\}$. We have

$$\frac{s_\beta}{t_\beta} = \frac{s_\alpha}{t_\alpha},$$

hence the quotient s/t is a global meromorphic function. Conversely, if f is a global meromorphic function, then $\{f \cdot s_\alpha\}$ defines another meromorphic section of L.

For a meromorphic section $s \not\equiv 0$, consider the divisor (s_α) associated to the local section s_α in U_α. Since

$$\frac{s_\beta}{s_\alpha} = g_{\alpha\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta),$$

$\{(s_\alpha)\}$ form a *global* divisor (s) on X. Conversely, suppose D is a divisor. Consider the construction of the associated line bundle $[D]$. Suppose the local defining equations of D are given by $\{s_\alpha\}$. Then the transition functions of $[D]$ are $\{g_{\alpha\beta} = s_\beta/s_\alpha\}$ and consequently the collection $\{s_\alpha\}$ gives rise to a meromorphic section of $[D]$. Note that for a section s of L, the divisor (s) is effective if and only if s is a holomorphic section. We thus obtain the following result.

Proposition 3.11. *For any section s of L, we have $L \cong [(s)]$. A line bundle L is associated to a divisor D if and only if it has a meromorphic section s such that $(s) = D$. In particular, L has a holomorphic section if and only if it is associated to an effective divisor.*

Now we treat a line bundle as a locally free sheaf of rank 1 and reinterpret the above correspondence. Let D be a divisor on X. Define a sheaf $\mathscr{O}_X(D)$ or simply $\mathscr{O}(D)$ by

$$\mathscr{O}(D)(U) = \{f \in \mathscr{M}(U) : (f) + D|_U \geq 0\}.$$

It has a vector space structure since $(f) + D|_U \geq 0$ and $(g) + D|_U \geq 0$ implies that $(af + bg) + D|_U \geq 0$ for any a, b in the base field.

Proposition 3.12. *The space of holomorphic sections of $[D]$ can be identified with $H^0(X, \mathscr{O}(D))$.*

Proof. A global section $s \in H^0(X, \mathscr{O}(D))$ is a meromorphic function satisfying

$$(s) + D \geq 0.$$

Suppose D is locally defined by $\{f_\alpha\}$. The associated line bundle $[D]$ has transition functions

$$\{g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}\}.$$

Then the collection $\{s \cdot f_\alpha\}$ defines a section σ of $[D]$. Since

$$(s) + (f_\alpha) \geq 0$$

in every U_α, σ is a holomorphic section of $[D]$. Moreover, the associated divisor $D' = (s) + D$ of the section is linearly equivalent to D, since $D' - D = (s)$ is principal.

Conversely, given a holomorphic section σ of $[D]$, i.e. a collection $\{h_\alpha \in \mathscr{O}(U_\alpha)\}$ such that

$$\frac{h_\beta}{h_\alpha} = g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

Then $\{h_\alpha/f_\alpha\}$ defines a global meromorphic function g. Since $(h_\alpha) \geq 0$ in every U_α, we have

$$(g|_{U_\alpha}) + (f_\alpha) = (h_\alpha) \geq 0,$$

hence $(g) + D \geq 0$ globally on X and $g \in H^0(X, \mathscr{O}(D))$. □

Remark 3.13. Replacing X by any open subset U, the proposition implies that the sheaf $\mathscr{O}(D)$ can be regarded as gathering local holomorphic sections of the line bundle $[D]$. If $D \sim D'$, i.e. $D' - D = (f)$ for a global meromorphic function f, then for any $g \in \mathscr{O}(D')(U)$, we have

$$0 \leq (g) + D' = (g) + (f) + (D) = (fg) + (D)$$

restricted to U. So we obtain an isomorphism

$$\mathscr{O}(D')(U) \xrightarrow{\cdot f} \mathscr{O}(D)(U)$$

for any open subset U, compatible with the sheaf restriction maps. In this sense, the sheaf $\mathscr{O}(D)$ and the line bundle $[D]$ have a one-to-one correspondence up to isomorphism and linear equivalence, respectively, assuming that every line bundle can be associated to a divisor.

Let $|D|$ be the set of effective divisors that are linearly equivalent to D. We call $|D|$ the *linear system* associated to D.

Proposition 3.14. *Let X be compact and D a divisor on X. Then we have*

$$\mathbb{P}H^0(X, \mathscr{O}(D)) = |D|,$$

i.e. an effective divisor in $|D|$ and a holomorphic section of $[D]$ (up to scalar) determine each other.

Proof. For any $D' \in |D|$, by definition $D' - D = (f)$ is principal for some $f \in \mathscr{M}(X)$, hence $(f) + D = D' \geq 0$ and $f \in H^0(X, \mathscr{O}(D))$. Since X is compact, if g is another function such that $D' - D = (g)$, then $(f/g) = 0$, i.e. f/g is holomorphic, hence it is a constant.

Conversely, any $f \in H^0(X, \mathscr{O}(D))$ defines an effective divisor $D' = (f) + D$. If $(f) + D = (g) + D$, then $(f/g) = 0$ and f/g is a constant, since X is compact. □

Exercise 3.15. Let $D = \sum a_i p_i$ be a divisor on $\mathbb{P}^1$ with $a_i \in \mathbb{Z}$ and $p_i \in \mathbb{P}^1$. Define the *degree* of D by $\deg(D) = \sum a_i$.

(1) Prove that $D \sim D'$ if and only if $\deg(D) = \deg(D')$.

(2) Calculate the cohomology $H^*(\mathbb{P}^1, \mathscr{O}(D))$ in terms of $\deg(D)$.

4. Classical results

In this section, we discuss some fundamental results on Riemann surfaces (and in general on algebraic curves).

4.1. **The Riemann-Roch formula.** Let X be a compact Riemann surface (or a closed algebraic curve). Define its *(arithmetic) genus* by

$$g := h^1(\mathscr{O}_X) = \dim_{\mathbb{C}} H^1(\mathscr{O}_X).$$

Theorem 4.1 (Riemann-Roch Formula)**.** *Let D be a divisor on X and $\mathscr{O}(D)$ the associated line bundle or invertible sheaf. Then we have*

$$h^0(\mathscr{O}(D)) - h^1(\mathscr{O}(D)) = 1 - g + \deg(D).$$

Remark 4.2. Define the (holomorphic) *Euler characteristic* of a sheaf $\mathscr{F}$ by

$$\chi(\mathscr{F}) := \sum_{i\geq 0}(-1)^i h^i(\mathscr{F}).$$

The Riemann-Roch formula can be written as

$$\chi(\mathscr{O}(D)) - \chi(\mathscr{O}_X) = \deg(D).$$

Proof. Let us first prove it for effective divisors of degree ≥ 0. Do induction on n. The formula obviously holds for $\mathscr{O}_X$. Suppose it is true for $\deg(D) < n$. Consider $D = p + D'$ with D' an effective divisor of degree $n-1$. We have the short exact sequence

$$0 \to \mathscr{O}(D') \to \mathscr{O}(D) \to \mathbb{C}_p \to 0,$$

where $\mathbb{C}_p$ is the skyscraper sheaf with one-dimensional stalk supported at p. The exactness can be easily checked. The map $\mathscr{O}(D') \to \mathscr{O}(D)$ is an inclusion, since

$$(f) + D' \geq 0$$

implies that

$$(f) + D = (f) + D' + p \geq 0.$$

The quotient corresponds to germs of functions f at p such that

$$(f)|_U + D'|_U = -p$$

in arbitrarily small neighborhoods U of p. In other words, if $\mathrm{ord}_p(D') = m \geq 0$, we can write

$$f = z^{-m-1}h(z),$$

where $h \in \mathscr{O}^*(U)$. So the quotient sheaf is given by $\mathbb{C}\cdot\{z^{-m-1}\} \cong \mathbb{C}$ supported at p. Since the associated cohomology sequence is long exact, we have

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}(D')) + 1 = 1 - g + (n-1) + 1 = 1 - g + n.$$

In general, write a divisor $D = D_1 - D_2$, where D_1 and D_2 are both effective divisors of degree d_1 and d_2, respectively, and $d_1 - d_2 = \deg(D)$. By the same token, we have the short exact sequence

$$0 \to \mathscr{O}(D) \to \mathscr{O}(D_1) \to \mathbb{C}^{d_2} \to 0.$$

Then we obtain that

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}(D_1)) - d_2 = 1 - g + d_1 - d_2 = 1 - g + \deg(D).$$

□

Remark 4.3. Assuming the Serre duality

$$H^1(\mathscr{O}(D)) \cong H^0(K \otimes \mathscr{O}(-D)),$$

where K is the canonical line bundle of X, then we can rewrite the Riemann-Roch formula as

$$h^0(L) - h^0(K \otimes L^*) = 1 - g + \deg(L),$$

where L is a line bundle on X. Note that K is a degree $2g-2$ line bundle (to be discussed later). We conclude that

$$h^0(K) = g, \quad h^1(K) = h^0(\mathscr{O}) = 1.$$

It implies that the space of holomorphic one-forms on a genus g Riemann surface is g-dimensional.

4.2. **Serre Duality.** Let D be a divisor on a closed Riemann surface X, treated as a projective complex algebraic curve. Let Ω be the sheaf of holomorphic one-forms on X (the corresponding line bundle is the canonical line bundle K).

Theorem 4.4. *There is a natural perfect paring*

$$H^0(X, \Omega(-D)) \times H^1(X, \mathscr{O}(D)) \to \mathbb{C}.$$

In particular, $H^0(X, \Omega(-D)) \cong H^1(X, \mathscr{O}(D))^*$ *and* $h^0(X, \Omega(-D)) = h^1(X, \mathscr{O}(D))$.

For the proof, we follow Ravi Vakil's notes (originally due to Weil and elaborated by Serre) for an algebraic version. It consists of four steps.

Step 1. Let F be the *function field* of X, i.e., an element of F is a collection of compatible rational functions $f_\alpha \in U_\alpha$ for an open covering $\{U_\alpha\}$ where f_α is an element in the field of fractions of the ring of regular functions on U.

Denote by $I(D) := H^1(X, \mathscr{O}(D))$. Define a *repartition* $r = (f_p)_{p \in X}$ as a collection of $f_p \in F$ indexed by every point $p \in X$ such that f_p is holomorphic at p for all but finitely many p (here f_p and f_q do not need to be the same function for $p \neq q$). Denote by R the set of all repartitions. For $r_1, r_2 \in R$, $r_1 + r_2 \in R$ and $r_1 r_2 \in R$, hence R is called the *ring of repartitions*. Note that F is a *subring* of R, because every meromorphic function on X has finitely many poles. Hence R is also an F-algebra.

For a divisor D on X, let $v_p(D)$ be the multiplicity of p in D. In particular, $v_p(D) = 0$ for all but finitely many p. Define

$$R(D) = \{(f_p) \in R : v_p(f_p) + v_p(D) \geq 0 \text{ for all } p \in X\}.$$

Then $R(D)$ is an additive subgroup of R, because $v_p(f+g) \geq \min\{v_p(f), v_p(g)\}$. Note that if $D \leq D'$, then $R(D) \subset R(D')$.

Lemma 4.5. $I(D) = H^1(X, \mathscr{O}(D)) \cong R/(R(D) + F)$.

Proof. Let $\mathscr{F}$ be the sheaf of rational functions on X. For an open set U and $f \in \mathscr{F}(U)$, consider the maximal subset V of X on which f extends. Then $X \backslash V$ consists of finitely many points. Hence f can be identified with an element in F. In other words, one can treat $\mathscr{F}$ as the locally constant sheaf associated to the group F. Hence there are no obstructions to glue sections, and $H^1(X, \mathscr{F}) = 0$. Alternatively, $\mathscr{F}(V) \to \mathscr{F}(U)$ is onto for any $U \subset V$, hence $\mathscr{F}$ is *flasque* and $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.

We have the following short exact sequence

$$0 \to \mathscr{O}(D) \to \mathscr{F} \to S \to 0,$$

where S is the cokernel sheaf. Take the long exact sequence of cohomology:

$$\cdots H^0(X, \mathscr{F}) = F \to H^0(X, S) \to H^1(X, \mathscr{O}(D)) \to H^1(X, \mathscr{F}) = 0 \cdots$$

It suffices to prove that $H^0(X, S) \cong R/R(D)$. Note that for each $p \in X$, the stalk

$$S_p = (F/\mathscr{O}(D))_p = F/\{f \in F : v_p(f) + v_p(D) \geq 0\}.$$

Hence a section of S is a selection of values of S_p over all $p \in X$, where all but finitely many choices are zero, i.e., it is an element in $\bigoplus_{p \in X} S_p$. On the other hand, by definition S_p is the p-part of $R/R(D)$, hence $R/R(D) = \bigoplus_{p \in X} S_p$. We thus obtain the desired isomorphism. $\square$

Step 2. Denote by $J(D) := I(D)^* = (R/(R(D)+F))^*$ (so we will prove that $J(D) \cong H^0(X, \omega(-D)))$. An element of $J(D)$ is a $\mathbb{C}$-linear form on R that vanishes on $R(D)$ and F. Recall that for $D \leq D'$, $R(D) \subset R(D')$, hence $J(D) \supset J(D')$. Define

$$J := \bigcup_D J(D).$$

An element of J is a linear functional on R that vanishes on F and also on $R(D)$ for *some* divisor D.

Lemma 4.6. *J is a vector space over F.*

Proof. Take $f \in F$ and $\alpha \in J$. Define the linear functional $f\alpha : R \to \mathbb{C}$ by

$$\langle f\alpha, r\rangle = \langle \alpha, fr\rangle$$

for $r \in R$. If $r \in F$, then $fr \in F$. Since α vanishes on F, so does $f\alpha$. Moreover, take $\alpha \in J(D)$ for some divisor D and suppose $(f) = D'$. If $r \in R(D-D')$, then $fr \in R(D)$ and α vanished on fr, hence $f\alpha$ vanishes on r, i.e., $f\alpha \in J(D-D') \subset J$. Therefore, under this assignment J is a vector space over F. □

Lemma 4.7. $\dim_F J \leq 1$.

Proof. Prove by contradiction. Suppose $\alpha, \beta \in J$ are linearly independent over F. If $\alpha \in J(A)$ and $\beta \in J(B)$, take D such that $D \leq A, B$, and then $\alpha, \beta \in J(D)$. Denote by $d = \deg(D)$. Let D_n be any divisor with $\deg(D_n) = n$. For $f, g \in H^0(X, \mathscr{O}(D_n))$, by the above proof we have $f\alpha, g\beta \in J(D-D_n)$. Since α, β are linearly independent over F, we obtain an injection

$$H^0(X, \mathscr{O}(D_n)) \oplus H^0(X, \mathscr{O}(D_n)) \to J(D-D_n)$$

by $(f, g) \mapsto f\alpha + g\beta$. It follows that

$$\dim_{\mathbb{C}} J(D-D_n) \geq 2h^0(X, \mathscr{O}(D_n)).$$

On one hand,

$$\begin{aligned} \dim_{\mathbb{C}} J(D-D_n) &= \dim_{\mathbb{C}} I(D-D_n) \\ &= h^1(X, \mathscr{O}(D-D_n)) \\ &= h^0(X, \mathscr{O}(D-D_n)) - (1 - g_X + (d-n)) \\ &= n + g_X - 1 - d \end{aligned}$$

for $n > d$. On the other hand,

$$\begin{aligned} 2h^0(X, \mathscr{O}(D_n)) &\geq 2(1 - g_X + n) \\ &= 2n + 2 - 2g_X. \end{aligned}$$

Comparing them for $n \gg 0$ leads to a contradiction. □

Later on we will see that $\dim_F J = 1$.

Step 3. Let M be the set of rational differentials on X. If $\{f(z)dz\}$ and $\{g(z)dz\} \not\equiv 0$ are two differentials, then $\{f/g\} \in F$, hence M is a one-dimensional vector space over F. Suppose a differential ω has a pole at p of order n. Expand w at p as

$$a_{-n}t^{-n} + \cdots + a_{-1}t^{-1} + \text{ regular part.}$$

Define the *residue* of ω at p by

$$\operatorname{Res}_p(\omega) = a_{-1}.$$

Alternatively,

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i}\int_\gamma \omega$$

for a small loop γ surrounding p. Recall the *residue theorem* that

$$\sum_{p\in X} \mathrm{Res}_p(\omega) = 0,$$

which can be proved by Stokes' theorem.

Step 4. For a differential $\omega \in M$, let

$$(\omega) = \sum_{p\in X} v_p(\omega)p$$

be the associated divisor of ω. Then a section of $H^0(X, \Omega(-D))$ is a differential ω such that $(\omega) \geq D$. Define a *pairing*

$$\langle -, - \rangle : M \times R \to \mathbb{C}$$

by

$$\langle \omega, r\rangle = \sum_{p\in X} \mathrm{Res}_p(r_p\omega).$$

Lemma 4.8. *The above pairing satisfies*
(1) $\langle \omega, r\rangle = 0$ *if* $r \in F$*;*
(2) $\langle \omega, r\rangle = 0$ *if* $r \in R(D)$ *and* $\omega \in H^0(X, \Omega(-D))$*;*
(3) $\langle f\omega, r\rangle = \langle \omega, fr\rangle$ *if* $f \in F$*.*

Proof. (1) follows from the residue theorem. For (2), $r\omega$ has no pole. For (3), both sides are equal to the sum of residues over $fr\omega$. □

For $\omega \in M$, let $\theta(\omega)$ be the linear functional on R induced by the above pairing, i.e., we have a map

$$\theta : M \to J.$$

The above lemma says that if $\omega \in H^0(X, \Omega(-D))$, then $\theta(\omega)$ vanishes on $R(D)$ and F, hence $\theta(\omega)$ can be regarded as a linear functional on $R/(R(D) + F)$. In particular, we obtain

$$\theta_D : H^0(X, \Omega(-D)) \to J(D) = (R/(R(D) + F))^*.$$

Lemma 4.9. *For* $\omega \in M$*, if* $\theta(\omega) \in J(D)$*, i.e., if* $\theta(\omega)$ *vanishes on* $R(D)$*, then* $\omega \in H^0(X, \Omega(-D))$*.*

Proof. Prove by contradiction. Suppose $\omega \notin H^0(X, \Omega(-D))$. Then there exists $p \in X$ such that

$$v_p(\omega) < v_p(D).$$

Take a repartition $r \in R(D)$ by $r_q = 0$ for $q \neq p$ and $r_p = t^{-v_p(\omega)-1}$. Because

$$v_p(r_p\omega) = -1,$$

we have

$$\theta(\omega)(r) = \sum_{q\in X} \mathrm{Res}_q(r_q\omega) = \mathrm{Res}_p(r_p\omega) \neq 0.$$

It means that $\theta(\omega)$ does not vanish on $R(D)$, leading to a contradiction. □

Proof of Serre Duality. It suffices to prove that

$$\theta_D : H^0(X, \Omega(-D)) \to J(D) = (R/(R(D) + F))^*$$

is an isomorphism.

To see that θ_D is injective, take $\omega \in H^0(X, \Omega(-D))$ such that $\theta_D(\omega) = 0$. Then $\theta(\omega)$ vanishes entirely on R, hence $\theta(\omega) \in J(D')$ for every divisor D'. By the above lemma, $\omega \in \Omega(-D')$ for every divisor D', which is impossible if $\omega \neq 0$, e.g., by taking $D' > (\omega)$. This also shows that $\theta : M \to J$ is injective.

To see that θ_D is surjective, because $\dim_F M = 1$ and $\dim_F J \leq 1$, the injection $\theta : M \to J$ is also onto. Hence for $\alpha \in J(D) \subset J$, there exists $\omega \in M$ such that $\theta(\omega) = \alpha$. Applying the above lemma again, we obtain that $\omega \in H^0(X, \Omega(-D))$ and $\theta_D(\omega) = \alpha$. □

Remark 4.10. There is a much more general version of Serre Duality for higher dimensional varieties. One can first derive it on $\mathbb{P}^n$ by using Ext-groups, and then prove it for subvarieties of $\mathbb{P}^n$. See [H, III.7] for more details.

4.3. **The Riemann-Hurwitz formula.** A *branched cover* $\pi : X \to Y$ between two (compact, connected) Riemann surfaces is a (surjective) regular morphism. For a general point $q \in Y$, $\pi^{-1}(q)$ consists of d distinct points. Call d the *degree* of π. Locally around $p \mapsto q$, if the map is given by

$$x \mapsto y = x^m,$$

where x, y are local coordinates of p, q, respectively, call m the *vanishing order* of π at p and denote it by

$$\mathrm{ord}_p(\pi) = m.$$

If $\mathrm{ord}_p(\pi) > 1$, we say that p is a *ramification point*. If $\pi^{-1}(q)$ contains a ramification point, then q is called a *branch point*. Define the *pullback*

$$\pi^*(q) = \sum_{p \in \pi^{-1}(q)} \mathrm{ord}_p(\pi) \cdot p.$$

Note that $\pi^*(q)$ is a degree d effective divisor on X.

Theorem 4.11 (Riemann-Hurwitz Formula)**.** *Let $\pi : X \to Y$ be a branched cover between two Riemann surfaces. Then we have*

$$K_X \sim \pi^* K_Y + \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1) \cdot p,$$

where K_X and K_Y are canonical divisors on X and Y, respectively.

Proof. Take a one-form ω on Y locally expressed as $f(w)dw$ around a point $q = \pi(p)$. Suppose the covering at p is given by

$$z \mapsto w = z^m,$$

then we have

$$\pi^*(f(w)dw) = mf(z^m)z^{m-1}dz.$$

Namely, the associated divisors satisfy the relation

$$(\pi^*\omega)|_U = (\pi^*(\omega))|_U + (\mathrm{ord}_p(\pi) - 1) \cdot p$$

in a local neighborhood U of p. So globally it implies that

$$(\pi^*\omega) = \pi^*(\omega) + \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1) \cdot p.$$

Since $\pi^*\omega$ is a one-form on X, $(\pi^*\omega)$ is a canonical divisor of X and the claimed formula follows. □

We can interpret the (numerical) Riemann-Hurwitz formula from a topological viewpoint. Let $\chi(X)$ denote the *topological Euler characteristic* of X. If X is a Riemann surface of genus g, take a triangulation of X and suppose the number of k-dimensional edges is c_k for $k = 0, 1, 2$. Then we have

$$\chi(X) = c_0 - c_1 + c_2 = 2 - 2g.$$

Proposition 4.12. *Let $\pi : X \to Y$ be a degree d branched cover between two Riemann surfaces. Then we have*

$$\chi(X) = d \cdot \chi(Y) - \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1).$$

Proof. Take a triangulation of Y such that every branch point is a vertex. Pull it back as a triangulation of X. Note that it pulls back a face to d faces, an edge to d edges and a vertex v to $|\pi^{-1}(v)|$ vertices. Note that if

$$\pi^{-1}(v) = \sum_{i=1}^{k} m_i p_i$$

for distinct points p_i, then $|\pi^{-1}(v)| = m$. In other words, we have

$$|\pi^{-1}(v)| = d - \sum_{p \in \pi^{-1}(v)} (\mathrm{ord}_p(\pi) - 1).$$

Then the claimed formula follows right away. □

Corollary 4.13 (Numerical Riemann-Hurwitz)**.** *Let $\pi : X \to Y$ be a degree d branched cover between two Riemann surfaces of genus g and h, respectively. Then we have*

$$2g - 2 = d(2h - 2) + \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1).$$

In particular, if $g < h$, such branched covers do not exist.

Corollary 4.14. *The canonical line bundle of a genus g Riemann surface X has degree equal to $2g - 2$.*

Proof. Every Riemann surface X possesses a nontrivial meromorphic function, say by the Riemann-Roch formula. It induces a branched cover $\pi : X \to \mathbb{P}^1$ of degree d. By the Riemann-Hurwitz Formula we know

$$\deg(K_X) = d(-2) + \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1),$$

since we have seen that $\deg(K_{\mathbb{P}^1}) = -2$. By the Numerical Riemann-Hurwitz we have

$$2 - 2g = 2d - \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1).$$

Then the claim follows immediately. □

Exercise 4.15. Let X be a Riemann surface or algebraic curve of genus g. If X admits a branched cover of degree 2 to $\mathbb{P}^1$, we say that X is a *hyperelliptic curve.* Prove that every $g \leq 2$ curve is hyperelliptic. For $g \geq 2$, can you calculate the dimension of the parameter space of genus g hyperelliptic curves?

Remark 4.16. The moduli space of genus g curves has dimension $3g-3$, which is bigger than $2g-1$ for $g>2$. Hence a general $g>2$ curve is *not* hyperelliptic.

4.4. **Genus formula of plane curves.** Suppose $F(Z_0, Z_1, Z_2)$ is a general degree d homogeneous polynomial whose vanishing locus is a plane curve $C \subset \mathbb{P}^2$. Since F is general, C is smooth. In other words, the singularities of C locate at the common zeros of $F=0$ and $\partial F/\partial Z_i = 0$ for all i, which are empty for a general F.

Theorem 4.17. *In the above setting, the genus g of C is given by*

$$g = \frac{(d-1)(d-2)}{2}.$$

Proof. We give two proofs. The first one is more algebraic. Note that all degree d curves are linearly equivalent in $\mathbb{P}^2$. Hence it makes sense to use $\mathscr{O}_{\mathbb{P}^2}(d)$ to denote the associated line bundle. In particular, $\mathscr{O}(1)$ is the associated line bundle of a line L. Then we have the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_L \to 0.$$

Tensor it with $\mathscr{O}_{\mathbb{P}^2}(1-m)$. We obtain that

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-m) \to \mathscr{O}_{\mathbb{P}^2}(-(m-1)) \to \mathscr{O}_{\mathbb{P}^2}(1-m)|_L \to 0.$$

Since $\mathscr{O}_{\mathbb{P}^2}(1-m)|_L$ is the line bundle associated to a degree $1-m$ divisor on L and $L \cong \mathbb{P}^1$, we conclude that

$$\chi(\mathscr{O}_{\mathbb{P}^2}(-(m-1))) - \chi(\mathscr{O}_{\mathbb{P}^2}(m)) = \chi(\mathscr{O}_{\mathbb{P}^1}(1-m)) = 2-m,$$

where we apply the Riemann-Roch formula to $\mathbb{P}^1$ in the last equality. Then we obtain that

$$\chi(\mathscr{O}_{\mathbb{P}^2}) - \chi(\mathscr{O}_{\mathbb{P}^2}(-d)) = \sum_{m=1}^{d}(2-m) = -\frac{d(d-3)}{2}.$$

Now by the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-d) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_C \to 0,$$

we have

$$1-g = \chi(\mathscr{O}_C) = -\frac{d(d-3)}{2},$$

hence the genus formula follows.

The other proof is an application of the Riemann-Hurwitz formula. Without loss of generality, suppose $o=[0,0,1] \notin C$. Let L be the line $Z_2 = 0$ and project C to L from o, i.e.,

$$[Z_0, Z_1, Z_2] \mapsto [Z_0, Z_1].$$

In affine coordinates $x = Z_1/Z_0$ and $y = Z_2/Z_0$, this map is given by

$$(x, y) \mapsto x,$$

i.e., we project C vertically to the x-axis. This yields a degree d branched cover

$$\pi : C \to L \cong \mathbb{P}^1.$$

A point p is a ramification point of π if and only if there exists a vertical line tangent to C at p, i.e., p is a common zero of F and $\partial F/\partial Z_2$. Since F and $\partial F/\partial Z_2$ have degree d and $d-1$, respectively, they intersect at $d(d-1)$ points. By Riemann-Hurwitz, we have

$$2g-2 = d(-2) + d(d-1),$$

hence the genus formula follows. In order to make sure all the ramification points are simple, we can choose a general projection direction such that it is different from those given by the (finitely many) lines with higher tangency order to C. □

Remark 4.18. In the first proof, indeed we did not use the smoothness of C. So the (arithmetic) genus formula holds for an arbitrary plane curve, even if it is singular. Similarly in the second proof, even if the projection has higher ramification points, a detailed local study plus Riemann-Hurwitz can provide the same formula.

4.5. **Base point free and very ample line bundles.** Let L be a line bundle on a complete variety X. We say that L has a *base point* at $p \in X$ if p belongs to the vanishing locus of every regular section of L. If the base locus of L is empty, then L is called *base point free*.

For a base point free line bundle L, let $\sigma_0, \ldots, \sigma_n$ be a basis of the space $H^0(X, L)$ of regular sections. Locally around a point $p \in X$, consider σ_i as a regular function and associate to p the point

$$[\sigma_0(p), \ldots, \sigma_n(p)] \in \mathbb{P}^n.$$

This is well-defined, since if we take a different chart, then we get $\sigma_i'(p) = g_{\alpha\beta}\sigma_i(p)$ where $\{g_{\alpha\beta} \in \mathscr{O}^*\}$ are transition functions of L. Therefore, we obtain a regular map

$$\phi_L : X \to \mathbb{P}^n.$$

We can give a more conceptual and coordinate free description of ϕ_L. Since L is base point free, the space of regular sections σ vanishing at p forms a hyperplane $H_p \subset H^0(X, L) \cong \mathbb{C}^{n+1}$. Then one can define $\phi_L(p) = [H_p] \in (\mathbb{P}^n)^*$ in the dual projective space parameterizing hyperplanes.

Proposition 4.19. *In the above setting, there is a one-to-one correspondence between (the pullback of) hyperplane sections of X and effective divisors in the linear system $|L|$.*

Proof. This is just a reformulation of the one-to-one correspondence

$$|L| = \mathbb{P}H^0(X, L),$$

which we proved before. In other words, an effective divisor in $|L|$ uniquely determines a regular section $\sigma = \sum a_i \sigma_i$ mod scalar. □

Example 4.20. If $L = \mathscr{O}$, then $\phi_{\mathscr{O}}$ maps X to a single point.

Example 4.21. Let $X = \mathbb{P}^1$ and $L = \mathscr{O}(2p)$ where $p = [0, 1]$. Then $H^0(\mathbb{P}^1, L)$ is 3-dimensional and we can choose a basis by

$$1, \quad \frac{Y}{X}, \quad \frac{Y^2}{X^2}.$$

Recall that around p the sections of L are given by $f \cdot (X/Y)^2$. Hence we obtain that

$$\phi_L([X, Y]) = [X^2, XY, Y^2],$$

which is a smooth conic in $\mathbb{P}^2$. The genus formula for a plane curve of degree 2 also implies that the image has $g = 0$.

Exercise 4.22. A variety $X \subset \mathbb{P}^n$ is called *non-degenerate* if it is not contained in any hyperplane.

(1) Show that any non-degenerate smooth rational curve in $\mathbb{P}^n$ has degree $\geq n$.

(2) For $d \geq n \geq 3$, show that there exist non-degenerate smooth degree d rational curves in $\mathbb{P}^n$.

Example 4.23. Let E be an elliptic curve and $L = \mathscr{O}(2p)$. By Riemann-Roch, $h^0(E, L) = 2$. Moreover, L is base point free. Otherwise if q is a base point, then q has to be p and there exists another effective divisor $p+r \in |2p|$ such that $p+r \sim 2p$. But this implies $r - p$ is principal and $E \cong \mathbb{P}^1$, leading to a contradiction. Now $\phi_L : E \to \mathbb{P}^1$ is a branched cover of degree 2. Two points s and t lie in the same fiber if and only if $s + t \sim 2p$.

The above example indicates that ϕ_L is *not* always an embedding. We say that L is *very ample* if ϕ_L is an embedding and that L is *ample* if $L^{\otimes m}$ is very ample for some $m > 0$.

Example 4.24. The line bundle $\mathscr{O}(d)$ is very ample on $\mathbb{P}^1$ if and only if $d > 0$. The induced map ϕ embeds $\mathbb{P}^1$ into $\mathbb{P}^d$ as a degree d smooth rational curve, i.e., a rational normal curve.

Let us give a criterion for base point free and very ample line bundles.

Proposition 4.25. *Let L be a line bundle on a Riemann surface X.*

(1) L is base point free if and only if

$$h^0(X, L \otimes \mathscr{O}(-p)) = h^0(X, L) - 1$$

for any $p \in X$.

(2) L is very ample if and only L is base point free and for any $p, q \in X$ (not necessarily distinct)

$$h^0(X, L \otimes \mathscr{O}(-p-q)) = h^0(X, L \otimes \mathscr{O}(-p)) - 1 = h^0(X, L \otimes \mathscr{O}(-q)) - 1.$$

Proof. Treat L as a locally free sheaf of rank 1. By the short exact sequence

$$0 \to L \otimes \mathscr{O}(-p) \to L \to \mathbb{C}_p \to 0,$$

we have

$$h^0(X, L) - 1 \leq h^0(X, L \otimes \mathscr{O}(-p)) \leq h^0(X, L).$$

Then L has a base point at p if and only if all sections of L vanish at p, i.e., $H^0(X, L \otimes \mathscr{O}(-p)) = H^0(X, L)$. This proves (1).

For (2), a very ample line bundle is necessarily base point free by definition. If $p \neq q \in X$ have the same image under ϕ_L, it is equivalent to saying that the subspace of sections vanishing at p is the same as the subspace of sections vanishing at q, which is further equivalent to

$$h^0(X, L \otimes \mathscr{O}(-p)) = h^0(X, L \otimes \mathscr{O}(-p-q)) = h^0(X, L \otimes \mathscr{O}(-q)).$$

Moreover, ϕ_L induces an injection restricted to the tangent space $T_p(X)$ if and only if there exists a hyperplane such that it cuts out X locally a simple point at p, namely, if and only if there is a section vanishing at p with multiplicity 1, i.e.,

$$h^0(X, L \otimes \mathscr{O}(-2p)) < h^0(X, L \otimes \mathscr{O}(-p)).$$

But we have

$$h^0(X, L \otimes \mathscr{O}(-2p)) \geq h^0(X, L \otimes \mathscr{O}(-p)) - 1.$$

Hence (2) follows from combining the two cases. ∎

Remark 4.26. In (2), for $p \neq q$ the condition geometrically means that the sections of L separate any two points. When $p = q$, it says that the sections of L separate tangent vectors at p.

Example 4.27. Let E be an elliptic curve, i.e., a Riemann surface of genus one. Fix a point $p \in E$. The morphism

$$\tau : E \to J(E) \cong \mathrm{Pic}^0(E)$$

by $\tau(q) = [q - p]$ is an isomorphism. This defines a group law on E with respect to p, i.e., $q + r = s$, where $s \in E$ is the unique point satisfying

$$(q - p) + (r - p) \sim s - p.$$

Now consider the linear system $|3p|$ on E. Since

$$h^0(E, \mathscr{O}(3p)) = 3, \quad h^0(E, \mathscr{O}(2p)) = 2, \quad h^0(E, \mathscr{O}(p)) = 1,$$

$\mathscr{O}(3p)$ is very ample. It induces an embedding of E into $\mathbb{P}^2$ as a plane cubic. A line cuts out a divisor of degree 3 in E, say, $q + r + s$ (not necessarily distinct) if and only if

$$q + r + s \sim 3p.$$

Note that the tangent line L of E at p is a *flex line*, i.e., the tangency multiplicity $(L \cdot E)_p = 3$. Such p is called a *flex point*.

Exercise 4.28. Show that there are in total 9 flex points on a smooth plane cubic.

Let $V \subset |L|$ be a linear subspace. We say that V is a *linear series* of L. The linear system $|L|$ is also called a *complete linear series*. The above definitions and properties go through similarly for the induced map ϕ_V.

Exercise 4.29. Write down a linear series of $|\mathscr{O}(3)|$ on $\mathbb{P}^1$ such that it maps $\mathbb{P}^1$ into $\mathbb{P}^2$ as a *singular* plane cubic. How many different types of such singular plane cubics can you describe?

4.6. **Canonical maps.** Let K be the canonical line bundle a Riemann surface X. If X is $\mathbb{P}^1$, $\deg(K) = -2$ and K is not effective. If X is an elliptic curve, then $K \cong \mathscr{O}$ and the induced map ϕ_K is onto a point. From now on, assume that the genus of X satisfies $g \geq 2$. We say that X is *hyperelliptic* if it admits a degree 2 branched cover of $\mathbb{P}^1$. Two points $p, q \in X$ are called *conjugate* if they have the same image in $\mathbb{P}^1$. A ramification point of the double cover is called a *Weierstrass point* of X, i.e., it is self conjugate. By Riemann-Hurwitz, a genus $g \geq 2$ hyperelliptic curve possesses $2g + 2$ Weierstrass points.

Lemma 4.30. *If X is a hyperelliptic curve of genus ≥ 2, then X admits a unique double cover of $\mathbb{P}^1$.*

Proof. Otherwise suppose $h^0(X, \mathscr{O}(p + q)) = 2$ and $h^0(X, \mathscr{O}(p + r)) = 2$ for $q \neq r$. Let $L = \mathscr{O}_X(p + q + r)$. If $h^0(X, L) = 3$, since $h^0(X, L(-x - y)) \leq 1$ and $h^0(X, L(-x)) \leq 2$ by degree reason, ϕ_L would map X as a plane cubic of genus one, leading to a contradiction. Then we conclude that

$$H^0(X, \mathscr{O}(p + q)) = H^0(X, \mathscr{O}(p + r)) = H^0(X, \mathscr{O}(p + q + r)),$$

which implies that both q, r are base points of $|p + q + r|$ and $h^0(X, \mathscr{O}(p)) = 2$, $X \cong \mathbb{P}^1$, leading to a contradiction. ∎

Proposition 4.31. *Let X be a curve of genus $g \geq 2$. Then the canonical line bundle K is base point free. The induced map*

$$\phi_K : X \to \mathbb{P}^{g-1}$$

is an embedding if and only if X is not hyperelliptic. If X is hyperelliptic, ϕ_K is a double cover of a rational normal curve in $\mathbb{P}^{g-1}$.

Proof. First, let us show that K is base point free. For any point $p \in X$, by Riemann-Roch we have

$$h^0(X, K \otimes \mathscr{O}(-p)) - h^0(X, \mathscr{O}(p)) = 1 - g + (2g-3),$$
$$h^0(X, K \otimes \mathscr{O}(-p)) = g - 1 = h^0(X, K) - 1.$$

Hence K satisfies the criterion of base point freeness.

Next, K fails to separate p, q (not necessarily distinct) if and only if

$$h^0(X, K \otimes \mathscr{O}(-p-q)) = h^0(X, K \otimes \mathscr{O}(-p)) = g - 1,$$

which is equivalent to, by Riemann-Roch again, that

$$h^0(X, \mathscr{O}(p+q)) = 2.$$

In other words, the linear system $|p+q|$ induces a double cover $X \to \mathbb{P}^1$.

Finally, if X is hyperelliptic of genus ≥ 2, it admits a unique double cover of $\mathbb{P}^1$. By the above analysis, two points p, q have the same image under the canonical map if and only if $h^0(X, \mathscr{O}(p+q)) = 2$, i.e., p, q are conjugate. Then the canonical map is a double cover of a rational curve of degree $\deg(K)/2 = g - 1$ in $\mathbb{P}^{g-1}$, i.e., a rational normal curve. A hyperplane section of $\phi_K(X)$ pulls back to X a divisor

$$\sum_{i=1}^{g-1} (p_i + q_i),$$

where p_i, q_i are conjugate or $p_i = q_i$ a Weierstrass point. □

Remark 4.32. For a non-hyperelliptic curve X, ϕ_K is called the *canonical embedding* of X and its image is called a *canonical curve*.

Example 4.33. Let X be a curve of genus two. Then $h^0(X, K) = 2$, hence X is hyperelliptic and the double cover of $\mathbb{P}^1$ is induced by the canonical line bundle, as we have seen.

Example 4.34. A non-hyperelliptic curve of genus three admits a canonical embedding to $\mathbb{P}^2$ as a plane quartic. An effective canonical divisor corresponds to a line section of the quartic. By the genus formula, any smooth plane quartic also has genus equal to 3. Moreover, a smooth plane quartic X gives rise to a line bundle L of degree 4 on X by restricting $\mathscr{O}_{\mathbb{P}^2}(1)$. By Riemann-Roch, $h^0(X, K \otimes L^*) \geq 1$, but $\deg(K \otimes L^*) = 0$, hence $L \cong K$. So any plane quartic is a canonical embedding of a non-hyperelliptic curve of genus three.

Example 4.35. Let X be a non-hyperellipic curve of genus four. Then its canonical embedding is a curve of degree six in $\mathbb{P}^3$. Let $\mathscr{O}_{\mathbb{P}^3}(1)$ denote the line bundle on $\mathbb{P}^3$ associated to a hyperplane class. Its restriction to X is the canonical line bundle K_X. We have the exact sequence

$$0 \to \mathscr{I}_X \otimes \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_X(2) \to 0.$$

Since $h^0(\mathbb{P}^3, \mathscr{O}(2)) = 10 > h^0(X, \mathscr{O}(2)) = 9$, we conclude that X is contained in a quadric surface Q in $\mathbb{P}^3$. Indeed Q is unique, because otherwise $\deg(X) \leq 4$.

If Q is smooth, it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Up to projective equivalence, it has coordinates

$$[XZ, XW, YZ, YW],$$

i.e., its equation is

$$Z_0Z_3 - Z_1Z_2 = 0.$$

Note that

$$Z_0 - aZ_1 = Z_2 - aZ_3 = 0$$

defines a family A of lines in Q parameterized by the value of a. Any two lines in A are disjoint. Similarly,

$$Z_0 - bZ_2 = Z_1 - bZ_3 = 0$$

defines another family B of lines in Q parameterized by b, and any two lines in B are disjoint. Moreover, a line L_1 in A and a line L_2 in B intersect at a unique point $[ab, b, a, 1]$. Hence they span a plane H in $\mathbb{P}^3$. Suppose $H \cdot X = D_1 + D_2$, where D_i is a divisor in L_i. Since L_i varies in Q in a $\mathbb{P}^1$ family, we have $h^0(X, \mathscr{O}(D_i)) \geq 2$ for $i = 1, 2$. Moreover, because $D_1 + D_2 \sim K_X$, we have $\deg(D_1) + \deg(D_2) = 6$. Since X is not hyperelliptic, we conclude that $\deg(D_i) = 3$ for $i = 1, 2$. This can also be seen since X is a complete intersection of Q with a cubic surface, hence its class in Q is linearly equivalent to $3L_1 + 3L_2$. Therefore, we obtain two distinct line bundles of degree 3 given by $\mathscr{O}_X(D_1)$ and $\mathscr{O}_X(D_2)$ such that $h^0(\mathscr{O}_X(D_i)) = 2$. In other words, X admits two triple covers of $\mathbb{P}^1$. This can be seen by projecting X along the direction of L_i for $i = 1, 2$.

If Q is singular, since X is non-degenerate, Q is a quadric cone whose equation is given by

$$Z_0^2 - Z_1Z_2 = 0.$$

It has a unique singular point $v = [0, 0, 0, 1]$ as the vertex and Q can be viewed as the cone over a plane conic C defined by the same equation. Every line L contained in Q passes through v, hence C parameterizes a $\mathbb{P}^1$-family of lines. Similarly, one checks that $L \cdot X = 3$ and X admits a unique triple cover of $\mathbb{P}^1$ by projecting from v to C.

Exercise 4.36. Suppose $g \geq 3$. Let $X \subset \mathbb{P}^{g-1}$ be a non-degenerate smooth genus g curve of degree $2g - 2$. Show that X is a canonical curve.

Exercise 4.37. Let X be a non-hyperelliptic curve of genus four. If $D = p + q + r$ is an effective divisor of degree three on X such that $h^0(X, \mathscr{O}(D)) = 2$, show that p, q and r are collinear in the canonical embedding of X in $\mathbb{P}^3$.

Exercise 4.38. A hyperelliptic curve X of genus g can be explicitly written as the locus of (z, w) satisfying

$$w^2 = (z - a_1) \cdots (z - a_{2g+2}).$$

One can treat $(w, z) \mapsto z$ as the double cover of $\mathbb{P}^1$ which is branched at $a_1, \ldots, a_{2g+2}$. Prove that a basis for the space $H^0(X, K)$ of holomorphic one-forms on X is

$$\frac{dz}{w},\ z\frac{dz}{w}, \ldots,\ z^{g-1}\frac{dz}{w}.$$

4.7. **Dimensions of linear systems.** Let $D = p_1 + \cdots + p_d$ be an effective divisor of degree d on a genus g curve X. Recall that the linear system $|D|$ can be identified with $\mathbb{P}H^0(X, \mathscr{O}(D))$ parameterizing effective divisors linearly equivalent to D. Suppose as a projective space

$$r = \dim |D| = h^0(X, \mathscr{O}(D)) - 1.$$

By Riemann-Roch, we have

$$\dim |K \otimes \mathscr{O}(-D)| = r + g - d - 1.$$

Note that $|K \otimes \mathscr{O}(-D)|$ is the linear system of canonical divisors that contain D. By the canonical map

$$\phi_K : X \to \mathbb{P}^{g-1},$$

it says that the space of hyperplanes of $\mathbb{P}^{g-1}$ that contain $\phi_K(p_1), \ldots, \phi_K(p_d)$ is $(r + g - d - 1)$-dimensional. In other words, the linear span of $\phi_K(p_1), \ldots, \phi_K(p_d)$ is a

$$(g - 2) - (r + g - d - 1) = (d - 1) - r$$

dimensional subspace in $\mathbb{P}^{g-1}$. Since we expect d points to span a $(d-1)$-dimensional linear subspace, geometrically it says that $\phi_K(D)$ fails to impose

$$r = \dim |D|$$

independent conditions. We summarize the discussion as a geometric version of the Riemann-Roch formula.

Theorem 4.39 (Geometric Riemann-Roch)**.** *In the above setting, let $\overline{\phi_K(D)}$ be the linear span of the image of D under the canonical map. Then we have*

$$\dim |D| = \deg(D) - 1 - \dim \overline{\phi_K(D)}.$$

Remark 4.40. Even if D contains points with higher multiplicity, the formulation still holds. Say, if D contains $2p$, then $2p$ span the tangent line at p. If D contains $3p$, then $3p$ spans an osculating 2-plane at p etc.

Example 4.41. Let us revisit the canonical embedding of a non-hyperelliptic curve X of genus four in $\mathbb{P}^3$. Recall that X is contained in a unique quadric surface Q and we showed that X admits a triple cover of $\mathbb{P}^1$ corresponding to a family of lines in Q. Conversely, if $D = p + q + r$ induces a triple cover $X \to \mathbb{P}^1$, i.e., if $\dim |D| = 1$, by Geometric Riemann-Roch, we have

$$\dim \overline{\phi_K(D)} = 3 - 1 - 1 = 1,$$

i.e., p, q and r are collinear in a line L in $\mathbb{P}^3$. Because $L \cdot Q = 2$ unless L is contained in Q, any triple cover of $\mathbb{P}^1$ on X corresponds to a family of lines in Q. We have seen that if Q is smooth, there are two such families of lines, i.e., X admits two distinct triple covers of $\mathbb{P}^1$, while if Q is singular, such a triple cover is unique.

Let us study in detail the dimension of a linear system.

Lemma 4.42. *Let D be a divisor on a curve X. Then $\dim |D| \geq k$ if and only if for every k points $p_1, \ldots, p_k \in X$ there exists an effective divisor in $|D|$ containing all of them.*

Proof. Suppose for every k points $p_1, \dots, p_k \in X$ there exists an effective divisor in $|D|$ containing all of them. Since $\sum_{i=1}^k p_i$ varies in a k-dimensional family, then $\dim |D| \geq k$ is obvious. Alternatively, we can prove it by induction. Suppose it holds for $\leq k$. Assume for every $p_1, \dots, p_{k+1}$, there exists $D' \in |D|$ containing all of them. Then we conclude that $\dim |D - p| \geq k$ for any $p \in X$. Choose a point p not in the base locus of $|D|$. Consequently we have

$$\dim |D| = \dim |D - p| + 1 \geq k + 1.$$

Now suppose $\dim |D| \geq k$. Then we have

$$h^0\left(X, \mathscr{O}\left(D - \sum_{i=1}^k p_i\right)\right) \geq h^0(X, \mathscr{O}(D)) - k \geq 1.$$

It implies that there exists a non-zero meromorphic function f such that

$$(f) + D - \sum_{i=1}^k p_i \geq 0,$$

hence $(f) + D = D'$ is an effective divisor in $|D|$ containing $p_1, \dots, p_k$. □

Corollary 4.43. *For any two effective divisors D_1 and D_2 on X, we have*

$$\dim |D_1| + \dim |D_2| \leq \dim |D_1 + D_2|.$$

Proof. Suppose $\dim |D_i| = k_i$ for $i = 1, 2$. Take any $k_1 + k_2$ points

$$p_1, \dots, p_{k_1}, q_1, \dots, q_{k_2}$$

in X. By the above lemma, there exist $D_1' \in |D_1|$ and $D_2' \in |D_2|$ such that D_1' contains all the p_i and D_2' contains all the q_j. Then $D_1' + D_2' \in |D_1 + D_2|$ contains all the p_i, q_j, hence we obtain

$$\dim |D_1 + D_2| \geq k_1 + k_2$$

by the lemma again. □

Note that if $h^0(X, K \otimes (-D)) = 0$, then Riemann-Roch determines that

$$h^0(X, \mathscr{O}(D)) = 1 - g + \deg(D).$$

Some subtlety may occur if

$$h^0(X, K \otimes (-D)) > 0$$

and we call such a divisor D a *special divisor* and the associated linear system $|D|$ a *special linear system*. By Riemann-Roch, any D with $\deg(D) > 2g - 2$ is non-special. By Geometric Riemann-Roch, D is non-special if and only if the linear span of $\phi_K(D)$ is the entire space $\mathbb{P}^{g-1}$.

Theorem 4.44 (Clifford's Theorem)**.** *Let D be an effective divisor such that* $\deg(D) \leq 2g - 2$ *on X. Then we have*

$$\dim |D| \leq \frac{1}{2} \cdot \deg(D).$$

Proof. If D is non-special, we have

$$\dim |D| = \deg(D) - g < \frac{1}{2}\deg(D).$$

If D is special, there exists an effective divisor D' such that $D + D' \sim K$. By the above lemma we have

$$\dim |D| + \dim |D'| \leq \dim |K| = g - 1.$$

By Riemann-Roch, we have

$$\dim |D| - \dim |D'| = 1 - g + \deg(D).$$

The desired inequality follows by combining the two relations. □

Remark 4.45. Indeed, the equality holds only if $D = 0$, $D = K$ or X is hyperelliptic. If $D = 0$ or $D = K$, one easily checks that the equality holds. If X is hyperelliptic, we can take $D = p + q$, where p, q are conjugate and $\dim |p + q| = 1$. To prove that these are the only possibilities, we need the *uniform position theorem* regarding a general hyperplane section of a non-degenerate space curve, see e.g., [GH, p. 249].

Exercise 4.46. Let X be a hyperelliptic curve of genus ≥ 2. For $0 < 2k \leq g$, find an effective divisor D of degree $2k$ on X such that $\dim |D| = k$. Classify all such divisors up to linear equivalence.

5. Weierstrass points

We want to generalize the concept of Weierstrass points on a hyperelliptic curve to an arbitrary curve. Let X be a curve of genus $g \geq 2$ and $p \in X$ a point.

5.1. **Weierstrass semigroups and gap sequences.** Set $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$ and define $H_p \subset \mathbb{N}$ by

$$H_p = \{n : \exists f \in \mathscr{M}(X) \text{ such that } (f)_\infty = np\}.$$

Note that if $(f)_\infty = np$ and $(h)_\infty = mp$, then $(fh)_\infty = (m+n)p$. We say that H_p is the *Weierstrass semigroup* of p. Define $G_p \subset \mathbb{N}$ as the complement of H_p, i.e.

$$G_p = \{n : \nexists f \in \mathscr{M}(X) \text{ such that } (f)_\infty = np\}.$$

We say that G_p is the *Weierstrass gap sequence* of p.

Lemma 5.1. *We have $n \in H_p$ if and only if*

$$h^0(X, \mathscr{O}(np)) = h^0(X, \mathscr{O}((n-1)p)) + 1$$

and $n \in G_p$ if and only if

$$h^0(X, \mathscr{O}(np)) = h^0(X, \mathscr{O}((n-1)p)).$$

Namely, $n \in G_p$ (resp. H_p) if and only if p is (resp. is not) a base point of the linear system $|np|$. Moreover, G_p is a subset of $\{1, \dots, 2g-1\}$ with cardinality g.

Proof. By the exact sequence

$$0 \to \mathscr{O}((n-1)p) \to \mathscr{O}(np) \to \mathbb{C}_p \to 0,$$

we conclude that

$$h^0(X, \mathscr{O}((n-1)p)) \leq h^0(X, \mathscr{O}(np)) \leq h^0(X, \mathscr{O}((n-1)p)) + 1.$$

The right hand side equality holds if and only if p is not a base point of $|np|$, namely, if and only if there exists $f \in \mathscr{M}(X)$ such that $(f)+np \geq 0$ but $(f)+(n-1)p \not\geq 0$, which is equivalent to saying that $(f)_\infty = np$, i.e., if and only if $n \in H_p$.

For any $n \geq 2g$, by Riemann-Roch we have

$$h^0(X, \mathscr{O}(np)) = h^0(X, \mathscr{O}(n-1)p) + 1.$$

Hence we conclude that $G_p \subset \{1, \ldots, 2g-1\}$. Moreover, we have

$$\begin{aligned} g-1 &= h^0(X, \mathscr{O}((2g-1)p)) - h^0(X, \mathscr{O}) \\ &= \sum_{n=0}^{2g-1} \Big(h^0(X, \mathscr{O}(np)) - h^0(X, \mathscr{O}((n-1)p)) \Big) \end{aligned}$$

So there are $g-1$ elements of $\{1, \ldots, 2g-1\}$ belonging to H_p. In other words, the cardinality of G_p is

$$|G_p| = (2g-1) - (g-1) = g.$$

□

Let us give an alternative interpretation of the above result.

Lemma 5.2. *We have $n \in G_p$ if and only if there exists a section of the canonical line bundle, i.e., a holomorphic one-form ω, such that* $\mathrm{ord}_p(\omega) = n-1$. *Consequently we conclude that G_p is a subset of $\{1, \ldots, 2g-1\}$ of cardinality g.*

Proof. By Riemann-Roch, $h^0(X, \mathscr{O}(np)) = h^0(X, \mathscr{O}((n-1)p))$ if and only if

$$h^0(X, K \otimes \mathscr{O}(-(n-1)p)) = h^0(X, K \otimes \mathscr{O}(-np)) + 1.$$

Note that $H^0(X, K \otimes \mathscr{O}(-mp))$ parameterizes holomorphic one-forms ω such that $(\omega) \geq mp$. So the above equality holds if and only if there exists ω such that $(\omega) \geq (n-1)p$ but $(\omega) \not\geq np$, namely, if and only if $\mathrm{ord}_p(\omega) = (n-1)p$.

Since $h^0(X, K) = g$, one can choose a basis $\omega_1, \ldots, \omega_g$ such that $\mathrm{ord}_p(\omega_i) = a_i$ and

$$a_1 < a_2 < \cdots < a_g.$$

Then we obtain that

$$G_p = \{a_1+1, a_2+1, \ldots, a_g+1\}.$$

Because $\deg(K) = 2g-2$, we have $0 \leq a_i \leq 2g-2$ for all i, hence G_p is a subset of $\{1, \ldots, 2g-1\}$. □

We say that p is a *Weierstrass point* of X if $G_p \neq \{1, 2, \ldots, g\}$ and p is a *normal Weierstrass point* if $G_p = \{1, 2, \ldots, g-1, g+1\}$. Define the *weight* of p by

$$\begin{aligned} w(p) &= \Big(\sum_{n \in G_p} n \Big) - (1 + 2 + \cdots + g) \\ &= \Big(\sum_{n \in G_p} n \Big) - \frac{g(g+1)}{2}. \end{aligned}$$

Then p is a Weierstrass point if and only if $w(p) > 0$ and p is a normal Weierstrass point if and only if $w(p) = 1$.

Now we introduce some basic properties of a semigroup of $\mathbb{N}$. In general, if $H \subset \mathbb{N}$ is a semigroup whose complement $G = \mathbb{N} - H$ consists of g elements, define the *weight* of H by

$$w(H) = \Big(\sum_{n\in G} n\Big) - \frac{g(g+1)}{2}.$$

Lemma 5.3. *In the above setting, we have*

$$w(H) \le \frac{(g-1)g}{2}.$$

The equality holds if and only if $2 \in H$.

Proof. Let $H = \{a_1, a_2, \dots,\}$ such that $a_1 < a_2 < \cdots$. Suppose $b \in G$ is the smallest element such that $b > a_n$. Then for $1 \le i \le n$, we have

$$b - a_i \in G, \quad b - a_i \le a_n.$$

The two disjoint sets

$$\{b - a_n, b - a_{n-1}, \dots, b - a_1\}, \quad \{a_1, \dots, a_n\}$$

are both contained in $\{1, 2, \dots, a_n\}$. Hence we conclude that

$$a_n \ge n + n = 2n.$$

Now suppose $G = \{b_1, \dots, b_g\}$. Then H can be expressed as

$$\{1, \dots, b_1 - 1; b_1 + 1, \dots, b_2 - 1; b_2 + 1, \dots, b_3 - 1; \dots\}.$$

We have $b_1 - 1 = a_{b_1 - 1}$, $b_2 - 1 = a_{b_2 - 2}$, $\dots$, and in general, that is,

$$b_k - 1 = a_{b_k - k} \ge 2(b_k - k)$$

by the above inequality. Hence we conclude that $b_k \le 2k - 1$.

Now by definition, we have

$$\begin{aligned} w(H) &= \sum_{k=1}^{g} (b_k - k) \\ &\le \sum_{k=1}^{g} (k-1) \\ &= \frac{(g-1)g}{2}. \end{aligned}$$

Moreover, the equality holds if and only if $b_k = 2k - 1$ for $1 \le k \le g$, hence $G = \{1, 3, \dots, 2g - 1\}$ and $2 \in H$. □

Proposition 5.4. *Let X be a curve of genus $g \ge 2$. Then we have*

$$\sum_{p\in X} w(p) = (g-1)g(g+1).$$

Proof. Choose a basis $\omega_1, \dots, \omega_g$ of $H^0(X, K)$ and write $\omega_i = f_i(z)dz$ for $i = 1, \dots, g$ in terms of a local coordinate z around p. Consider the Wronskian

$$W(z) = \det \begin{pmatrix} f_1(z) & \cdots & f_g(z) \\ f_1'(z) & \cdots & f_g'(z) \\ \vdots & & \vdots \\ f_1^{(g-1)}(z) & \cdots & f_g^{(g-1)}(z) \end{pmatrix}$$

Since $f_1, \dots, f_g$ are linearly independent and analytic, $W(z)$ is not identically zero.

Suppose $\tilde{z}$ is another coordinate around p such that $\omega_i = \tilde{f}_i(\tilde{z})d\tilde{z}$. Let

$$\psi = \frac{d\tilde{z}}{dz} \in \mathscr{O}^*(U \cap \tilde{U}).$$

Since $f_i(z)dz = \tilde{f}_i(\tilde{z})d\tilde{z}$, we have $f = \psi\tilde{f}$, and hence

$$\begin{aligned} \frac{df}{dz} &= \psi\frac{d\tilde{f}}{d\tilde{z}}\frac{d\tilde{z}}{dz} + \frac{d\psi}{dz}\tilde{f} \\ &= \psi^2\frac{d\tilde{f}}{d\tilde{z}} + \frac{d\psi}{dz}\tilde{f} \end{aligned}$$

In general, we have

$$\frac{d^n f}{dz^n} = \psi^{n+1}\frac{d^n\tilde{f}}{d\tilde{z}^n} + \cdots$$

for higher derivatives. Denote by

$$N = 1 + 2 + \cdots + g = g(g+1)/2.$$

Then we conclude that

$$W(z) = \psi^N W(\tilde{z}) = \Big(\frac{d\tilde{z}}{dz}\Big)^N W(\tilde{z}).$$

For example, if $g = 2$, we have

$$\begin{aligned} \det\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} &= \det\begin{pmatrix} \psi\tilde{f}_1 & \psi\tilde{f}_2 \\ \psi^2\tilde{f}_1' + \psi'\tilde{f}_1 & \psi^2\tilde{f}_2' + \psi'\tilde{f}_2 \end{pmatrix} \\ &= \psi^{1+2}\det\begin{pmatrix} \tilde{f}_1 & \tilde{f}_2 \\ \tilde{f}_1' & \tilde{f}_2' \end{pmatrix}. \end{aligned}$$

Consequently it implies that $W(z)(dz)^N$ defines a global section of $H^0(X, K^{\otimes N})$.

Moreover, from the expression of $W(z)$ we have

$$\begin{aligned} \mathrm{ord}_p(W(z)) &= \Big(\sum_{i=1}^g a_i\Big) - (0 + 1 + \cdots + (g-1)) \\ &= \Big(\sum_{i=1}^g (a_i + 1)\Big) - (1 + 2 + \cdots + g) \\ &= \Big(\sum_{n \in G_p} n\Big) - g(g+1)/2 \\ &= w(p). \end{aligned}$$

Since $\deg(K^{\otimes N}) = (g-1)g(g+1)$, the desired formula follows right away. □

Corollary 5.5. *Let X be a genus g curve for $g \geq 2$. Then X has finitely many Weierstrass points and the number of distinct Weierstrass points is at least $2g+2$. This lower bound can be attained if and only if X is hyperelliptic.*

Proof. By definition, a Weierstrass point has strictly positive weight, so X has finitely many Weierstrass points. Since $w(p) \leq (g-1)g/2$, we conclude that X has at least $2g+2$ distinct Weierstrass points. This lower bound is attained if and only if each Weierstrass point p satisfies $2 \in H_p$, i.e., X admits a double cover of $\mathbb{P}^1$ with p as a ramification point, hence if and only if X is hyperelliptic. □

Exercise 5.6. Let X be a hyperelliptic curve of genus ≥ 2 with a double cover $\pi : X \to \mathbb{P}^1$ and $p \in X$ a point. Prove directly that

(1) p is a ramification point of π if and only if $G_p = \{1, 3, \ldots, 2g-1\}$;

(2) p is not a ramification point of π if and only if $G_p = \{1, 2, \ldots, g\}$.

Exercise 5.7. Determine all possible Weierstrass gap sequences for a point on a genus 3 curve.

5.2. **Weierstrass points on genus four curves.** Let X be a non-hyperelliptic curve of genus four. Use g^r_d to denote a r-dimensional linear system associated to a divisor of degree d. Recall that the canonical embedding of X is contained in a quadric surface Q in $\mathbb{P}^3$. If Q is smooth, X has two g^1_3's, residual to each other with respect to K_X. If Q is singular, X has a unique g^1_3 induced by a half-canonical divisor.

Lemma 5.8. *In the above setting, if* $3p$ *induces a* g^1_3 *and if* $6p$ *is a canonical divisor, then* X *has a unique* g^1_3.

Proof. Otherwise, we have $3p$ and $p+q+r$ as the two g^1_3's of X, where $q, r \neq p$. Then $6p \sim 4p+q+r$ and $2p \sim q+r$, contradicting that X is non-hyperelliptic. □

For a Weierstrass point on X, all possible gap sequences G are $\{1,2,4,7\}$, $\{1,2,4,5\}$, $\{1,2,3,7\}$, $\{1,2,3,6\}$ and $\{1,2,3,5\}$ of weights 4, 2, 3, 2 and 1, respectively. Let n_1, n_2 and n_3 be the numbers of Weierstrass points whose gap sequences are $\{1,2,4,7\}$, $\{1,2,4,5\}$ and $\{1,2,3,5\}$, respectively.

Theorem 5.9. *(i)* $n_1 \leq 6$. *(ii) If* $n_1 > 0$*, then* $n_2 = 0$ *and* $2n_1 + n_3 \geq 12$. *(iii)* $n_2 \leq 12$.

Proof. (i) Suppose p has gap sequence $\{1,2,4,7\}$. Then we have $6p \sim K_X$ and $3p$ induces a g^1_3. By the lemma, X has a unique g^1_3. If another point q also has gap sequence $\{1,2,4,7\}$, then $3q$ induces the same g^1_3. Then the unique triple cover $X \to \mathbb{P}^1$ has total ramification order 12 by Riemann-Hurwitz. Each triple ramification point contributes 2, hence we get $n_1 \leq 6$.

(ii) If $n_1 > 0$, let p be a point with gap sequence $\{1,2,4,7\}$. By the lemma, X has a unique g^1_3 and $3p$ induces the unique triple cover $X \to \mathbb{P}^1$. Suppose q has gap sequence $\{1,2,4,5\}$. Then $3 \in H_q$ implies that $3q$ also induces the unique g^1_3, hence $6q$ is a canonical divisor. Then we have $7 \in G_q$, leading to a contradiction.

Next, suppose q is a simple ramification point of the triple cover, i.e., $2q+r \sim 3p$ for some $r \neq q$. Then $4q+2r$ is a canonical divisor, hence $5 \in G_q$. Moreover, $3 \notin H_q$ for otherwise $3q$ would also induce the same g^1_3, contradicting that $q \neq r$. Therefore, we conclude that $G_q = \{1,2,3,5\}$. By Riemann-Hurwitz, we have $2n_1 + n_3 \geq 12$.

(iii) Let p_i be the points with gap sequence $\{1,2,4,5\}$ for $i = 1, \ldots, n_2$. Since $3 \in H_{p_i}$, it implies that $3p_i$ induces a g^1_3. Then at least half of the $3p_i$'s belong to the same g^1_3 of X for $1 \leq i \leq n_2$. Using Riemann-Hurwitz associated to that triple cover, we conclude that $2([(n_2+1)/2]) \leq 12$, hence $n_2 \leq 12$. □

Corollary 5.10. *Every non-hyperelliptic curve of genus four admits a* 4*-sheeted covering of* $\mathbb{P}^1$ *such that* $\infty \in \mathbb{P}^1$ *has a unique inverse image.*

Proof. It is equivalent to show that there exists $p \in X$ such that $4 \in H_p$. Note that the total Weierstrass weight equals 60. By Theorem 5.9, we have $4n_1 + 2n_2 \leq 24 < 60$. The claim follows right away. Indeed, there are at least 12 Weierstrass points on X whose first non-gap equals 4. □

This discussion motivates a question: which numerical semigroups occur as Weierstrass semigroups? A necessary condition can be easily imposed as follows.

Proposition 5.11. *For a semigroup H, let G be its gap set of cardinality g, and nG the set of sums of arbitrary n elements in G. If for some $n > 1$ we have $|nG| > (2n-1)(g-1)$, then H cannot be a Weierstrass semigroup.*

Proof. Suppose $G = \{a_1, \ldots, a_g\}$ is the Weierstrass gap sequence at $p \in X$. Then there exist a basis $\omega_1, \ldots, \omega_g$ of $H^0(X, K)$ such that ω_i has vanishing order $a_i - 1$ at p. Note that $h^0(X, K^{\otimes n}) = (2n-1)(g-1)$ for $n > 1$. Therefore, $nG - n$ has cardinality at most $(2n-1)(g-1)$, so does $|nG|$. □

Remark 5.12. The above criterion is due to Buchweitz. He also found the first example of a numerical semigroup H in $g = 16$ that violates this criterion.

5.3. **Automorphisms of a curve.** As an application of Weierstrass points, we have the following result.

Proposition 5.13. *The automorphism group of a genus $g \geq 2$ curve is finite.*

Proof. Let X be a genus g curve. Since $g \geq 2$, X has finitely many Weierstrass points. In particular, an automorphism τ of X sends a Weierstrass point to a Weierstrass point.

If X is hyperelliptic, there are $2g+2$ Weierstrass points, corresponding to the $2g+2$ ramification points of the associated double cover $\pi : X \to \mathbb{P}^1$. Modulo the hyperelliptic involution, τ is induced by an automorphism of $\mathbb{P}^1$ that sends the $2g+2$ branch points to themselves. In other words, $\mathrm{Aut}(X)$ is a $(\mathbb{Z}/2)$-extension of the automorphism group of a $(2g+2)$-pointed $\mathbb{P}^1$. Since $2g+2 > 3$, such automorphisms of $\mathbb{P}^1$ are of finitely many, because any automorphism of $\mathbb{P}^1$ fixing three points must be the identity.

Suppose X is non-hyperelliptic. It is sufficient to show that if an automorphism τ fixes all Weierstrass points of X, then τ is the identity. Suppose τ is not identity and fixes all Weierstrass points of X. Take general points $p_1, \ldots, p_{g+1}$ on X such that any g of them are not contained in a hyperplane in the canonical embedding of X, none of them is fixed by τ, and $\tau\{p_1, \ldots, p_{g+1}\} \neq \{p_1, \ldots, p_{g+1}\}$. By Geometric Riemann-Roch, there exists $f \in \mathscr{M}(X)$ such that

$$(f)_\infty = p_1 + \cdots + p_{g+1}.$$

Consider $h = f - \tau^* f \in \mathscr{M}(X)$. By the assumption on p_i, h is not identically zero. Since f has $g+1$ poles, then h can have at most $2g+2$ poles, and hence at most $2g+2$ zeros. If p is a fixed point of τ, then $h(p) = f(p) - f(\tau(p)) = 0$, hence p is a zero of h. So the number of fixed points of τ is bounded from above by $2g+2$. But X has more than $2g+2$ Weierstrass points because it is not hyperelliptic, leading to a contradiction. □

Remark 5.14. Indeed, Hurwitz's automorphisms theorem says that $|\mathrm{Aut}(X)| \leq 84(g-1)$ for a curve X of genus $g \geq 2$ in the case of characteristic zero.

Let us give another proof of Proposition 5.13 for the case of Riemann surfaces. Let $f : X \to X$ be a continuous map of a compact triangulable space X. Define the *Lefschetz number*

$$\Lambda_f = \sum_{k \geq 0} (-1)^k \operatorname{Tr}(f^* : H^k(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})).$$

If f has finitely many fixed points, then the Lefschetz fixed point theorem says that

$$\Lambda_f = \sum_{x=f(x)} i(f,x),$$

where $i(f,x)$ is the index of a fixed point x.

In our situation, since X is a complex manifold, all the indices are positive, coming from the intersection multiplicity of the graph of ϕ and the diagonal in $X \times X$ for $\phi \in \mathrm{Aut}(X)$. In particular, if X is a Riemann surface, suppose $X \to X/\phi$ locally is given by $x \mapsto x^n$ at p for $n > 1$. Then the graph of ϕ at $(p,p) \in X \times X$ is locally given by $y - \mu x$, where μ is a primitive nth root of unity. Since the diagonal at (p,p) is locally given by $y - x$, we conclude that $i(\phi, p) = 1$.

Proof of Proposition 5.13. It suffices to show that if $\phi \in \mathrm{Aut}(X)$ fixes all the Weierstrass points for a non-hyperelliptic Riemann surface X, then ϕ is the identity. Suppose it is not. Then ϕ has only finitely many fixed points, since X is compact. Moreover, it is easy to see that ϕ is of finite order, hence its eigenvalues are unit roots. By the Lefschetz Fixed Point Theorem, we have

$$\Lambda_\phi \leq \sum_k \dim H^k(X, \mathbb{Q}) = 2g+2,$$

i.e., ϕ has at most $2g+2$ fixed points, contradicting that X is not hyperelliptic. □

6. Curve Jacobians

6.1. **Periods.** Let X be a compact Riemann surface of genus g. Take a symplectic basis of $H_1(X, \mathbb{Z})$:

$$\delta_1, \ldots, \delta_g; \delta_{g+1}, \ldots, \delta_{2g}$$

such that $\delta_i \cdot \delta_{g+i} = 1$ and $\delta_i \cdot \delta_j = 0$ for $j \neq g+i$. Such a basis can be chosen from the standard polygon presentation of X with $4g$ sides labeled as follows:

$$\delta_1, \delta_{g+1}, \delta_1^{-1}, \delta_{g+1}^{-1}; \delta_2, \delta_{g+2}, \delta_2^{-1}, \delta_{g+2}^{-1}; \cdots$$

Let $\omega_1, \ldots, \omega_g$ be a basis of $H^0(X, K)$. Define the *period matrix* of X to be the $g \times 2g$ matrix:

$$\Omega = \begin{pmatrix} \int_{\delta_1} \omega_1 & \cdots & \int_{\delta_{2g}} \omega_1 \\ \vdots & & \vdots \\ \int_{\delta_1} \omega_g & \cdots & \int_{\delta_{2g}} \omega_g \end{pmatrix}.$$

The column vectors

$$\Pi_i = \left(\int_{\delta_i} \omega_1, \cdots, \int_{\delta_i} \omega_g \right)^t$$

for $i = 1, \ldots, 2g$ are called the *periods* of X.

Remark 6.1. $\Pi_1, \ldots, \Pi_g$ are linearly independent. Otherwise if $\sum_{i=1}^{2g} k_i \int_{\delta_i} \omega_j = 0$ for all $j = 1, \ldots, g$, then $\sum_{i=1}^{2g} k_i \int_{\delta_i} \overline{\omega}_j = 0$ for all j, hence $\sum_{i=1}^{2g} [\delta_i] = 0 \in H_1(X, \mathbb{Z})$, because $\{\omega_i, \overline{\omega}_i\}_{i=1}^g$ span $H^1_{\mathrm{DR}}(X)$ which is dual to $H_1(X, \mathbb{Z})$ via integrating ω over δ. But it contradicts that $\delta_1, \ldots, \delta_{2g}$ form a basis of $H_1(X, \mathbb{Z})$.

The periods $\Pi_1, \ldots, \Pi_{2g} \in \mathbb{C}^g$ generate a lattice

$$\Lambda = \{m_1\Pi_1 + \cdots + m_{2g}\Pi_{2g} \mid m_i \in \mathbb{Z}\}$$

in $\mathbb{C}^g$. Define the *Jacobian variety* of X to be the complex torus

$$J(X) = \mathbb{C}^g/\Lambda.$$

For two points $p, q \in X$, the vector

$$\left(\int_p^q \omega_1, \ldots, \int_p^q \omega_g\right)$$

is well defined up to modulo Λ, because the difference of two paths joining p to q is a closed loop, whose class can be represented by a linear combination of $\delta_1, \ldots, \delta_{2g}$. Fix a base point p_0. We have a natural map $\mu : X \to J(X)$ given by

$$\mu(p) = \left(\int_{p_0}^p \omega_1, \cdots, \int_{p_0}^p \omega_g\right) \in J(X).$$

This map generalizes to the group $\mathrm{Div}^0(X)$ of divisors of degree 0 on X. For a divisor $D = \sum_{i=1}^n p_i - \sum_{i=1}^n q_i \in \mathrm{Div}^0(X)$, we define

$$\mu(D) = \left(\sum_{i=1}^n \int_{q_i}^{p_i} \omega_1, \cdots, \sum_{i=1}^n \int_{q_i}^{p_i} \omega_g\right) \in J(X).$$

Let Δ be the standard polygon presentation of X such that its boundary $\partial\Delta$ consists of

$$\sum_{i=1}^g (\delta_i + \delta_{g+i} + \delta_i^{-1} + \delta_{g+i}^{-1}).$$

Let ω be a holomorphic one-form, and η be a meromorphic one-form with at worst simple poles $s_1, \ldots, s_n$ contained in the interior of Δ. Denote by

$$\Pi^i = \int_{\delta_i} \omega,$$

$$N^i = \int_{\delta_i} \eta$$

for $i = 1, \ldots, 2g$. Fix a point b in the interior of Δ. Since Δ is simply connected, let

$$f(z) = \int_b^z \omega$$

for all $z \in \Delta$, which is well defined, and $df = \omega$.

Suppose $p \in \delta_i$ is identified with $p' \in \delta_i^{-1}$ on X for $i = 1, \ldots, g$. Then

$$\begin{aligned} f(p') - f(p) &= \int_p^{p'} \omega \\ &= \int_p^{v_i} \omega + \int_{\delta_{g+i}} \omega + \int_{v_i}^{p'} \omega \\ &= \int_{\delta_{g+i}} \omega \\ &= \Pi^{g+i}, \end{aligned}$$

where v_i is the vertex where δ_i and δ_{g+i} intersect. Similarly if $p \in \delta_{g+i}$ is identified with $p' \in \delta_{g+i}^{-1}$ for $i = 1, \ldots, g$, then

$$f(p') - f(p) = -\Pi^i.$$

Now consider the form $f(z)\eta$ for all $z \in \Delta$. Since η has only simple poles at $s_1, \ldots, s_n$, by the residue theorem

$$\begin{aligned}
\int_{\partial\Delta} f\eta &= 2\pi i \sum_{i=1}^{n} \mathrm{Res}_{s_i}(f\eta) \\
&= 2\pi i \sum_{i=1}^{n} (\mathrm{Res}_{s_i}\,\eta) \cdot f(s_i) \\
&= 2\pi i \sum_{i=1}^{n} (\mathrm{Res}_{s_i}\,\eta) \cdot \int_b^{s_i} \omega.
\end{aligned}$$

Note that for $p \in \delta_i$ identified with $p' \in \delta_i^{-1}$ for $i = 1, \ldots, g$, the difference $f(p') - f(p) = \Pi^{g+i}$ is a constant, as we showed above. Hence

$$\int_{\delta_i + \delta_i^{-1}} f\eta = -\Pi^{g+i} \int_{\delta_i} \eta = -\Pi^{g+i} \cdot N^i.$$

Similarly

$$\int_{\delta_{g+i} + \delta_{g+i}^{-1}} f\eta = \Pi^i \int_{\delta_{g+i}} \eta = \Pi^i \cdot N^{g+i}.$$

Comparing the two expressions for $\int_{\partial\Delta} f\eta$, we conclude the following.

Theorem 6.2 (Reciprocity Law I)**.** *Under the above setting,*

$$\sum_{i=1}^{g} (\Pi^i \cdot N^{g+i} - \Pi^{g+i} \cdot N^i) = 2\pi i \sum_{i=1}^{n} (\mathrm{Res}_{s_i}\,\eta) \cdot \int_b^{s_i} \omega,$$

where the integrals on the right are taken in the interior of Δ*.*

Take η to be a holomorphic one-form ω', which has no pole. Let Π'^i be the period of ω' over δ_i for $i = 1, \ldots, 2g$. Then the reciprocity law implies the following.

Corollary 6.3 (Riemann Bilinear Relation I)**.**

$$\sum_{i=1}^{g} (\Pi^i \cdot \Pi'^{g+i} - \Pi^{g+i} \cdot \Pi'^i) = 0.$$

Let us draw some further consequences. Note that

$$d(f\overline{\omega'}) = df \wedge \overline{\omega'} = \omega \wedge \overline{\omega'}.$$

Hence by Stokes' theorem

$$\int_{\partial\Delta} f\overline{\omega'} = \int_X \omega \wedge \overline{\omega'}.$$

The line integral on the left can be evaluated as in the proof of the theorem above, which gives

$$\int_X \omega \wedge \overline{\omega'} = \sum_{i=1}^{g} (\Pi^i \cdot \overline{\Pi'^{g+i}} - \Pi^{g+i} \cdot \overline{\Pi'^i}). \tag{1}$$

Note that

$$ih(z)(dx+idy)\wedge\overline{h(z)}(dx-idy)=|h|^2dx\wedge dy$$

which has positive integral over X. Hence $\int_X \omega\wedge\overline{\omega}$ is positive. In the above if we take $\omega'=\omega$, then we get

$$0<i\int_X \omega\wedge\overline{\omega}=i\sum_{i=1}^{g}(\Pi^i\cdot\overline{\Pi^{g+i}}-\Pi^{g+i}\cdot\overline{\Pi^i}).$$

It implies that any holomorphic one-form ω whose first g periods all vanish must be identically zero. In other words, the first $g\times g$ minor of the period matrix Ω is nonsingular. Therefore, we can choose a *normalized basis* $\omega_1,\ldots,\omega_g$ (the standard dual basis from the nondegenerate bilinear form $\int_\delta \omega$) such that

$$\int_{\delta_i}\omega_j=\delta_{ij},\quad 1\le i,j\le g,$$

where δ_{ij} is the Kronecker symbol. Under this basis the period matrix Ω has the form

$$\Omega=(I_g,Z)$$

where I_g is the $g\times g$ identity matrix and $Z=(\int_{\delta_{g+i}}\omega_j)$ for $1\le i,j\le g$.

If we set $\omega=\omega_i$ and $\omega'=\omega_j$ in the first Riemann bilinear relation, we obtain that

$$\int_{g+i}\omega_j-\int_{g+j}\omega_i=0, \tag{2}$$

namely, the block Z is *symmetric*. In addition, consider the quadratic form on $H^0(X,K)$ defined by

$$\langle\omega,\omega'\rangle=i\int_X\omega\wedge\overline{\omega'}.$$

By Equations (1) and (2),

$$\begin{aligned}\langle\omega_i,\omega_j\rangle &= i\int_X\omega_i\wedge\overline{\omega}_j\\ &= i\left(\overline{\int_{\delta_{g+i}}\omega_j}\right)-i\int_{g+j}\omega_i\\ &= 2\cdot\operatorname{Im}\int_{\delta_{g+i}}\omega_j.\end{aligned}$$

Since $\langle\omega,\omega\rangle>0$, the quadratic form on $H^0(X,K)$ is positive definite, hence the imaginary part Im Z of Z is *positive definite*. We summarize the above discussion as follows.

Corollary 6.4 (Riemann Bilinear Relation II)**.** *Under the normalized basis, the period matrix* Ω *has the form*

$$\Omega=(I_g,Z)$$

where Z *is symmetric and* Im Z *is positive definite.*

6.2. **Abel's theorem.** We first study the residues of a meromorphic one-form with simple poles only.

Lemma 6.5. *Given a finite set of points $P = \{p_1, \dots, p_n\}$ on a compact Riemann surface X and a number r_i at each p_i such that $r_1 + \cdots + r_n = 0$, there exists a one-form η which is holomorphic on $X \backslash P$ and has a simple pole at each p_i with residue equal to r_i. Moreover, any two such η differ by a holomorphic one-form.*

Proof. Consider η as an element in $H^0(X, K(p_1 + \cdots + p_n))$. We have the short exact sequence

$$0 \to K \to K(p_1 + \cdots + p_n) \to \bigoplus_{i=1}^{n} \mathbb{C}_{p_i} \to 0,$$

where the right-side map is given by taking the residue at each p_i. Since $H^1(X, K) = \mathbb{C}$, the image of the map

$$H^0(X, K(p_1 + \cdots + p_n)) \to \bigoplus_{i=1}^{n} \mathbb{C}_{p_i}$$

has codimension at most one. On the other hand, for any one-form the total sum of residues is zero, so the image is contained in the hyperplane cut out by the condition $r_1 + \cdots + r_n = 0$, and hence equal to this hyperplane.

Now suppose η_1 and η_2 both have the specified residues at the p_i and are holomorphic elsewhere. Then $\omega = \eta_1 - \eta_2$ has no pole, hence ω is a holomorphic one-form. □

Now we can state and prove Abel's theorem.

Theorem 6.6 (Abel's Theorem)**.** *Given $D = \sum_{i=1}^{d}(p_i - q_i) \in \mathrm{Div}^0(X)$ and $\omega_1, \dots, \omega_g$ a basis of $H^0(X, K)$, then $D = (f)$ is a principal divisor if and only if*

$$\mu(D) = \left(\sum_{i=1}^{d} \int_{q_i}^{p_i} \omega_1, \dots, \sum_{i=1}^{d} \int_{q_i}^{p_i} \omega_g \right) = 0 \in J(X).$$

Remark 6.7. Here we do not require p_i and q_i are distinct. If we need to emphasize their multiplicities, we write $D = \sum a_j p_j + \sum b_j q_j$ where all p_j and q_j are distinct and $\sum a_j + \sum b_j = 0$. Both notations will be used in the proof below, and we distinguish them by using indices i and j, respectively.

Proof. One direction is easier. Suppose $D = (f)$ for a meromorphic function f. Define a map $\psi : \mathbb{P}^1 \to J(X)$ by

$$\psi([u, v]) \to \mu((uf - v)).$$

Since $J(X) = \mathbb{C}^g/\Lambda$, the holomorphic one-forms dz_i descends to $J(X)$ for $i = 1, \dots, g$ and they span the tangent space of $J(X)$ at every point. Hence $\psi^* dz_i$ is holomorphic on $\mathbb{P}^1$, and then $\psi^* dz_i \equiv 0$, which implies that ψ is a constant map. Therefore,

$$\mu(D) = \mu([1, 0]) = \mu([0, 1]) = 0.$$

The other direction is harder. Suppose $D \in \mathrm{Div}^0(X)$ satisfies that $\mu(D) = 0 \in J(X)$. We want to show that D is the associated divisor of a meromorphic function f. Note that if $(f) = \sum a_j p_j + \sum b_j q_j$, then the differential

$$\eta = \frac{1}{2\pi i} d \log f = \frac{1}{2\pi i} \frac{df}{f}$$

satisfies that

$$(\eta)_\infty = -\left(\sum_j p_j + \sum_j q_j\right),$$

$$\mathrm{Res}_{p_j}\,\eta = \frac{a_j}{2\pi i}, \quad \mathrm{Res}_{q_j}\,\eta = \frac{b_j}{2\pi i}.$$

Moreover, if γ is a closed loop in $X\backslash D$, then

$$\int_\gamma \eta \in \mathbb{Z}$$

because $\log f/2\pi i$ is an antiderivative of η. Conversely, if η is a meromorphic one-form satisfying the above properties, then

$$f(p) = e^{2\pi i \int_{p_0}^{p} \eta}$$

with $p_0 \in X$ fixed is a well-defined meromorphic function such that $(f) = D$. Therefore, it suffices to show that there exists a meromorphic one-form η such that

(1) η is holomorphic on $X\backslash D$ and has a simple pole at each p_j and q_j with residues a_j and b_j, respectively;
(2) All periods of η are integers.

Choose a symplectic basis $\delta_1, \ldots, \delta_g; \delta_{g+1}, \ldots, \delta_{2g}$ disjoint with the support of D. Let $\omega_1, \ldots, \omega_g$ be a normalized basis with respect to the δ_i. By the preceding lemma, there exists a η satisfying condition (1), i.e.,

$$\mathrm{Res}_{p_j}\,\eta = a_j, \quad \mathrm{Res}_{q_j}\,\eta = b_j.$$

Moreover, adding holomorphic one-forms to η does not effect the residues, hence we may further assume that

$$N^i = \int_{\delta_i} \eta = 0, \quad i = 1, \ldots, g.$$

Now the question reduces to adjust η such that the remaining periods N^{g+i} are integers for $i = 1, \ldots, g$. Apply the Reciprocity Law to ω_k and η:

$$\sum_{i=1}^{g}\left(\int_{\delta_i} \omega_k \cdot N^{g+i} - \int_{\delta_{g+i}} \omega_k \cdot N^i\right) = 2\pi i \left(\sum_j (\mathrm{Res}_{p_j}\,\eta) \cdot \int_b^{p_j} \omega_k + \sum_j (\mathrm{Res}_{q_j}\,\eta) \cdot \int_b^{q_j} \omega_k\right).$$

Since $N^i = 0$ for $i = 1, \ldots, g$ and $\int_{\delta_i} \omega_k = \delta_{ik}$, we conclude that

$$\begin{aligned} N^{g+k} &= \sum_j a_j \int_{p_0}^{p_j} \omega_k + \sum_j b_j \int_{p_0}^{q_j} \omega_k \\ &= \sum_{i=1}^{d} \int_{q_i}^{p_i} \omega_k \\ &= \sum_{i=1}^{d} \int_{\alpha_i} \omega_k \end{aligned}$$

for some choices of paths α_i joining q_i to p_i.

By assumption

$$\mu(D) = \left(\sum_{i=1}^{d} \int_{q_i}^{p_i} \omega_1, \ldots, \sum_{i=1}^{d} \int_{q_i}^{p_i} \omega_g \right) \in \Lambda,$$

hence there exists a cycle

$$\gamma = \sum_{k=1}^{2g} m_k \delta_k, \quad m_k \in \mathbb{Z}$$

such that

$$\sum_{i=1}^{d} \int_{\alpha_i} \omega_k = \int_{\gamma} \omega_k, \quad k = 1, \ldots, g.$$

Therefore, we conclude that

$$N^{g+k} = \sum_{i=1}^{d} \int_{\alpha_i} \omega_k = \int_{\gamma} \omega_k, \quad k = 1, \ldots, g.$$

Finally define

$$\eta' = \eta - \sum_{k=1}^{g} m_{g+k} \omega_k.$$

The periods of η' are given by

$$N'^i = \int_{\delta_i} \eta - \sum_{k=1}^{g} m_{g+k} \int_{\delta_i} \omega_k = N_i - m_{g+i} = -m_{g+i}, \quad i = 1, \ldots, g,$$

$$\begin{aligned} N'^{g+i} &= N^{g+i} - \sum_{k=1}^{g} m_{g+k} \int_{\delta_{g+i}} \omega_k \\ &= \sum_{k=1}^{2g} m_k \int_{\delta_k} \omega_i - \sum_{k=1}^{g} m_{g+k} \int_{\delta_{g+i}} \omega_k \\ &= m_i + \sum_{k=1}^{g} m_{g+k} \left(\int_{\delta_{g+k}} \omega_i - \int_{\delta_{g+i}} \omega_k \right) \\ &= m_i, \end{aligned}$$

where in the last step we used Equation (2) for a normalized basis. Then η' has all the desired properties to define $f(p) = e^{2\pi i \int_{p_0}^{p} \eta'}$ and hence $D = (f)$. □

Remark 6.8. Recall that two divisors are linearly equivalent if and only if their difference is a principal divisor. Then Abel's theorem implies that the map μ factors through $\mathrm{Pic}^0(X)$ and the induced map $\tilde{\mu} : \mathrm{Pic}^0(X) \to J(X)$ is an injection.

6.3. **Jacobi Inversion.** Following the preceding section, it is natural to ask whether $\mu : \mathrm{Div}^0(X) \to J(X)$ is surjective. If it is the case, then the induced map $\tilde{\mu} : \mathrm{Pic}^0(X) \to J(X)$ is an isomorphism.

Theorem 6.9 (Jabobi Inversion)**.** *Let X be a compact Riemann surface of genus g, $p_0 \in X$, and $\omega_1, \ldots, \omega_g$ a basis of $H^0(X, K)$. Then for any $\lambda \in J(X)$, there exist g points $p_1, \ldots, p_g \in X$ such that*

$$\mu\left(\sum_{i=1}^{g}(p_i - p_0)\right) = \lambda.$$

In other word, for any $\lambda = (\lambda_1, \ldots, \lambda_g) \in \mathbb{C}^g$, there exist g points $p_1, \ldots, p_g$ and some paths α_i joining p_0 to p_i such that

$$\sum_{i=1}^{g} \int_{\alpha_i} \omega_j = \lambda_j, \quad j = 1, \ldots, g.$$

Moreover, for general $\lambda \in \mathbb{C}^g$, such a divisor $\sum_{i=1}^{g} p_i$ is unique.

Proof. Let $X^{(d)}$ denote the dth symmetric product of X, which is a d-dimensional complex manifold. Define $\mu^{(g)} : X^{(g)} \to J(X)$ by

$$\sum_{i=1}^{g} p_i \mapsto \left(\sum_{i=1}^{g} \int_{p_0}^{p_i} \omega_1, \ldots, \sum_{i=1}^{g} \int_{p_0}^{p_i} \omega_g\right).$$

It suffices to show that $\mu^{(g)}$ is surjective and generically one-to-one.

Let $D = \sum_{i=1}^{g} p_i \in X^{(g)}$ with all p_i distinct, and z_i a local coordinate of X at p_i where p_i corresponds to by $z_i = 0$. Then $(z_1, \ldots, z_g)$ provide local coordinates of $X^{(g)}$ near D. Let $D' = \sum_{i=1}^{g} z_i$ be a divisor near D, then

$$\begin{aligned} \frac{\partial}{\partial z_i}(\mu^{(g)})(D') &= \frac{\partial}{\partial z_i}\left(\int_{p_0}^{z_i} \omega_j\right) \\ &= \omega_j / dz_i, \end{aligned}$$

where $\omega_j/dz_i = h_j(z_i)$, if $\omega_j = h_j(z_i)dz_i$ under the coordinate z_i at p_i.

The Jacobian matrix of $\mu^{(g)}$ near D is thus given by

$$J = \begin{pmatrix} \omega_1/dz_1 & \cdots & \omega_1/dz_g \\ \vdots & & \vdots \\ \omega_g/dz_1 & \cdots & \omega_g/dz_g \end{pmatrix}.$$

Note that changing the local coordinate z_i amounts to multiplying the ith column by a nonzero factor, hence it does not affect the rank of J.

We can choose p_1 such that $\omega_1(p_1) \neq 0$. Subtracting suitable copies of ω_1 from $\omega_2, \ldots, \omega_g$, we may arrange that $\omega_2(p_1) = \ldots = \omega_g(p_1) = 0$. Next, we choose p_2 such that $\omega_2(p_2) \neq 0$, and then arrange $\omega_3(p_2) = \ldots = \omega_g(p_2) = 0$, so on and so forth. Eventually J will be triangular with zeros below the diagonal and nonzero on the diagonal, hence J has full rank at such D. In general, if a holomorphic map between two connected, equi-dimensional complex manifolds at one point has a non-degenerate Jacobian, it follows from the proper mapping theorem that the map is surjective. Here the domain and the target of $\mu^{(g)}$ are both g-dimensinal, hence $\mu^{(g)}$ is surjective.

It remains to show that $\mu^{(g)}$ is generically one-to-one. By Abel's theorem, the fiber of $\mu^{(g)}$ over $\lambda \in J(X)$ consists of a linear system $|D|$ where $\mu^{(g)}(D) = \lambda$, hence the fiber is a projective space. But $X^{(g)}$ and $J(X)$ have the same dimension, so the generic fiber can only be a point, i.e., $\mu^{(g)}$ is generically one-to-one. □

6.4. **Another reciprocity law.** Let ω be a holomorphic differential on X. Suppose η is a meromorphic differential with *zero* residues. Let p be a pole of η of order n. Choose a local coordinate z at p, and write

$$\eta(z) = \left(\sum_{i=-n}^{\infty} a_i^p z^i \right) dz,$$

$$\omega(z) = \left(\sum_{j=0}^{\infty} b_j^p z^j \right) dz,$$

where $a_{-1}^p = \operatorname{Res}_p \eta = 0$.

Take the symplectic basis $\delta_1, \ldots, \delta_g; \delta_{g+1}, \ldots, \delta_{2g}$ as before, disjoint with the poles of η. Let Δ be the corresponding $4g$-gon presentation of X, and b the base point of all the δ_i. Denote by Π^i and N^i the periods of ω and η over δ_i, respectively. For $z \in \Delta$, define the function

$$f(z) = \int_b^z \omega$$

with $df = \omega$.

Now integrate $f\eta$ along $\partial\Delta$. As we did in the first reciprocity law, we may integrate it in two ways. The first way is to pair δ_i with δ_i^{-1} and pair δ_{g+i} with δ_{g+i}^{-1}, which gives

$$\int_{\partial\Delta} f\eta = \sum_{i=1}^{g} (\Pi^i N^{g+i} - \Pi^{g+i} N^i).$$

On the other hand, near a pole p of η, we have

$$f(z) = \int_b^z \omega = \sum_{j=0}^{\infty} \frac{b_j^p}{j+1} z^{j+1},$$

hence by the residue theorem

$$\int_{\partial\Delta} f\eta = 2\pi i \sum_p \operatorname{Res}_p(f\eta) = 2\pi i \sum_p \left(\sum_{j=2}^{n} \frac{a_{-j}^p b_{j-2}^p}{j-1} \right),$$

where we use $\operatorname{Res}_p \eta = 0$ hence we do not care about the constant term in f.

We thus conclude the Reciprocity Law II:

$$\sum_{i=1}^{g} (\Pi^i N^{g+i} - \Pi^{g+i} N^i) = 2\pi i \sum_p \left(\sum_{j=2}^{n} \frac{a_{-j}^p b_{j-2}^p}{j-1} \right).$$

There are a number of reciprocity laws that can be obtained similarly. Let us mention another such example.

Theorem 6.10 (Weil)**.** *Let f and g be two meromorphic functions on X such that the associated divisors (f) and (g) are disjoint. Then*

$$\prod_{p \in X} f(p)^{\operatorname{ord}_p(g)} = \prod_{p \in X} g(p)^{\operatorname{ord}_p(f)}.$$

Example 6.11. Let $f(x) = \frac{x^2-1}{x^2}$ and $g(x) = \frac{x-2}{x+2}$ be two functions on $\mathbb{P}^1$. Then f has two simple zeros at ± 1 and has a double pole at 0, and g has a simple zero at 2 and a simple pole at -2. We have

$$\prod_{p\in\mathbb{P}^1} f(p)^{\mathrm{ord}_p(g)} = \frac{f(2)}{f(-2)} = 1,$$

$$\prod_{p\in\mathbb{P}^1} g(p)^{\mathrm{ord}_p(f)} = \frac{g(1)g(-1)}{g(0)^2} = 1,$$

thus confirming the theorem in this special case.

An elementary proof for $X \cong \mathbb{P}^1$. In this case, we may assume that the supports of (f) and (g) do not contain ∞. Suppose

$$f(z) = \prod_i (z - a_i)^{m_i}, \quad g(z) = \prod_j (z - b_j)^{n_j},$$

where all a_i and b_j are distinct, and $\sum_i m_i = \sum_j n_j = 0$. Then we have

$$\begin{aligned}\prod_{p\in\mathbb{P}^1} f(p)^{\mathrm{ord}_p(g)} &= \prod_j f(b_j)^{n_j} \\ &= \prod_{i,j} (b_j - a_i)^{m_i n_j}.\end{aligned}$$

Similarly

$$\prod_{p\in\mathbb{P}^1} g(p)^{\mathrm{ord}_p(f)} = \prod_{i,j} (a_i - b_j)^{m_i n_j}.$$

It suffices to show that

$$\prod_{i,j} (-1)^{m_i n_j} = 1,$$

which is equivalent to show that $\sum_{i,j} m_i n_j$ is even. By assumption,

$$\sum_{i,j} m_i n_j = \left(\sum_i m_i\right) \cdot \left(\sum_j n_j\right) = 0,$$

thus proving the desired equality. □

Proof of the general case. Suppose $\{p_i\}$ is the support of (f) and $\{q_i\}$ is the support of (g). It suffices to prove that

$$\sum_i \mathrm{ord}_{q_i}(g) \log f(q_i) - \sum_i \mathrm{ord}_{p_i}(f) \log g(p_i) \in 2\pi i \cdot \mathbb{Z}.$$

Use the previous setting on the δ_i and Δ. Draw a path α_i from b to p_i. Let Δ' be the complement of the paths α_i in Δ. Then Δ' can be considered as a polygon with edges $\ldots, \alpha_i, \alpha_i^{-1}, \ldots$ besides the δ_i. Since Δ' is simply connected and f is holomorphic and nonzero in the interior of Δ', we can choose a single branch of $\log f$ in Δ'.

Consider the meromorphic differential

$$\phi = \log f \cdot d \log g = \log f \cdot \frac{dg}{g}$$

in Δ'. We have seen that dg/g has a simple pole with residue equal to $\mathrm{ord}_{q_i}(g)$ at each q_i. Hence by the residue theorem

$$\int_{\partial\Delta'} \phi = 2\pi i \sum_{q_i} \mathrm{Res}_{q_i} \phi = 2\pi i \sum_{q_i} \mathrm{ord}_{q_i}(g) \log f(q_i).$$

Alternatively, we may integrate along $\partial\Delta'$ by pairing δ_i with δ_i^{-1}, δ_{g+i} with δ_{g+i}^{-1}, and α_i with α_i^{-1}. For $p \in \delta_i$ identified with $p' \in \delta_i^{-1}$, we have

$$\log f(p') - \log f(p) = \int_{\delta_{g+i}} d\log f,$$

hence

$$\int_{\delta_i + \delta_i^{-1}} \phi = \left(\int_{\delta_{g+i}} d\log f \right) \left(\int_{\delta_i} d\log g \right).$$

Similarly we have

$$\int_{\delta_{g+i} + \delta_{g+i}^{-1}} \phi = \left(- \int_{\delta_i} d\log f \right) \left(\int_{\delta_{g+i}} d\log g \right).$$

For $p \in \alpha_i$ identified with $p' \in \alpha_i^{-1}$, we have

$$\log f(p') - \log f(p) = -2\pi i \, \mathrm{ord}_{p_i}(f),$$

hence

$$\int_{\alpha_i + \alpha_i^{-1}} \phi = 2\pi i \, \mathrm{ord}_{p_i}(f) \int_b^{p_i} d\log g.$$

Therefore, we have

$$\sum_i \int_{\alpha_i + \alpha_i^{-1}} \phi = 2\pi i \left(\sum_i \mathrm{ord}_{p_i}(f)(\log g(p_i) - \log g(b)) \right) = 2\pi i \sum_i \mathrm{ord}_{p_i}(f) \log g(p_i),$$

because $\sum_i \mathrm{ord}_{p_i}(f) = 0$.

Comparing the two expressions of the integral of ϕ over $\partial\Delta'$, we conclude that

$$2\pi i \left(\sum_{q_i} \mathrm{ord}_{q_i}(g) \log f(q_i) - \sum_i \mathrm{ord}_{p_i}(f) \log g(p_i) \right)$$

$$= \sum_{i=1}^{g} \left(\left(\int_{\delta_{g+i}} d\log f \right) \left(\int_{\delta_i} d\log g \right) - \left(\int_{\delta_{g+i}} d\log f \right) \left(\int_{\delta_i} d\log g \right) \right).$$

Since

$$\int_{\delta_i} d\log f \in 2\pi i \cdot \mathbb{Z}, \quad \int_{\delta_i} d\log g \in 2\pi i \cdot \mathbb{Z},$$

the right side of the above equality is an integral multiple of $(2\pi i)^2$, which implies that

$$\sum_{q_i} \mathrm{ord}_{q_i}(g) \log f(q_i) - \sum_i \mathrm{ord}_{p_i}(f) \log g(p_i) \in 2\pi i \cdot \mathbb{Z}$$

as desired. $\square$

6.5. **Riemann-Roch revisit.** Let us prove a simpler version of the Riemann-Roch formula by the reciprocity laws. Suppose $D = \sum_{i=1}^d p_i$ is a degree d effective divisor on X, where the p_i are distinct. If $f \in \mathcal{M}(X)$ with $(f) + D \geq 0$, then $\eta = df$ is a meromorphic differential which is holomorphic on $X \backslash D$, with zero periods, no residues, and a pole of order at most two at each p_i. Conversely given such η, the function

$$f(p) = \int_b^p \eta$$

is well-defined and $(f) + D \geq 0$. Since $df = df'$ if and only if f and f' differ by a constant, it follows that

$$h^0(X, \mathscr{O}(D)) = 1 + \dim V,$$

where V is the space of all such meromorphic differentials η.

By the exact sequence

$$0 \to K(p) \to K(2p) \to \mathbb{C}_p \to 0,$$

there exists a meromorphic differential on X which is holomorphic away from p and has a double pole at p with zero residue. Therefore, given a sequence of complex numbers $a = (a_1, \ldots, a_d)$, there exists a meromorphic differential η_a on X which is holomorphic on $X \backslash D$ and of the form

$$\eta_a(z) = (a_i z_i^{-2} + \text{positive degree terms}) \cdot dz_i,$$

where z_i is a local coordinate at p_i. Moreover, any two such η_a differ by a holomorphic one-form on X, hence there exists a *unique* such differential, denoted by ϕ_a, whose first g periods are all zero.

Let $W \cong \mathbb{C}^d$ denote the vector space of such forms ϕ_a by varying $a \in \mathbb{C}^d$. Define a linear map

$$\psi : W \to \mathbb{C}^g$$

by

$$\phi_a \mapsto \left(\int_{\delta_{g+1}} \phi_a, \ldots, \int_{\delta_{2g}} \phi_a \right).$$

It follows that $V = \ker(\psi)$. To describe ψ explicitly, let $\omega_1, \ldots, \omega_g$ be a normalized basis. By the Reciprocity Law II applied to ω_j and ϕ_a, we have

$$\int_{\delta_{g+j}} \psi_a = 2\pi i \sum_{i=1}^d a_i (\omega_j / dz_j)(p_j).$$

Therefore, the map ψ is given by the matrix

$$\begin{pmatrix} \omega_1/dz_1(p_1) & \cdots & \omega_1/dz_d(p_d) \\ \vdots & & \vdots \\ \omega_g/dz_1(p_1) & \cdots & \omega_g/dz_g(p_d) \end{pmatrix}.$$

Note that the number of (independent) relations among the row vectors of this matrix equals the number of linearly independent holomorphic one-forms vanishing at p_i for all i, i.e., the dimension of $H^0(X, K(-D))$. We thus conclude that

$$\begin{aligned} h^0(X, \mathscr{O}(D)) &= \dim\ker(\psi) + 1 \\ &= d - \operatorname{rank}\psi + 1 \\ &= d - (g - h^0(X, K(-D))) + 1 \\ &= 1 - g + d + h^0(X, K(-D)), \end{aligned}$$

recovering the Riemann-Roch formula in this case.

Exercise 6.12. Generalize the above proof to the case when $D = \sum_i d_i p_i$ where $d_i \in \mathbb{Z}$ and the p_i are distinct.

6.6. **Theta divisors.** Recall that $\delta_1, \ldots, \delta_{2g}$ form a standard basis of $H_1(X, \mathbb{Z})$. Take $\omega_1, \ldots, \omega_g$ the normalized basis with respect to $\{\delta_i\}$. Let $\Lambda \subset \mathbb{C}^g$ be the lattice generated by the periods

$$e_\alpha = \lambda_\alpha = \left(\int_{\delta_\alpha} \omega_1, \ldots, \int_{\delta_\alpha} \omega_g \right), \quad \alpha = 1, \ldots, g,$$

$$Z_\alpha = \lambda_{g+\alpha} = \left(\int_{\delta_{g+\alpha}} \omega_1, \ldots, \int_{\delta_{g+\alpha}} \omega_g \right), \quad \alpha = 1, \ldots, g.$$

Then the period matrix $\Omega = (\int_{\delta_j} \omega_i)$ can be written as

$$\Omega = (I, Z),$$

where $Z = (Z_1, \ldots, Z_g)$ is symmetric and $\operatorname{Im} Z$ is positive definite. Denote by

$$Z_{\alpha\beta} = \int_{\delta_{g+\alpha}} \omega_\beta, \quad \alpha, \beta = 1, \ldots, g.$$

Let $x_1, \ldots, x_{2g}$ be the real coordinates on $\mathbb{C}^g$ dual to the real basis $\lambda_1, \ldots, \lambda_{2g}$. Then the differential form

$$\omega = \sum_{\alpha=1}^{g} dx_\alpha \wedge dx_{g+\alpha}$$

gives a principal polarization of $J(X) = \mathbb{C}^g/\Lambda$. Let L be the line bundle on $J(X)$ with $c_1(L) = \omega$, translated so that a global section $\tilde{\theta}$ of L is represented by the *Riemann theta function* $\theta \in \mathscr{O}(\mathbb{C}^g)$ satisfying

$$\theta(z + e_\alpha) = \theta(z), \quad \theta(z + Z_\alpha) = e^{-2\pi i (z_\alpha + Z_{\alpha\alpha}/2)} \theta(z).$$

Let $\Theta \subset J(X)$ be the (theta) divisor of the section $\tilde{\theta}$.

Fix a point $b \in X$. Consider the Abel-Jacobi map $\mu : X \to J(X)$ given by

$$\mu(z) = \left(\int_b^z \omega_1, \ldots, \int_b^z \omega_g \right).$$

We want to compute the intersection number of $\mu(X)$ with Θ in $J(X)$. We do it by computing the degree of $\mu^* L$ on X, or equivalently, by computing the zeros of the section $\mu^* \tilde{\theta}$ on X (counting with multiplicity).

Let Δ be the $4g$-gon presentation of X with respect to the basis $\{\delta_i\}$. Let $\tilde{\mu} : \Delta \to \mathbb{C}^g$ be the lifting of μ given by

$$\tilde{\mu}(z) = \left(\int_b^z \omega_1, \ldots, \int_b^z \omega_g \right) \in \mathbb{C}^g$$

where the path of integration is contained in Δ, and let $\mu_\alpha(z)$ be the α-th entry, i.e., the integral of ω_α from b to z. Recall that if $\mathrm{ord}_p(f) = n$ for a function f at a point p, then $d\log f = df/f$ has a simple pole at p with residue $2\pi i n$. Therefore, what we aim to count is equal to

$$\frac{1}{2\pi i}\int_{\partial\Delta} d\log(\theta(\tilde{\mu}(z))).$$

To evaluate this integral, we apply the same trick used a few times earlier. Suppose $p \in \delta_\beta$ is identified with $p' \in \delta_\beta^{-1}$ for $\beta = 1, \ldots, g$. Then

$$\tilde{\mu}_\alpha(p') - \tilde{\mu}_\alpha(p) = \int_{\delta_{g+\beta}} \omega_\alpha = Z_{\beta\alpha}.$$

It follows that

$$\tilde{\mu}(p') = \tilde{\mu}(p) + Z_\beta.$$

Hence

$$\theta(\tilde{\mu}(p')) = \theta(\tilde{\mu}(p) + Z_\beta) = e^{-2\pi i(\mu_\beta(z) + Z_{\beta\beta}/2)}\theta(\tilde{\mu}(p)),$$

which implies that

$$\begin{aligned} d\log\theta(\tilde{\mu}(p')) - d\log\theta(\tilde{\mu}(p)) &= d\log e^{-2\pi i(\mu_\beta(z) + Z_{\beta\beta}/2)} \\ &= -2\pi i \cdot d\mu_\beta(z). \end{aligned}$$

Then we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{\delta_\beta} d\log\theta(\tilde{\mu}(z)) + \frac{1}{2\pi i}\int_{\delta_\beta^{-1}} d\log\theta(\tilde{\mu}(z)) & \\ &= \frac{1}{2\pi i}\int_{\delta_\beta} 2\pi i \cdot d\mu_\beta(z) \\ &= \int_{\delta_\beta} \omega_\beta = 1. \end{aligned}$$

Similarly if $p \in \delta_{g+\beta}$ is identified with $p' \in \delta_{g+\beta}^{-1}$, then

$$\tilde{\mu}_\alpha(p') - \tilde{\mu}_\alpha(p) = \int_{\delta_\beta^{-1}} \omega_\alpha = -\delta_{\alpha\beta}.$$

It follows that

$$\tilde{\mu}(p') = \tilde{\mu}(p) - e_\beta.$$

Hence

$$\theta(\tilde{\mu}(p')) = \theta(\tilde{\mu}(p),$$

which implies that

$$\frac{1}{2\pi i}\int_{\delta_{g+\beta}} d\log\theta(\tilde{\mu}(z)) + \frac{1}{2\pi i}\int_{\delta_{g+\beta}^{-1}} d\log\theta(\tilde{\mu}(z)) = 0.$$

Adding up the above contributions, we conclude that

$$\deg \mu^* L = \frac{1}{2\pi i}\int_{\partial\Delta} d\log(\theta(\tilde{\mu}(z))) = g.$$

Alternatively, we can compute this degree by the following topological method. Since $c_1(L) = \omega$, we have

$$\deg \mu^* L = \int_X \mu^* \left(\sum_{\alpha=1}^{g} dx_\alpha \wedge dx_{g+\alpha}\right).$$

By the setting of μ, we have

$$\mu_* \delta_i = \lambda_i, \quad i = 1, \dots, 2g.$$

Hence

$$\int_{\delta_j} \mu^* dx_i = \int_{\lambda_j} dx_i = \delta_{ij}, \quad i, j = 1, \dots, 2g.$$

It implies that $\mu^* dx_\alpha$ is the Poincare dual of δ_α. Since $\delta_\alpha \cdot \delta_{g+\alpha} = 1$, we have

$$\deg \mu^* L = \int_X \mu^* \left(\sum_{\alpha=1}^{g} dx_\alpha \wedge dx_{g+\alpha} \right) = g.$$

For $\lambda \in J(X)$, define $\theta_\lambda(z) = \theta(z - \lambda)$ and $\Theta_\lambda = \Theta + \lambda = (\theta_\lambda)$. If $\tau_\lambda : J(X) \to J(X)$ is the translation by λ, then Θ_λ is the theta divisor associated to $\tau_\lambda^* L$, hence $[\Theta_\lambda] = c_1(\tau_\lambda^* L) = c_1(L) = [\Theta]$. By the properties of the theta function, if $\lambda' - \lambda \in \Lambda$, then $\theta_{\lambda'}$ and θ_λ differ by nonzero scalars everywhere, hence $\Theta_{\lambda'} = \Theta_\lambda$. The previous calculation shows that either $\mu(X) \subset \Theta_\lambda$ or $\mu(X)$ intersects Θ_λ at g points, counting with multiplicity.

Suppose $\mu(X) \not\subset \Theta_\lambda$. Write the divisor

$$(\mu^* \theta_\lambda) = z_1(\lambda) + \cdots + z_g(\lambda),$$

where some of the z_i may coincide.

Lemma 6.13. *For all $\lambda \in J(X)$ with $\mu(X) \not\subset \Theta_\lambda$, there is a suitable constant $\kappa \in J(X)$ such that*

$$\sum_{i=1}^{g} \mu(z_i(\lambda)) + \kappa = \lambda.$$

Proof. Recall the lift of μ as $\tilde{\mu} : \Delta \to \mathbb{C}^g$. Since $\tilde{\mu}^* \theta_\lambda$ vanished at the $z_i(\lambda)$, by the residue theorem

$$\sum_{i=1}^{g} \tilde{\mu}_\alpha(z_i(\lambda)) = \frac{1}{2\pi i} \int_{\partial \Delta} \tilde{\mu}_\alpha(z) \cdot d \log \theta_\lambda(\tilde{\mu}(z))$$

for $\alpha = 1, \dots, g$.

Suppose $z' \in \delta_\beta^{-1}$ is identified with $z \in \delta_\beta$ for $\beta = 1, \dots, g$. We have seen that

$$\tilde{\mu}_\alpha(z') - \tilde{\mu}_\alpha(z) = \delta_{g+\beta} \omega_\alpha = Z_{\alpha\beta},$$

hence by the properties of the theta function

$$\begin{aligned} \theta_\lambda(\tilde{\mu}(z')) &= \theta((\tilde{\mu}(z) - \lambda) + Z_{\alpha\beta}) \\ &= e^{-2\pi i(\tilde{\mu}_\beta(z) - \lambda_\beta + Z_{\beta\beta}/2)} \theta(\tilde{\mu}(z) - \lambda) \\ &= e^{-2\pi i(\tilde{\mu}_\beta(z) - \lambda_\beta + Z_{\beta\beta}/2)} \theta_\lambda(\tilde{\mu}(z)). \end{aligned}$$

It follows that

$$d \log \theta_\lambda(\tilde{\mu}(z')) - d \log \theta_\lambda(\tilde{\mu}(z)) = -2\pi i \cdot d\tilde{\mu}_\beta(z) = -2\pi i \cdot \omega_\beta(z).$$

Consequently

$$\frac{1}{2\pi i}\left(\int_{\delta_\beta}\tilde{\mu}_\alpha(z)d\log\theta_\lambda(\tilde{\mu}(z))+\int_{\delta_{\beta^{-1}}}\tilde{\mu}_\alpha(z)d\log\theta_\lambda(\tilde{\mu}(z))\right)=$$

$$\frac{1}{2\pi i}\int_{\delta_\beta}\tilde{\mu}_\alpha(z)d\log\theta_\lambda(\tilde{\mu}(z))-(d\log\theta_\lambda(\tilde{\mu}(z))-2\pi i\cdot\omega_\beta(z))(\tilde{\mu}_\alpha(z)+Z_{\alpha\beta})$$

$$=-\frac{Z_{\alpha\beta}}{2\pi i}\int_{\delta_\beta}d\log\theta_\lambda(\tilde{\mu}(z))+Z_{\alpha\beta}\int_{\delta_\beta}\omega_\beta(z)+\int_{\delta_\beta}\tilde{\mu}_\alpha(z)\omega_\beta(z).$$

The last two terms are independent of λ, hence they can be absorbed in the constant κ_α. For the first term, if z_1 and z_2 are the start and end points of δ_β in Δ, then $\tilde{\mu}(z_2)-\tilde{\mu}(z_1)=e_\beta$, hence $\theta_\lambda(\tilde{\mu}(z_1))=\theta_\lambda(\tilde{\mu}(z_2))$, and

$$\frac{1}{2\pi i}\int_{\delta_\beta}d\log\theta_\lambda(\tilde{\mu}(z))\in\mathbb{Z}.$$

Therefore, the first term in the above expression cannot continuously vary with λ, hence it must be a constant, which can be absorbed in κ_α.

Next, suppose $z'\in\delta_{g+\beta}^{-1}$ is identified with $z\in\delta_{g+\beta}$ for $\beta=1,\ldots,g$. We have seen that

$$\tilde{\mu}(z')=\tilde{\mu}-e_\beta,$$

hence

$$\theta_\lambda(\tilde{\mu}(z'))=\theta_\lambda(\tilde{\mu}(z)).$$

Consequently

$$\frac{1}{2\pi i}\left(\int_{\delta_{g+\beta}}\tilde{\mu}_\alpha(z)d\log\theta_\lambda(\tilde{\mu}(z))+\int_{\delta_{g+\beta}^{-1}}\tilde{\mu}_\alpha(z)d\log\theta_\lambda(\tilde{\mu}(z))\right)=$$

$$\frac{\delta_{\alpha\beta}}{2\pi i}\int_{\delta_{g+\beta}}d\log\theta_\lambda(\tilde{\mu}(z)).$$

If z_1 and z_2 are the start and end points of $\delta_{g+\beta}$ in Δ, then $\tilde{\mu}(z_2)=\tilde{\mu}(z_1)+Z_\beta$, hence

$$\theta_\lambda(\tilde{\mu}(z_2))=e^{-2\pi i(\tilde{\mu}_\beta(z))-\lambda_\beta+Z_{\beta\beta}/2}\theta_\lambda(\tilde{\mu}(z_1)),$$

and

$$\frac{1}{2\pi i}\int_{\delta_{g+\beta}}d\log\theta_\lambda(\tilde{\mu}(z))=\lambda_\beta-\tilde{\mu}_\beta(z))-Z_{\beta\beta}/2\mod\mathbb{Z},$$

$$\frac{1}{2\pi i}\int_{\delta_{g+\beta}}d\log\theta_\lambda(\tilde{\mu}(z))-\lambda_\beta=-\tilde{\mu}_\beta(z))-Z_{\beta\beta}/2\mod\mathbb{Z},$$

where the right hand side is independent of λ. Therefore, the left side must be a constant, which can be absorbed in κ_α.

In summary, we obtain that

$$\sum_{i=1}^{g}\tilde{\mu}_\alpha(z_i(\lambda))=\sum_{\beta=1}^{g}\delta_{\alpha\beta}\lambda_\beta+\kappa_\alpha=\lambda_\alpha+\kappa_\alpha,$$

thus proving the desired relation entry by entry for $\alpha=1,\ldots,g$. □

6.7. **Riemann's theorem.** In what follows we determine more explicitly about the constant κ and when $\mu(X) \subset \Theta_\lambda$ in the above. For an effective divisor $D = p_1 + \cdots + p_d \in X^{(d)}$, recall the general Abel-Jacobi map $\mu : X^{(d)} \to J(X)$ defined by

$$D \mapsto \left(\sum_{i=1}^{d} \int_b^{p_i} \omega_1, \ldots, \sum_{i=1}^{d} \int_b^{p_i} \omega_g \right)$$

where $b \in X$ is a fixed base point.

If z_i is a local coordinate of X at p_i, write $\omega_\alpha(p) = \Omega_\alpha(p) dz_i$ nearby p_i. Then the Jacobian of μ at D is

$$\begin{pmatrix} \Omega_1(p_1) & \cdots & \Omega_1(p_d) \\ \vdots & & \vdots \\ \Omega_g(p_1) & \cdots & \Omega_g(p_d) \end{pmatrix}.$$

For $d \leq g$ and general $p_1, \ldots, p_d \in X$, $p_1, \ldots, p_d$ are linearly independent on the canonical curve of X, hence in this case the matrix has maximal rank d. It follows that the image

$$W_d \coloneqq \mu(X^{(d)})$$

is a d-dimensional subvariety of $J(X)$. Since the fibers of μ are projective spaces, the map μ is generically one-to-one for $d \leq g$. In particular, W_{g-1} forms an effective divisor in $J(X)$. It satisfies a special relation with the theta divisor.

Theorem 6.14 (Riemann's Theorem)**.** $\Theta = W_{g-1} + \kappa$, *where* κ *is the constant appearing in the preceding lemma.*

Proof. The theta divisor translated by κ is $\Theta_{-\kappa} = \Theta - \kappa$. We first show that $W_{g-1} \subset \Theta_{-\kappa}$. Take a general $D = p_1 + \cdots + p_g \in X^{(g)}$ such that p_i's are all distinct, μ is one-to-one at D, and $\mu(X) \not\subset \Theta_{\kappa + \mu(D)}$.

Set $\lambda = \mu(D) + \kappa$. By the preceding lemma,

$$\Theta_\lambda \cap \mu(X) = \mu(p_1) + \cdots + \mu(p_g),$$

where $\mu^* \theta_\lambda$ vanishes at $p_1, \ldots, p_g$. It follows that

$$\theta(\mu(p_1) + \cdots + \mu(p_{g-1}) + \kappa) = \theta(\lambda - \mu(p_g)) = \theta(\mu(p_g) - \lambda) = \theta_\lambda(\mu(p_g)) = 0,$$

where we use the fact that θ is an even function. Therefore,

$$\theta_{-\kappa}(\mu(p_1) + \cdots + \mu(p_{g-1})) = 0,$$

i.e., $\mu^* \theta_{-\kappa}$ vanishes on an open dense subset of $X^{(g-1)}$, and hence on all of $X^{(g-1)}$. It implies that $W_{g-1} \subset \Theta_{-\kappa}$.

Write $\Theta_{-\kappa} = a W_{g-1} + \Theta'$, where $a \in \mathbb{Z}^+$ and Θ' is an effective divisor on $J(X)$. We want to show that $a = 1$ and $\Theta' = 0$. Take a general $\lambda = \mu(p_1) + \cdots + \mu(p_g)$ such that $-\mu(X) \not\subset W_{g-1} - \lambda$. Then $-\mu(X)$ and $W_{g-1} - \lambda$ intersect at isolated points. For each $i = 1, \ldots, g$, we have

$$-\mu(p_i) = \mu\left(\sum_{j \neq i} p_j \right) - \lambda \in W_{g-1} - \lambda,$$

Hence $(-\mu(X)) \cdot W_{g-1} \geq g$. By the fact that $-\mu(X)$ and $\mu(X)$ are homologous in $H_2(J(X))$, we conclude that $\mu(X) \cdot W_{g-1} \geq g$. Since we proved earlier that

$\mu(X) \cdot \Theta = g$, we have

$$a(\mu(X) \cdot W_{g-1}) + \mu(X) \cdot \Theta' = g.$$

Since we can translate Θ' to avoid containing entirely $\mu(X)$ (because the theta divisor is base point free), and preserve the intersection number, it follows that $\mu(X) \cdot \Theta' \geq 0$, hence $a = 1$, $\mu(X) \cdot W_{g-1} = g$, and $\mu(X) \cdot \Theta' = 0$.

It remains to show that $\Theta' = 0$. Since $\mu(X) \cdot \Theta' = 0$, then $\mu(X) \cdot \Theta'_\lambda = 0$ for any $\lambda \in J(X)$. If $\mu(X) \cap \Theta'_\lambda \neq \emptyset$, then $\mu(X) \subset \Theta'_\lambda$. If $\Theta'_\lambda \cap W_2 \neq \emptyset$, there exist $p_1, p_2 \in X$ such that $\mu(p_1) + \mu(p_2) \in \Theta'_\lambda$, $\mu(p_2) \in \Theta'_{\lambda - \mu(p_1)}$, and hence $\mu(X) \subset \Theta'_{\lambda-\mu(p_1)}$, i.e., for all $p'_2 \in X$, $\mu(p_1) + \mu(p'_2) \in \Theta'_\lambda$. By symmetry, for all $p'_1 \in X$, $\mu(p'_1) + \mu(p'_2) \in \Theta'_\lambda$, which implies that $W_2 = \mu(X^{(2)}) \subset \Theta'_\lambda$. Applying this argument inductively, one can show that if $W_d \cap \Theta'_\lambda \neq \emptyset$, then $W_d \subset \Theta'_\lambda$ for all d. Now take $d = g$. Since $W_g = J(X)$, if $\Theta' \neq 0$, then $W_g \cap \Theta'_\lambda = \Theta'_\lambda \neq \emptyset$, but $W_g \not\subset \Theta'_\lambda$, leading to a contradiction. Therefore, we conclude that $\Theta' = 0$. □

By Riemann-Roch, if D is an effective divisor of degree $g-1$, so is $K - D$. It follows that

$$W_{g-1} = \mu(K) - W_{g-1}.$$

By Riemann's theorem,

$$W_{g-1} + \kappa = \Theta = -\Theta = -W_{g-1} - \kappa = W_{g-1} - \mu(K) - \kappa,$$

$$2\kappa = -\mu(K),$$

because W_{g-1} is the theta divisor of a principal polarization, hence it cannot be fixed under any nonzero translation in $J(X)$.

We can also determine when $\mu(X) \subset \Theta_\lambda$. First note that for $\lambda \in J(X)$, by definition $\lambda - \mu(p) \in W_{g-1}$ if and only if $\lambda = \mu(D)$ for some $D \in X^{(g)}$ containing p. Hence $\lambda - \mu(X) \subset W_{g-1}$ if and only if $\lambda = \mu(D)$ for some $D \in X^{(g)}$ with $h^0(X, D) \geq 2$, where the elements in $|D|$ all map to λ under μ.

Now for any $\lambda = \mu(D)$ with $D \in X^{(g)}$, we have

$$\begin{aligned} \lambda - \mu(X) \subset W_{g-1} \quad &\Longleftrightarrow \quad \lambda - \mu(X) + \kappa \subset \Theta \\ &\Longleftrightarrow \quad \mu(X) - \lambda - \kappa \subset \Theta \\ &\Longleftrightarrow \quad \mu(X) \subset \Theta_\kappa + \lambda. \end{aligned}$$

In summary, $\mu(X) \subset \Theta_\kappa + \lambda$ if and only if $\lambda = \mu(D)$ for some $D \in X^{(g)}$ with $h^0(X, D) \geq 2$, i.e., if and only if λ belongs to the exceptional locus of $\mu : X^{(g)} \to J(X)$ parameterizing degree g line bundles L with $h^0(X, L) \geq 2$.

6.8. **Riemann's singularity theorem.** For $d \leq g - 1$, we proved that W_d is a d-dimensional subvariety in $J(X)$. We want to further study the geometry of W_d. By definition, at every point of $J(X)$, the projective tangent space can be identified with $\mathbb{P}H^0(X, K) \cong \mathbb{P}^{g-1}$, which contains the canonical curve of X. For $D \in X^{(d)}$ with $h^0(X, D) = 1$, the geometric Riemann-Roch says that the linear span $\overline{D}$ of D in the canonical curve is $(d-1)$-dimensional. Indeed we can identified it with the tangent space of W_d at $\mu(D)$.

Theorem 6.15. *For $D \in X^{(d)}$ with $h^0(X, D) = 1$, the projective tangent space $T_{\mu(D)} W_d = \overline{D}$. In particular, W_d is smooth at $\mu(D)$.*

Proof. We first prove a simple case when $D = p_1 + \cdots + p_d$, where all the p_i are distinct. Let $\omega_1, \ldots, \omega_g$ be a basis of $H^0(X, K)$. At each point p_i, write $\omega_\alpha(p) = \Omega_\alpha(p) dz_i$ for a local coordinate z_i of X for p nearby p_i. We have seen that the Jacobian matrix of μ is

$$\begin{pmatrix} \Omega_1(p_1) & \cdots & \Omega_g(p_1) \\ \vdots & & \vdots \\ \Omega_1(p_d) & \cdots & \Omega_g(p_d) \end{pmatrix}.$$

By assumption, the matrix has maximal rank d at D, because the row vectors

$$\Omega(p_i) = (\Omega_1(p_i), \cdots, \Omega_g(p_i))$$

are linearly independent, where Ω is the canonical map using $\omega_1, \ldots, \omega_g$ as a basis. Hence $W_d = \mu(X^{(d)})$ is smooth at $\mu(D)$ with tangent space $\overline{D}$ spanned by the points p_i in the canonical curve.

If D contains multiple points, let us treat the case $D = 2p_1 + p_2 + \cdots + p_{g-1}$ to illustrate what happens in general. In this case $\overline{D}$ is the space spanned by $\Omega(p_1), \Omega'(p_1), \Omega(p_2), \ldots, \Omega(p_g)$ in the canonical curve. One can choose $u_1 = z_1 + z_2$ and $u_2 = z_1 z_2$ along with $z_3, \ldots, z_n$ to be a local coordinate system of $X^{(n)}$ at D. Consider

$$F_\alpha(u_1, u_2) = \int_b^{z_1} \omega_\alpha + \int_b^{z_2} \omega_\alpha.$$

Then

$$\frac{dF_\alpha}{dz_1} = \frac{dF_\alpha}{du_1}\frac{du_1}{dz_1} + \frac{dF_\alpha}{du_2}\frac{du_2}{dz_1},$$

$$\Omega_\alpha(z_1) = \frac{dF_\alpha}{du_1} + z_2 \frac{dF_\alpha}{du_2}.$$

Similarly we have

$$\Omega_\alpha(z_2) = \frac{dF_\alpha}{du_1} + z_1 \frac{dF_\alpha}{du_2}.$$

It follows that

$$\frac{dF_\alpha}{du_2} = \frac{\Omega_\alpha(z_1) - \Omega_\alpha(z_2)}{z_2 - z_1},$$

$$\frac{dF_\alpha}{du_1} = \frac{z_1\Omega_\alpha(z_1) - z_2\Omega_\alpha(z_2)}{z_1 - z_2} = \Omega_\alpha(z_1) - z_2 \frac{dF_\alpha}{du_2}.$$

As $z_2 \to z_1$, we see that the row vectors of the Jacobian matrix of μ at $D = 2p_1 + p_2 + \cdots + p_{g-1}$ span the same space as $\Omega(p_1), \Omega'(p_1), \Omega(p_2), \ldots, \Omega(p_{g-1})$, which is exactly $\overline{D}$ spanned by $p_2, \ldots, p_{g-1}$ in the canonical curve along with the line $\overline{\Omega(p)\Omega'(p)}$.

Since $\dim \overline{D} = d - 1 = \dim W_d - 1$, we conclude that W_d is smooth at $\mu(D)$. □

Let X be an affine variety in $\mathbb{A}^n$ containing the origin. Suppose I is the defining ideal of X. For any $f \in I$, let $\mathrm{in}(f)$ be the homogeneous component of f of the lowest degree. Define the *initial ideal of* I by

$$\mathrm{in}(I) = \{\mathrm{in}(f) : f \in I\}.$$

The *tangent cone* to X at the origin is the variety cut out by $\mathrm{in}(I)$. Shifting the origin to $x \in X$, one can define the tangent cone to X at x. Geometrically speaking, the tangent cone is the locus of all tangent lines at x to analytic arcs in X.

At a smooth point of X, the tangent cone and the tangent space coincide. However at a singular point, the tangent cone carries refined information compared to the tangent space. For example, consider the plane cuspidal curve defined by $y^2 - x^3$. At the origin it has a cusp, hence the Zariski tangent space is the whole plane. The tangent cone at the origin is the x-axis, which is the geometric tangent line to the unique branch of the cusp. Another example is the plane nodal curve $y^2 - x^2(x+1)$. It has a node at the origin, whose Zariski tangent space is the whole plane. The tangent cone at the origin consists of two lines defined by $y \pm x$, which are tangent to the two branches of the node, respectively.

Suppose a degree d effective divisor D belongs to an r-dimensional linear system, i.e., $h^0(X, D) = r + 1$. Denote the divisors in $|D| \cong \mathbb{P}^r$ by

$$D_\lambda = p_1(\lambda) + \cdots + p_d(\lambda), \quad \lambda \in \mathbb{P}^r.$$

By the geometric Riemann-Roch, the linear span $\overline{D}_\lambda$ of $p_1(\lambda), \ldots, p_d(\lambda)$ is a $(d - r - 1)$-plane in $\mathbb{P}^{r-1}$ under the canonical map.

Theorem 6.16. *In the above setting, the projective tangent cone to W_d at the point $\mu(D)$ is the union*

$$\bigcup_{\lambda \in \mathbb{P}^r} \overline{D}_\lambda$$

of the $(d - r - 1)$-planes spanned by the divisors in $|D|$.

Proof. Let $D(t) = p_1(t) + \cdots + p_d(t)$ be a path in $X^{(d)}$ such that

$$D(0) = p_1 + \cdots + p_d = D.$$

Then $w(t) = \mu(D(t))$ forms an arc lying in W_d and passing through $w(0) = \mu(D)$. Conversely, any arc in W_d can be given in this fashion. For simplicity, suppose $p_1, \ldots, p_d$ are distinct. Let $z_i(t)$ be a local coordinate of X at $p_i(t)$. Then

$$w(t) = \mu(p_1(t)) + \cdots + \mu(p_d(t)) = \left(\cdots, \sum_{i=1}^{d} \int_b^{z_i(t)} \Omega_\alpha(z_i) dz_i, \cdots \right).$$

It follows that

$$\frac{dw}{dt} = \left(\cdots, \sum_{i=1}^{d} \Omega_\alpha(z_i(t)) z_i'(t), \cdots \right).$$

Setting $t = 0$, the tangent line to $w(t)$ at $\mu(D)$ is

$$\sum_{i=1}^{d} z_i'(0) \Omega(p_i),$$

belonging to $\overline{D}$ which is spanned by $\Omega(p_1), \ldots, \Omega(p_d)$. The numbers $z_i'(0)$ can be prescribed arbitrarily, thus proving the desired claim. □

Corollary 6.17. *For $d \leq g - 1$, W_d is singular at $\mu(D)$ if and only if* $\dim |D| > 0$.

Proof. We have shown that if $h^0(X, D) = 1$, then W_d is smooth at $\mu(D)$. If $h^0(X, D) = r + 1$ with $r > 0$, then every point of X is contained in some effective divisor in $|D|$. Therefore, the projective tangent cone to W_d at $\mu(D)$ contains the entire canonical curve, which is non-degenerate, i.e., cannot be contained in any proper linear subspace of $\mathbb{P}^{g-1}$. In particular, it cannot be a $(d - 1)$-dimensional projective space. Hence W_d is singular at $\mu(D)$. □

We conclude the study of W_d by the following result due to Kempf.

Theorem 6.18. *If* $\dim |D| = r$*, then the projective tangent cone to* W_d *at* $\mu(D)$ *has degree*

$$\binom{g-d+r}{r},$$

and is swept out once by the planes $\overline{D}_\lambda$*. In particular, the singular locus of the theta divisor* Θ *has dimension at least* $g-4$*.*

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