

MATH8811: COMPLEX ANALYSIS

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1. CLASSICAL TOPICS

1.1. Complex numbers. A *complex number* is expressed as $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = \sqrt{-1}$. We use \mathbb{C} to denote the set of complex numbers, which geometrically corresponds to the (x, y) -coordinate plane \mathbb{R}^2 . Define the *conjugate* of z to be $\bar{z} = x - iy$, and the *norm* of z satisfies $|z|^2 = z\bar{z} = x^2 + y^2$.

If $z \neq 0$, z can also be represented by the *polar coordinate* $z = r(\cos \theta + i \sin \theta)$, where $r = |z|$, $\cos \theta = x/r$ and $\sin \theta = y/r$. By comparing power series expansions, we conclude the Euler formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

for all $\theta \in \mathbb{R}$. In this form $z = re^{i\theta}$, which is convenient for multiplication:

$$z_1 z_2 = |z_1| |z_2| e^{i(\theta_1 + \theta_2)}.$$

Such θ is called the *argument* of z , which is well-defined modulo multiples of 2π .

Note that $z = e^{(\log r) + i\theta}$ (where $\log r$ is the real log function), hence one can define $\log z = \log r + i\theta$, which is a multi-valued function since θ and $\theta + 2\pi n$ for $n \in \mathbb{Z}$ are both arguments of z . Similarly for $z \neq 0$, $\sqrt[n]{z}$ has n different values, given by

$$r^{1/n} e^{i(\theta + 2k\pi)/n}$$

for $0 \leq k \leq n - 1$.

1.2. Differentiability. Let U be an open subset of \mathbb{R}^2 . Let $f(z) : U \rightarrow \mathbb{C}$ be a complex-valued function. We may also write

$$f(z) = u(z) + iv(z) = u(x, y) + iv(x, y),$$

where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$. Recall the difference-quotient definition of derivative

$$f'(z) = \lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z}.$$

Definition 1.1. If $f'(z)$ exists and continuous for all $z \in U$, we say that f is *holomorphic* on U , and denote it by $f \in \mathcal{H}(U)$. A function $f \in \mathcal{H}(\mathbb{C})$ is called *entire*.

Example 1.2. Polynomials and e^z are entire functions. A rational function (quotient of two polynomials) is holomorphic away from the zeros of its denominator.

Definition 1.3. We say that f is *analytic* if f can be represented by a convergent power series expansion on a neighborhood of every point in U , and we denote it by $f \in \mathcal{A}(U)$.

Later on we will show that $\mathcal{H}(U) = \mathcal{A}(U)$ and the condition that $f'(z)$ is continuous can be dropped from the definition of being holomorphic. Hence one can freely interchange the notion of being analytic and holomorphic. Below we verify one direction of their equivalence.

Lemma 1.4. *In the above setting, we have $\mathcal{A}(U) \subset \mathcal{H}(U)$.*

Proof. Suppose $z_0 \in U$. For $0 < r \ll 1$ (e.g. being the radius of convergence),

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

converges absolutely and uniformly on (compact subsets of) the disk $|z - z_0| < r$, so does the formal derivative series

$$\sum_{n=0}^{\infty} n a_n(z - z_0)^{n-1}.$$

As in calculus, under these conditions differentiation is interchangeable with summation, hence

$$f'(z) = \sum_{n=0}^{\infty} n a_n(z - z_0)^{n-1},$$

which is convergent and continuous for $|z - z_0| < r$.

By the same argument we have

$$f^{(k)}(z) = \sum_{n=0}^{\infty} n(n-1)\cdots(n-k+1)a_n(z - z_0)^{n-k}$$

for $|z - z_0| < r$, which shows that $a_n = \frac{f^{(n)}(z_0)}{n!}$ for all $n \geq 0$. \square

1.3. Cauchy-Riemann Equations. Abuse notation slightly by writing $f(z) = f(x + iy) = f(x, y)$. If $f'(z)$ exists for some $z = x + iy \in U$, then one can approach z along the x or y directions, respectively, and obtain that

$$f'(z) = \lim_{\substack{h \in \mathbb{R} \\ h \rightarrow 0}} \frac{f(z+h) - f(z)}{h} = \lim_{\substack{h \in \mathbb{R} \\ h \rightarrow 0}} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial f}{\partial x},$$

$$f'(z) = \lim_{\substack{h \in \mathbb{R} \\ h \rightarrow 0}} \frac{f(z+ih) - f(z)}{ih} = \lim_{\substack{h \in \mathbb{R} \\ h \rightarrow 0}} -i \frac{f(x, y+h) - f(x, y)}{h} = -i \frac{\partial f}{\partial y}.$$

Hence we conclude one version of the *Cauchy-Riemann equation*

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.$$

If we write $f(x, y) = u(x, y) + iv(x, y)$, then similarly we obtain that

$$f'(z) = u_x + iv_x = -iu_y + v_y,$$

where $u_x = \frac{\partial u}{\partial x}$, etc. The Cauchy-Riemann equation is equivalent to the following equations

$$u_x = v_y, \quad u_y = -v_x.$$

If we treat f as a function from \mathbb{R}^2 to \mathbb{R}^2 , then the Jacobian matrix of f is

$$df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

In this setting the Cauchy-Riemann equation is equivalent to the property that

$$df = \rho A$$

for some $\rho \geq 0$ and $A \in \text{SO}(2, \mathbb{R})$. If $\rho \neq 0$, i.e. if $f'(z) \neq 0$, then df is the composition of rotation by A and dilation by ρ , hence as a map $U \rightarrow \mathbb{R}^2$, f preserves angles between smooth arcs through z and orientation. In general, a map preserving

angle and orientation at each point is called *conformal*. We thus conclude that for $f \in \mathcal{H}(U)$, if $f'(z) \neq 0$ for all $z \in U$, then f is conformal on U .

Conversely if the Cauchy-Riemann equations hold for $f = u + iv$, we can write

$$\begin{aligned} u(x + s, y + t) &= u(x, y) + u_x(x, y)s + u_y(x, y)t + o(|s|, |t|), \\ v(x + s, y + t) &= v(x, y) + v_x(x, y)s + v_y(x, y)t + o(|s|, |t|). \end{aligned}$$

Using $u_x = v_y$, $u_y = -v_x$ and writing $w = s + it$, we obtain

$$\begin{aligned} f(z + w) - f(z) &= (u(x + s, y + t) - u(x, y)) + i(v(x + s, y + t) - v(x, y)) + o(|w|) \\ &= u_x(x, y)s + u_y(x, y)t + i(v_x(x, y)s + v_y(x, y)t) + o(|w|) \\ &= u_x(x, y)s - v_x(x, y)t + i(v_x(x, y)s + u_x(x, y)t) + o(|w|) \\ &= (u_x(x, y) + iv_x(x, y))(s + it) + o(|w|) \\ &= (u_x(z) + iv_x(z))w + o(|w|). \end{aligned}$$

It follows that $f'(z)$ exists and equals $u_x(z) + iv_x(z) = v_y(z) - iv_y(z)$ as desired.

We have thus proved the following result.

Theorem 1.5. *A complex-valued function f (with continuous partials u_x, u_y, v_x, v_y) is holomorphic iff it satisfies the Cauchy-Riemann equation.*

Another way to record the Cauchy-Riemann equation is as follows. Treat z, \bar{z} as independent variables and write $x = (z + \bar{z})/2$, $y = (z - \bar{z})/(2i)$. Then we have

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \\ \frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right). \end{aligned}$$

Then the Cauchy-Riemann equation is equivalent to

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

which is also equivalent to

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}.$$

Example 1.6. $f(z) = \bar{z}$ does not satisfy the Cauchy-Riemann equation, hence it is not holomorphic.

1.4. The Riemann Sphere. Let \mathbb{C}_∞ be the one-point compactification of \mathbb{C} , by adding a point at ∞ . The neighborhoods of ∞ are the complements of compact sets in \mathbb{C} . It is clear that \mathbb{C}_∞ is homeomorphic to the 2-sphere S^2 .

A complex structure can be given to \mathbb{C}_∞ such that it becomes a complex one-dimensional manifold, i.e., a Riemann surface. Cover \mathbb{C}_∞ by two charts (\mathbb{C}, z) and $(\mathbb{C}_\infty - \{0\}, z^{-1})$, both of which are isomorphic to \mathbb{C} . On the overlap $\mathbb{C}^* = \mathbb{C} - \{0\}$, the transition function is given by the map $z \mapsto z^{-1}$, which is biholomorphic.

This description is equivalent to saying that \mathbb{C}_∞ is the complex projective line \mathbb{CP}^1 . By definition, $\mathbb{CP}^1 = (\mathbb{C}^2 - \{(0, 0)\})/\mathbb{C}^*$, with homogeneous coordinates $\{[x, y]\}$, where $x, y \in \mathbb{C}$, not both zero, and $[x, y]$ represents the equivalence class of (x, y) modulo simultaneous scaling. Note that \mathbb{CP}^1 can be covered by two charts $U = \{[x, y]\}$ with $y \neq 0$ and $V = \{[x, y]\}$ with $x \neq 0$. Then the coordinate on U is $u = x/y$ and on V is $v = y/x$, with the transition function $v = 1/u$, which is exactly the same as above.

From now on we will use \mathbb{C}_∞ , S^2 , or \mathbb{CP}^1 interchangeably for the notation of the *Riemann sphere*.

Using the above charts, one can extend the notion of holomorphic maps from $\mathbb{C} \rightarrow \mathbb{C}$ to $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$. There are three cases. If $f(z_0) = \infty$ for some $z_0 \in \mathbb{C}$, then we require that $1/f(z)$ is holomorphic around z_0 . To make sense of being holomorphic at $z = \infty$, if $f(\infty) \neq \infty$, we require that $f(1/z)$ is holomorphic around $z = 0$. Finally if $f(\infty) = \infty$, we require that $1/f(1/z)$ is holomorphic around $z = 0$.

Example 1.7. Any rational function on \mathbb{C} extends to a holomorphic map from $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$.

1.5. Möbius transformations. For a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{C}),$$

define the *Möbius transformation* associated to A by

$$T_A(z) = \frac{az + b}{cz + d}$$

as a map from $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$.

It is easy to check that $T_A \circ T_B = T_{A \cdot B}$ and $T_A^{-1} = T_{A^{-1}}$. In particular, T_A is biholomorphic and is thus an automorphism of \mathbb{C}_∞ .

Note that if $\tilde{A} = \lambda A$ for $\lambda \in \mathbb{C}^*$, then $T_{\tilde{A}} = T_A$. Conversely if $T_A = T_{\tilde{A}}$ for $A, \tilde{A} \in \mathrm{SL}(2, \mathbb{C})$, then

$$T'_A(z) = \frac{ad - bc}{(cz + d)^2} = T'_{\tilde{A}}(z) = \frac{\tilde{a}\tilde{d} - \tilde{b}\tilde{c}}{(\tilde{c}z + \tilde{d})^2},$$

hence $cz + d = \pm(\tilde{c}z + \tilde{d})$ since the numerators are 1 by assumption. It follows that A and \tilde{A} are the same matrices in $\mathrm{SL}(2, \mathbb{C})$ possibly up to a choice of sign. Therefore, we conclude the following.

Proposition 1.8. *The family of all Möbius transformations is the same as $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm \mathrm{Id}\}$.*

Lemma 1.9. *Every Möbius transformation is the composition of four elementary maps:*

- (1) *Translation* $z \mapsto z + z_0$.
- (2) *Dilation* $z \mapsto \lambda z$, $\lambda > 0$.
- (3) *Rotation* $z \mapsto e^{i\theta} z$, $\theta \in \mathbb{R}$.
- (4) *Inversion* $z \mapsto 1/z$.

Proof. If $c = 0$, then

$$T_A(z) = \frac{a}{d}z + \frac{b}{d},$$

which is the composition of translation, dilation and possibly rotation (if $a/d < 0$).

If $c \neq 0$, then

$$T_A(z) = \frac{bc - ad}{c^2} \frac{1}{z + \frac{d}{c}} + \frac{a}{c}.$$

□

Example 1.10. The Möbius transformation $z \mapsto \frac{z-1}{z+1}$ takes the right half-plane onto the unit disk. In particular, the imaginary axis $i\mathbb{R}$ maps onto the unit circle.

If we include all lines in \mathbb{R}^2 as circles (of infinite radius), then it is not hard to verify the following.

Lemma 1.11. *Möbius transformations take circles onto circles.*

Proof. This can be checked using the four elementary maps in the composition of a Möbius transformation. \square

Note that the equation $T(z) = z$ is (at most) a quadratic equation for any Möbius transformation T . Hence T can have at most two fixed points unless it is the identity. For instance, $z \mapsto z + 1$ has only one fixed point at ∞ , while $z \mapsto 1/z$ has two fixed points at ± 1 .

Lemma 1.12. *A Möbius transformation is determined completely by its action on three distinct points on \mathbb{C}_∞ . Moreover, given distinct $z_1, z_2, z_3 \in \mathbb{C}_\infty$, there exists a unique Möbius transformation T such that $T(z_1) = 0$, $T(z_2) = 1$, and $T(z_3) = \infty$.*

Proof. If S and T are two Möbius transformations that agree on three distinct points, then $S \cdot T^{-1}$ has at least three fixed points, hence $S \cdot T^{-1}$ is the identity transformation, and $S = T$.

For the other statement, one can first apply Möbius transformations to assume that z_i are not ∞ . Then define

$$T(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1},$$

which does the job. \square

Definition 1.13. The *cross ratio* of four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ is defined as

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4} = \frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{z_2 - z_4}{z_1 - z_4}.$$

Lemma 1.14. *The cross ratio is preserved under Möbius transformations. Moreover, four distinct points lie on a circle iff their cross ratio is real.*

Proof. For a Möbius transformation T , let $w_j = T(z_j)$ for distinct z_j for $j = 2, 3, 4$. Consider the cross ratios $S_1(z) = (z, z_2, z_3, z_4)$ and $S_2 \circ T(z) = S_2(w) = (w = T(z), w_2, w_3, w_4)$ as Möbius transformations. Since $S_1(z_2) = 1$, $S_1(z_3) = 0$, $S_1(z_4) = \infty$, $S_2 \circ T(z_2) = S_2(w_2) = 1$, $S_2(w_3) = 0$, $S_2(w_4) = \infty$, we conclude that $S_1 = S_2 \circ T$, hence

$$(z_1, z_2, z_3, z_4) = S_1(z_1) = S_2 \circ T(z_1) = (T(z_1), T(z_2), T(z_3), T(z_4)).$$

for four distinct $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$. \square

1.6. Integration. Recall that a smooth path is a C^1 -curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$, i.e., γ is differentiable with continuous derivatives. In many cases we will also consider piecewise smooth paths. If $\gamma(0) = \gamma(1)$, we say that γ is closed.

Definition 1.15. For any complex-valued continuous function f on $U \subset \mathbb{C}$ and a path γ in U , define

$$\int_\gamma f(z) dz = \int_0^1 f(\gamma(t)) \gamma'(t) dt.$$

If γ is closed, then we also write $\oint_\gamma f(z) dz$ for this integral.

By the chain rule, the above definition is independent of parameterizations of γ (as long as the orientation is preserved).

Example 1.16. Suppose $f \in \mathcal{H}(U)$. Then

$$\int_{\gamma} f'(z)dz = \int_0^1 f'(\gamma(t))\gamma'(t)dt = \int_0^1 \frac{d}{dt}f(\gamma(t))dt = f(\gamma(1)) - f(\gamma(0)).$$

In particular if γ is closed, then

$$\oint_{\gamma} f'(z)dz = 0.$$

Example 1.17. For $n \neq -1$, z^n has an antiderivative $(z^{n+1})/n+1$, hence for any closed path $\gamma \in \mathbb{C}$ we have

$$\oint_{\gamma} z^n dz = 0.$$

If $n = -1$, consider the circle parameterized by $\gamma(t) = re^{2\pi it}$ for $t \in [0, 1]$. Then

$$\oint_{\gamma} \frac{1}{z} dz = \int_0^1 \frac{1}{re^{2\pi it}} (re^{2\pi it})' dt = 2\pi i \int_0^1 dt = 2\pi i.$$

1.7. Cauchy's Theorems and Applications. The next few results form a cornerstone of complex integration.

Theorem 1.18 (Cauchy's Integration Theorem). *Let γ_1 and γ_2 be two C^1 -paths with same endpoints such that they are C^1 -homotopic in U . Then for any $f \in \mathcal{H}(U)$, we have*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof. We can triangulate the homotopy region into a union of small subregions with boundary loops η_j such that

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \sum_j \oint_{\eta_j} f(z)dz.$$

Hence it suffices to prove that $\oint_{\eta} f(z)dz = 0$ for sufficiently small loops η .

Recall that $d(f(z)dz) = 0$, hence $f(z)dz$ is a closed form. By Poincaré's lemma, any closed form is locally exact, hence on the region bounded by η , $f(z)dz = F'(z)$ for some F , and consequently $\oint_{\eta} f(z)dz = 0$.

Alternatively, write $f(z) = u(x, y) + iv(x, y)$. Let V be the region with (positive oriented) boundary curve η . Then

$$\begin{aligned} \oint_{\eta} f(z)dz &= \oint_{\partial V} f(z)dz \\ &= \oint_{\partial V} (u + iv)dx + (iu - v)dy \\ &= \iint_V (-u_y - iv_y + iu_x - v_x)dx dy \\ &= 0, \end{aligned}$$

where we used Green's theorem and the Cauchy-Riemann equations in the last two steps. \square

Denote by $D(z, r)$ the open disk centered at z of radius r .

Theorem 1.19 (Cauchy's Integration Formula). *Let $\overline{D(z_0, r)} \subset U$ and $f \in \mathcal{H}(U)$. Then*

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w-z} dw$$

for all $z \in D(z_0, r)$, where $\gamma(t) = z_0 + re^{2\pi it}$ is the boundary of $D(z_0, r)$.

Proof. Fix $z \in D(z_0, r)$ and consider the region $U_\varepsilon = D(z_0, r) - D(z, \varepsilon)$ for small ε . Then the boundary of U_ε consists of two circles that are homotopic in U . Applying Cauchy's Integration Theorem, we have

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \oint_{\partial D(z_0, r)} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{\partial D(z, \varepsilon)} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \oint_{\partial D(z_0, r)} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{\partial D(z, \varepsilon)} \frac{f(w) - f(z)}{w-z} dw - \frac{f(z)}{2\pi i} \oint_{\partial D(z, \varepsilon)} \frac{1}{w-z} dw \\ &= \frac{1}{2\pi i} \oint_{\partial D(z_0, r)} \frac{f(w)}{w-z} dw + O(\varepsilon) - f(z), \end{aligned}$$

where we used $\frac{f(w)-f(z)}{w-z}$ is close to $f'(z)$ and $\oint_{\partial D(z, \varepsilon)} \frac{1}{w-z} dw = 2\pi i$. Now letting $\varepsilon \rightarrow 0$, we thus conclude the desired formula. \square

Corollary 1.20. *Every $f \in \mathcal{H}(U)$ can be represented by a convergent power series on $D(z_0, r)$ where $r = \text{dist}(z_0, \partial U)$. Hence $\mathcal{A}(U) = \mathcal{H}(U)$. Moreover, the n th derivatives of f exist and they are all analytic on U .*

Proof. Previously we showed that $\mathcal{A}(U) \subset \mathcal{H}(U)$. Conversely for $f \in \mathcal{H}(U)$, let γ be a circle in U centered at z_0 that encloses a point z in the interior. By Cauchy's Integration Formula, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z_0) - (z-z_0)} dw \\ &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0} \right)^n dw \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z_0)^{n+1}} dw \right) (z-z_0)^n \end{aligned}$$

where the interchange of summation and integration is due to absolute and uniform convergence of the geometric series. Setting

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z_0)^{n+1}} dw$$

for any $n \geq 0$, we thus conclude that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

converges on $D(z_0, r)$ and f is analytic, with its n th derivative $f^{(n)}(z_0)$ given by the above expression, hence $f^{(n)} \in \mathcal{A}(U)$ for all n . \square

Corollary 1.21 (Cauchy's Estimates). *Let $f \in \mathcal{H}(U)$ with $|f(z)| \leq M$ on U . Then*

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{\text{dist}(z_0, \partial U)^n}$$

for all $n \geq 0$ and $z_0 \in U$.

Proof. Let $r = \text{dist}(z_0, \partial U)$ (or close to). By the previous proof, we have

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int_0^1 \frac{f(z_0 + re^{2\pi it})}{r^{n+1}e^{2(n+1)\pi it}} r(e^{2\pi it})' dt \right| \\ &= \frac{n!}{r^n} \left| \int_0^1 \frac{f(z_0 + re^{2\pi it})}{e^{2n\pi it}} dt \right| \\ &\leq \frac{n!}{r^n} \int_0^1 \left| \frac{f(z_0 + re^{2\pi it})}{e^{2n\pi it}} \right| dt \\ &\leq \frac{Mn!}{r^n} \int_0^1 dt \\ &= \frac{Mn!}{r^n}. \end{aligned}$$

□

1.8. Properties of Analytic Functions.

Theorem 1.22 (Liouville's Theorem). *If f is a bounded entire function, then f is constant.*

More generally, if $|f(z)| \leq C(1 + |z|^N)$ for all $z \in \mathbb{C}$ with a fixed constant C and integer $N \geq 0$, then f is a polynomial of degree at most N .

Proof. For the first statement, applying Cauchy's estimates, $|f'(z)| \leq C/r$ for a constant C and arbitrarily large r . Hence $f'(z) = 0$ for all z , and f is constant.

For the other statement, take $U = D(z, R)$. We have

$$|f^{(N+1)}(z)| \leq \frac{C(1 + R^N)(N + 1)!}{R^{N+1}}.$$

As $R \rightarrow \infty$, we conclude that $f^{(N+1)}(z) = 0$ for all z , hence f is a polynomial of degree at most N . □

Theorem 1.23 (Fundamental Theorem of Algebra). *Every non-constant complex polynomial has at least one zero in \mathbb{C} .*

Proof. Suppose $p(z)$ is a nowhere vanishing, non-constant polynomial. Define $f(z) = 1/p(z)$. Then $f \in \mathcal{H}(\mathbb{C})$. As $z \rightarrow \infty$, $|p(z)| \rightarrow \infty$, hence $|f(z)| \rightarrow 0$, and f is bounded on \mathbb{C} . By Liouville's Theorem, f is constant, leading to a contradiction. □

Theorem 1.24 (Uniqueness Theorem). *Let $f \in \mathcal{H}(U)$. Then the following are equivalent:*

- (1) $f \equiv 0$.
- (2) For some $z_0 \in U$, $f^{(n)}(z_0) = 0$ for all $n \geq 0$.
- (3) The set $\{z \in U \mid f(z) = 0\}$ has an accumulation point in U .

In particular, a non-constant holomorphic function has isolated zeros.

Proof. The implication (1) \implies (2), (3) is clear. Using the power series expansion of f at z_0 , we see that (2) \implies (1). Finally suppose (3) holds. Let $\{z_n\}$ be a sequence of zeros of f in U that converges to z_0 as $n \rightarrow \infty$. Prove by contradiction. Suppose N is the minimal nonnegative integer such that $f^{(N)}(z_0) \neq 0$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n = a_N(z - z_0)^N(1 + O(z - z_0))$$

as $z \rightarrow z_0$, where $a_N \neq 0$. It implies that $f(z) \neq 0$ in a small neighborhood of z_0 , contradicting that z_0 is a limit of zeros of f . Hence we conclude that (3) \implies (2). \square

Theorem 1.25 (Open Mapping Theorem). *For non-constant $f \in \mathcal{H}(U)$ and every $z_0 \in U$, there exists a positive integer n and a holomorphic function g locally at z_0 such that $g(z_0) \neq 0$ and*

$$f(z) = f(z_0) + (z - z_0)^n g(z).$$

We say that z_0 is a zero of order n . In particular, f maps open sets to open sets.

Proof. Take $n \geq 1$ to be the first nonzero derivative $f^{(n)}(z_0)$ and then we get the desired expression. Since $g(z_0) \neq 0$, locally around z_0 one can take a holomorphic n th root (not unique) h of g , i.e., $g(z) = (h(z))^n$. Moreover, the derivative of $(z - z_0)h(z)$ is nonzero at z_0 , hence $(z - z_0)h(z)$ is a local diffeomorphism by the usual inverse function theorem. Then $f(z) - f(z_0)$ is the composition of this diffeomorphism with $w \mapsto w^n$, which implies that f maps open sets to open sets. In fact, this proof shows that around a zero z_0 of order n , f behaves like a branched cover of degree n totally ramified at z_0 (to be discussed later). \square

Theorem 1.26 (Maximum Modulus Principle). *Let $U \subset \mathbb{C}$ be a connected open set and $f \in \mathcal{H}(U)$. If there exists $z_0 \in U$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in U$, then f is constant.*

Proof. If f is not constant, then f maps U to an open set, hence $f(z_0)$ lies in the interior of $f(U)$, contradicting that f has maximum modulus at z_0 . \square

Theorem 1.27 (Morera's Theorem). *Let f be a continuous function on U . If $\oint_{\gamma} f = 0$ for all closed paths γ in U , then $f \in \mathcal{A}(U)$.*

Proof. Fix $z_0 \in U$. Define

$$g(z) = \int_{z_0}^z f(w)dw$$

which is independent of the integration path by the assumption on f . Since $g'(z) = f(z)$, $g \in \mathcal{H}(U) = \mathcal{A}(U)$, hence all derivatives of g are analytic on U , including f as the first derivative of g . \square

Theorem 1.28. *If f is continuous on U , then f admits an antiderivative on U iff $\int_{\gamma} f = 0$ for all closed paths γ in U .*

Proof. If $F'(z) = f(z)$ on U , then $\int_{\gamma} f = 0$ by the Fundamental Theorem of Calculus. Conversely if $\int_{\gamma} f = 0$ for all smooth closed paths γ in U , define $F(z) = \int_{z_0}^z f(w)dw$ for a fixed $z_0 \in U$. Then $F(z)$ is well-defined and $F'(z) = f(z)$. \square

Finally we show that the condition of continuous derivatives can be dropped from the definition of holomorphic functions.

Theorem 1.29 (Goursat's Theorem). *Let f be a complex differentiable function on U . Then $f \in \mathcal{H}(U)$.*

Proof. By Morera's theorem, it suffices to show that $\oint_{\gamma} f(z)dz = 0$ for all closed paths γ in U . Such an integration can be approximated by summations of integrals over rectangles, hence it suffices to prove $\oint_{\gamma} f(z)dz = 0$ for γ being the boundary of a rectangle R . Divide R into four rectangles $R_1^{(i)}$ of equal size for $i = 1, 2, 3, 4$. Then

$$\oint_{\partial R} f(z)dz = \sum_i \oint_{\partial R_1^{(i)}} f(z)dz,$$

and there exists $R_1^{(i)}$, say $R_1^{(1)}$, such that

$$\left| \oint_{\partial R} f(z)dz \right| \leq 4 \cdot \left| \oint_{\partial R_1^{(1)}} f(z)dz \right|.$$

Now divide $R_1^{(1)}$ and repeat this process. We obtain a sequence of rectangles

$$R \supset R_1 \supset R_2 \supset \dots$$

(where we omit the upper scripts), such that

$$\left| \oint_{\partial R} f(z)dz \right| \leq 4^n \cdot \left| \oint_{\partial R_n} f(z)dz \right|$$

for all n . Note that $|\partial R_n| = 2^{-n}|\partial R|$.

Let z_0 be the limit point of R_n as $n \rightarrow \infty$. For z close to z_0 , by assumption

$$|(f(z) - f(z_0)) - f'(z_0)(z - z_0)| \leq \varepsilon|z - z_0|$$

for $\varepsilon \ll 1$. Since $\oint_{\partial R_n} (z - z_0)dz = 0$, it follows that

$$\left| \oint_{\partial R_n} f(z)dz \right| \leq \varepsilon \oint_{\partial R_n} |z - z_0|dz \leq \varepsilon |\partial R_n|^2 = \varepsilon 4^{-n} |\partial R|^2$$

where we used $|z - z_0| \leq |\partial R_n|$ for z on the boundary of R_n and z_0 inside R_n . We thus conclude that

$$\left| \oint_{\partial R} f(z)dz \right| \leq \varepsilon |\partial R|^2,$$

hence it must be zero as $\varepsilon \rightarrow 0$. \square

1.9. The Winding Number. Recall that

$$1 = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z - z_0} dz$$

for any circle γ centered at z_0 (or more generally any closed path homotopic to γ without passing through z_0). Such closed paths look like going around z_0 once. This motivates the following definition-theorem.

Theorem 1.30 (Winding Number). *Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a closed path. Then for any z_0 not contained in the image of γ , the integral*

$$n(\gamma; z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z - z_0} dz$$

is an integer, called the winding number or index of γ relative to z_0 . It is constant on each connected component of $\mathbb{C} - \gamma$ and is zero on the unbounded component (that includes ∞).

Proof. Define

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - z_0} ds$$

for $t \in [0, 1]$. One checks that

$$\frac{d}{dt} e^{-g(t)} (\gamma(t) - z_0) = 0,$$

hence

$$e^{-g(t)} (\gamma(t) - z_0) = e^{-g(0)} (\gamma(0) - z_0) = \gamma(0) - z_0.$$

Taking $t = 1$, we have

$$e^{-2\pi i n(\gamma; z_0)} (\gamma(1) - z_0) = \gamma(0) - z_0.$$

Since $\gamma(1) = \gamma(0)$, we conclude that $n(\gamma; z_0)$ is an integer. Apparently $n(\gamma; z_0)$ is locally constant, hence constant on each connected component of $\mathbb{C} - \gamma$. If z_0 is outside of the region R bounded by γ , then $1/(z - z_0)$ is holomorphic on R . It follows that $n(\gamma; z_0) = 0$ on the unbounded component of $\mathbb{C} - \gamma$. \square

Corollary 1.31. *If γ_1 and γ_2 are homotopic closed paths in $\mathbb{C} \setminus \{z_0\}$, then $n(\gamma_1; z_0) = n(\gamma_2; z_0)$.*

Theorem 1.32 (Winding Number Version of Cauchy's Integral Formula). *Let γ be a closed path homotopic to a point in U , and let $f \in \mathcal{H}(U)$. Then for any $z_0 \in U \setminus \gamma$, we have*

$$n(\gamma; z_0) \cdot f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz.$$

Proof. For fixed z_0 , define

$$g(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & z \neq z_0 \\ f'(z_0) & z = z_0. \end{cases}$$

Using the power series expansion of f at z_0 , one checks that $g \in \mathcal{H}(U)$, hence by Cauchy's Theorem, $\int_{\gamma} g(z) dz = 0$. We thus conclude the desired formula. \square

1.10. Singularities. Suppose f is analytic on a small disk except possibly at the center z_0 . We say that z_0 is an *isolated singularity* of f . If one can assign a value at z_0 such that f becomes analytic at z_0 , then z_0 is called *removable*. If $f(z) \rightarrow \infty$ as $z \rightarrow z_0$, we say that z_0 is a *pole* of f . For all other cases, z_0 is called an *essential singularity*.

Example 1.33. For an integer n , z^n has a removable singularity at 0 iff $n \geq 0$, and it has a pole at 0 iff $n < 0$.

Example 1.34. The function $e^{1/z}$ has an essential singularity at 0, which can be seen by taking $z \rightarrow 0$ along the real and imaginary axes, respectively.

Proposition 1.35. *Suppose $f \in \mathcal{H}(U \setminus \{z_0\})$. Then*

- (1) z_0 is removable iff $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$.
- (2) z_0 is a pole iff there exists a positive integer n and $h \in \mathcal{H}(U)$ with $h(z_0) \neq 0$ such that $f(z) = \frac{h(z)}{(z - z_0)^n}$. In this case n is called the pole order of f at z_0 .
- (3) z_0 is essential iff for all $0 < \varepsilon \ll 1$, the set $f(D(z_0, \varepsilon)^*)$ is (open) dense in \mathbb{C} , where $D(z_0, \varepsilon)^*$ denotes the punctured disk at z_0 contained in U .

Proof. For (1), if z_0 is removable, then the conclusion is clear. Conversely if $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$, define $g(z) = (z - z_0)^2 f(z)$ for $z \neq z_0$ and $g(z_0) = 0$. Then $g(z)$ is differentiable (including at z_0). Hence by Goursat's theorem $g \in \mathcal{H}(U)$. Moreover, $g'(z_0) = 0$ by assumption, hence g has at least a double zero at z_0 by its power series expansion. It follows that $h(z) = g(z)/(z - z_0)^2 \in \mathcal{H}(U)$. Now define $f(z_0) = h(z_0)$. Since $f = h$ on $U \setminus \{z_0\}$, it implies that z_0 is a removable singularity of f .

For (2), the “if” part is clear by definition. Conversely if z_0 is a pole, then $g(z) = 1/f(z)$ has a removable singularity at z_0 by (1), since $\lim_{z \rightarrow z_0} g(z) = 0$. Hence $g(z) = (z - z_0)^n \tilde{g}(z)$ for some $n \geq 1$ and \tilde{g} locally holomorphic with $\tilde{g}(z_0) \neq 0$. It follows that $f(z) = \frac{h(z)}{(z - z_0)^n}$ with $h(z) = 1/\tilde{g}(z)$, which is well-defined and nonzero at z_0 .

For (3), first suppose $f(D(z_0, \varepsilon)^*) \cap D(w_0, \delta) = \emptyset$ for some $w_0 \in \mathbb{C}$ and $\varepsilon, \delta > 0$. Then $\frac{1}{f(z) - w_0} \in \mathcal{H}(D(z_0, \varepsilon)^*)$ has a removable singularity at z_0 by (1), which implies that $f(z)$ has either a removable singularity or a pole at z_0 . Conversely, the density of $f(D(z_0, \varepsilon)^*)$ for every $\varepsilon > 0$ clearly violates the definition of having a removable singularity or a pole at z_0 . \square

Remark 1.36. One can strengthen (3) by the Great Picard's Theorem, which says that if f has an essential singularity at z_0 , then on any $D(z_0, \varepsilon)^*$, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.

Definition 1.37. If there exists a discrete set $P \subset U$ such that $f \in \mathcal{H}(U \setminus P)$ and such that each point in P is a pole of f , we say that f is a *meromorphic function* on U . The set of meromorphic functions on U is denoted by $\mathcal{M}(U)$.

1.11. **Laurent Series.** Let \mathcal{A} be an *annulus*

$$\mathcal{A} = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}, \quad 0 \leq r_1 < r_2 \leq \infty$$

where we allow the inner radius to be zero and the outer radius to be infinity.

Proposition 1.38 (Laurent Expansion). *Suppose $f \in \mathcal{H}(\mathcal{A})$. Then there exist unique $a_n \in \mathbb{C}$ such that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

which converges absolutely and uniformly on any compact subset of \mathcal{A} . Moreover,

$$a_n = \frac{1}{2\pi i} \oint_{|w - z_0| = r} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

for all n and any $r_1 < r < r_2$.

Such series is called the *Laurent series* of f around z_0 , and $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ is called its *principal part*.

Proof. For fixed $z \in \mathcal{A}$, denote by γ_r and η_r the circles centered at z_0 and at z of radius r , respectively. Take s_1 and s_2 such that $r_1 < s_1 < |z - z_0| < s_2 < r_2$, and take ε small. Let c be a cycle defined by

$$c = \gamma_{s_2} - \gamma_{s_1} - \eta_\varepsilon.$$

Since $\gamma_{s_2} - \gamma_{s_1}$ is homotopic to η_ε away from z , we have

$$\frac{1}{2\pi i} \oint_c \frac{f(w)}{w-z} dw = 0,$$

which combined with Cauchy's integration formula implies that

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_{s_2}} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{\gamma_{s_1}} \frac{f(w)}{w-z} dw.$$

Next we show that the integral over γ_{s_2} contributes to a_n for $n \geq 0$ and the integral over γ_{s_1} contributes to a_n for $n < 0$. Consider the expansion

$$\frac{1}{w-z} = \frac{1}{(w-z_0) - (z-z_0)} = \begin{cases} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} & \text{if } |w-z_0| > |z-z_0| \\ -\sum_{n=0}^{\infty} \frac{(w-z_0)^n}{(z-z_0)^{n+1}} & \text{if } |w-z_0| < |z-z_0|. \end{cases}$$

Since these power series converge absolutely and uniformly on the respective integration paths, inserting them into the above integral and interchanging the summation and integration gives the desired expansion as well as the expression of a_n . Alternatively, the value of a_n can be verified by dividing the Laurent series by $(z-z_0)^{n+1}$ and then integrating. \square

Note that if $r_1 = 0$, then z_0 becomes an isolated singularity of f .

Corollary 1.39. *Suppose $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is the Laurent expansion of f at an isolated singularity z_0 , which converges on $0 < |z-z_0| < \delta$ for some $\delta > 0$. Then*

- (1) z_0 is removable iff $a_n = 0$ for all $n < 0$.
- (2) z_0 is a pole of order $m > 0$ iff $a_n = 0$ for all $n < -m$ but $a_{-m} \neq 0$.
- (3) Otherwise z_0 is an essential singularity.

Proof. Apply the previous properties of isolated singularities. \square

Next we classify holomorphic functions from $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$.

Proposition 1.40. *Suppose $f : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is a holomorphic function and $f \not\equiv \infty$. Then f is a rational function, i.e. $f = p/q$ for two polynomials p, q and $q \not\equiv 0$.*

Proof. If $f(z) \in \mathbb{C}$ for all $z \in \mathbb{C}_\infty$, then f is entire and bounded, hence constant by Liouville's theorem. Suppose $f(z_0) = \infty$ for some $z_0 \in \mathbb{C}$ (we may apply Möbius transformations to assume that $f(\infty) \neq \infty$, which preserve the set of rational functions). Since $\lim_{z \rightarrow z_0} f(z) = \infty$, z_0 is a pole of f . Note that f has finitely many poles, since the poles are isolated (or equivalently the zeros of $1/f$ are isolated). Now subtracting the principal parts of the Laurent series of f at each pole, we obtain an entire function which is also bounded as its image is a compact subset of \mathbb{C} , hence is a constant. It follows that f is the sum of a constant and those principal parts, hence f is a rational function. \square

1.12. Residue Theory. Suppose $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ is the Laurent expansion of f at an isolated singularity z_0 . Then the coefficient a_{-1} is called the *residue* of f at z_0 , and denote it by $\text{Res}(f; z_0)$.

Example 1.41. Suppose $f(z) = \frac{g(z)}{z - z_0}$ where g is holomorphic around z_0 . Let

$$g(z) = g(z_0) + a_1(z - z_0) + \dots$$

by the power series expansion of g at z_0 . Then we see that

$$\text{Res}(f; z_0) = g(z_0).$$

Similarly if $f(z) = \frac{g(z)}{(z - z_0)^2}$ where g is holomorphic around z_0 , then

$$\text{Res}(f; z_0) = g'(z_0).$$

Theorem 1.42 (Residue Theorem). Let $z_1, \dots, z_m \in U$, $f \in \mathcal{H}(U \setminus \{z_i\}_{i=1}^m)$, and γ a closed path homotopic to a point in U such that γ does not pass through any z_i . Then

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{i=1}^m n(\gamma; z_i) \text{Res}(f; z_i).$$

Proof. Each z_i is an isolated singularity of f . Let f_i be the principal part of f at z_i . Then $g = f - \sum_{i=1}^m f_i \in \mathcal{H}(U)$, hence by Cauchy's theorem

$$\frac{1}{2\pi i} \oint_{\gamma} f(z) dz = \sum_{i=1}^m \frac{1}{2\pi i} \oint_{\gamma} f_i(z) dz.$$

Note that

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{(z - z_i)^k} dz = n(\gamma; z_i)$$

for $k = 1$ and the integral is 0 for $k \geq 2$. The desired formula follows from the definition of residue. \square

Example 1.43. Let γ be a circle with 1 and 2 inside γ . Then by the Residue Theorem,

$$\int_{\gamma} \frac{z + 1}{(z - 1)(z - 2)} dz = 2\pi i (\text{Res}(f; 1) + \text{Res}(f; 2)).$$

By the previous example, we have

$$\text{Res}(f; 1) = \frac{1 + 1}{1 - 2} = -2, \quad \text{Res}(f; 2) = \frac{2 + 1}{2 - 1} = 3.$$

Suppose f is meromorphic at z_0 . Then $f(z) = (z - z_0)^m g(z)$ for some integer m and $g(z_0) \neq 0$, where $m = \text{ord}_{z_0}(f)$ is the zero or pole order of f at z_0 (we use -3 for a triple pole, etc in this context). Note that

$$\frac{f'}{f} = \frac{m}{z - z_0} + \frac{g'}{g},$$

and the set of poles of f'/f is contained in the set of zeros and poles of f . In particular,

$$\text{Res}(f'/f; z_0) = \text{ord}_{z_0}(f).$$

Hence the residue theorem implies the following.

Theorem 1.44 (Argument Principle). *Suppose $f \in \mathcal{M}(U)$ and γ is a closed path homotopic to a point in U such that γ does not meet the zero set Z and the pole set P of f . Then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{z \in Z \cup P} n(\gamma; z) \text{ord}_z(f).$$

Remark 1.45. Note that

$$\frac{f'(z)}{f(z)} dz = \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) dt = \frac{(f \circ \gamma(t))'}{f \circ \gamma(t) - 0} dt.$$

By the definition of winding number, the LHS of the above theorem is equal to $n(f \circ \gamma; 0)$, which measures the change of argument of the composed path $f \circ \gamma$ around 0.

Theorem 1.46 (Rouche's Theorem). *Let $f, g \in \mathcal{H}(U)$, and γ a simple closed path in U such that its interior is contained in U . If f has no zero on γ and if $|f(z) - g(z)| < |g(z)|$ for all z on γ , then f and g have the same number of zeros (counting with orders) inside γ .*

Proof. Set $h = f/g$. Then $h \in \mathcal{H}(U)$ and h has no zeros or poles on γ . Since $|h \circ \gamma(t) - 1| < 1$ for all $t \in [0, 1]$ and the inequality $|w - 1| < 1$ cuts out the unit disk centered at 1, we conclude that 0 belongs to the unbounded component of $\mathbb{C} \setminus h \circ \gamma([0, 1])$, and hence the winding number $n(h \circ \gamma; 0) = 0$. Since γ is homotopic to a point in U , by the Argument Principle and the above remark, the number of zeros minus the number of poles of h in the interior of γ (counting with orders) is zero. Note that this is the number of zeros of f minus the number of zeros of g inside γ (counting with orders), hence they are equal. \square

Let us apply residues to evaluate more integrals. First recall

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Set $z = e^{i\theta}$. We also have

$$\int_{|z|=1} \frac{g(z)}{iz} dz = \int_0^{2\pi} \frac{g(e^{i\theta})}{ie^{i\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} g(e^{i\theta}) d\theta.$$

Example 1.47. Let us evaluate $\int_0^{2\pi} \frac{d\theta}{2 + \sin \theta}$. Replace $\sin \theta$ by $\frac{z - z^{-1}}{2i}$ where $z = e^{i\theta}$. The integrand becomes

$$\frac{1}{2 + (z - z^{-1})/2i} = \frac{2iz}{4iz + z^2 - 1}$$

which is the function g above. Hence we need to evaluate

$$\int_{|z|=1} \frac{2}{z^2 + 4iz - 1} dz = \int_{|z|=1} \frac{2}{(z - (\sqrt{3} - 2)i)(z - (-2 - \sqrt{3})i)} dz.$$

The only pole inside the unit circle is $(\sqrt{3} - 2)i$. Hence by the Residue Theorem the value of the integral is

$$2\pi i \cdot \frac{2}{(\sqrt{3} - 2)i - (-2 - \sqrt{3})i} = \frac{2\pi}{\sqrt{3}}.$$

Next we consider improper integrals. Denote by H the closed upper-half plane. Let us first make the following observation.

Theorem 1.48. *Suppose f is holomorphic away from a finite set of singularities, has no singularities on \mathbb{R} , and there exist $R, C > 0$ and $p > 1$ such that $|f(z)| \leq C|z|^{-p}$ for $\{z \in H : |z| > R\}$, then*

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{j=1}^m \text{Res}(f; z_j),$$

where z_1, \dots, z_m are the singularities of f in H .

Proof. Let γ_r be the semicircle $\{z \in H : |z| = r\}$ union the real line segment $-r \leq x \leq r$ (draw a picture), positively oriented. Then

$$\int_{\gamma_r} f(z)dz = \int_{-r}^r f(x)dx + \int_0^{\pi} f(re^{it})ire^{it}dt.$$

If $r > R$, then $|f(re^{it})ire^{it}| \leq Cr^{-p}$, and hence

$$\left| \int_0^{\pi} f(re^{it})ire^{it}dt \right| \leq \pi Cr^{1-p}$$

which tends to zero as $r \rightarrow \infty$. It follows that

$$\lim_{r \rightarrow \infty} \int_{\gamma_r} f(z)dz = \lim_{r \rightarrow \infty} \int_{-r}^r f(x)dx = \int_{-\infty}^{\infty} f(x)dx.$$

Then the desired formula follows from the Residue Theorem. \square

Remark 1.49. In the statement of the above theorem, if we replace the upper-half plane by the lower-half plane, then the conclusion still holds with the sign changed to $-$ on the right-hand side, because the lower semicircle used in the proof will have negative orientation (draw a picture).

Example 1.50. Let us evaluate $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$. Since the integrand $f(z)$ is asymptotically z^{-4} , the assumption in the above theorem holds. There are two poles at $\pm i$ and only i is contained in the upper-half plane. Write

$$f(z) = \frac{g(z)}{(z-i)^2},$$

where $g(z) = (z+i)^{-2}$. Then

$$\text{Res}(f; i) = g'(i) = -\frac{i}{4}.$$

It follows that

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx = 2\pi i \text{Res}(f; i) = \frac{\pi}{2}.$$

1.13. Normal Families. Let U be an open set in \mathbb{C} and \mathcal{F} a family of holomorphic functions defined on U .

Definition 1.51. If each sequence in \mathcal{F} has a subsequence which converges uniformly to a holomorphic function on every compact subset of U , then \mathcal{F} is called a *normal family*.

Definition 1.52. If there is an M such that $|f(z)| \leq M$ for every $z \in U$ and every $f \in \mathcal{F}$, then \mathcal{F} is called *uniformly bounded* on U .

Theorem 1.53 (Montel's Theorem). *Every uniformly bounded family on U is a normal family.*

Proof. We apply the idea of Arzela-Ascoli Theorem in real analysis. Let $\{f_n\}$ be a uniformly bounded sequence of functions on U with $|f_n(z)| \leq M$ with a fixed M for all $z \in U$ and all n . Choose points z_1, z_2, \dots in U with rational coordinates (hence they are countable). Then there is a convergent subsequence of complex numbers $\{f_{1n}(z_1)\}$, since they are bounded by M . Then $\{f_{1n}(z_2)\}$ is also a bounded sequence, so it has a convergent subsequence denoted by $\{f_{2n}(z_2)\}$. Continuing this process, we inductively construct a sequence of subsequences of $\{f_n\}$

$$\begin{array}{cccc} f_{11}, f_{12}, \dots, f_{1n}, \dots \\ f_{21}, f_{22}, \dots, f_{2n}, \dots \\ \vdots \quad \dots \quad \vdots \quad \dots \end{array}$$

each being a subsequence of the preceding one, such that $\{f_{kn}(z_j)\}_n$ is a convergent sequence for each k and each $j \leq k$. Then the diagonal sequence $\{f_{nn}\}$ converges at every z_j .

Set $g_n = f_{nn}$. We will show that $\{g_n\}$ converges uniformly on every compact subset of U . For $w \in U$, choose $r > 0$ such that $\overline{D}(w, 2r) \subset U$. For each $z \in \overline{D}(w, r)$, we have $\overline{D}(z, r) \subset \overline{D}(w, 2r) \subset U$. By Cauchy's Estimates applied to $\overline{D}(z, r)$, every $f \in \mathcal{F}$ satisfies

$$|f'(z)| \leq \frac{M}{r}.$$

Then for any $z, z' \in \overline{D}(z, r)$, we have

$$|f(z) - f(z')| = \left| \int_{z'}^z f'(u) du \right| \leq \frac{M}{r} |z - z'|$$

which in particular holds for all g_n .

Given $\varepsilon > 0$, set $\delta = \frac{r\varepsilon}{3M}$. If $z \in \overline{D}(w, r)$, there are points with rational coordinates arbitrarily close to z . Let z_j be such point with $|z - z_j| < \delta$. Then

$$|g_n(z) - g_n(z_j)| < \frac{M}{r} \delta = \frac{\varepsilon}{3}.$$

Next since $\{g_n\} = \{f_{nn}\}$ converges at z_j , we choose N such that

$$|g_n(z_j) - g_m(z_j)| < \frac{\varepsilon}{3}, \quad \forall m, n \geq N.$$

Then we have

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(z_j)| + |g_n(z_j) - g_m(z_j)| + |g_m(z_j) - g_m(z)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $m, n \geq N$. It proves that $\{g_n\}$ converges uniformly on these closed disks $\overline{D}(w, r)$, hence it converges uniformly on every compact subset K of U , as K can be covered by finitely many such disks. \square

1.14. The Riemann Mapping Theorem. Let U be a path-connected open subset of \mathbb{C} . Recall that U is *simply connected* if every loop contained in U is contractible (i.e. the fundamental group of U is trivial).

Lemma 1.54. *If U is simply connected and f is a nowhere vanishing holomorphic function on U , then f has a holomorphic logarithm and a holomorphic n th root.*

Proof. Suppose $f \in \mathcal{H}(U)$ is nowhere zero. Then $f'/f \in \mathcal{H}(U)$. Since U is simply connected, for a fixed $z_0 \in U$, we have

$$F(z) = \int_{z_0}^z \frac{f'(w)}{f(w)} dw \in \mathcal{H}(U).$$

In particular, $F' = f'/f$ and $F(z_0) = 0$. Since $f(z_0) \neq 0$, fix one value of $\log f(z_0)$ (not unique) and define

$$h(z) = e^{-F(z) - \log f(z_0)} f(z).$$

Then one checks that $h'(z) = 0$ and $h(z_0) = 1$, hence $h(z) = 1$ for all $z \in U$. It follows that

$$f(z) = e^{F(z) + \log f(z_0)}.$$

Hence the exponent can be used as $\log f$. Take $g(z) = e^{(\log f)/n}$. Then $g^n = f$. \square

A complex valued function is called *conformal* at z_0 (i.e. preserving angles of two intersecting arcs), if it has a nonzero derivative at z_0 .

Theorem 1.55 (Riemann Mapping Theorem). *Every proper, simply connected, non-empty open subset U of \mathbb{C} is conformally equivalent to the open unit disk D .*

We need some preparations before proving the theorem. First we determine the conformal automorphisms of D . For $w \in D$, define

$$h_w(z) = \frac{z - w}{1 - \overline{w}z}$$

as a special Möbius transformation, where $h_w(w) = 0$ and $h_w(0) = -w$. If $|u| = 1$, then $u^{-1} = \overline{u}$, and

$$\left| \frac{u - w}{1 - \overline{w}u} \right| = \left| \frac{u - w}{u^{-1} - \overline{w}} \right| = \left| \frac{u - w}{\overline{u} - \overline{w}} \right| = 1.$$

It implies that h_w maps the unit circle to itself. Since w is inside D and $h_w(w) = 0$, h_w maps D onto itself, hence h_w is a conformal automorphism of D .

Theorem 1.56. *Every conformal automorphism h of the unit disk is of the form*

$$h(z) = uh_w(z) = u \frac{z - w}{1 - \overline{w}z}$$

for some u and w satisfying that $|u| = 1$ and $|w| < 1$.

Proof. Suppose $h(w) = 0$ for some $w \in D$. Consider $h \circ h_w^{-1}$, which is a conformal automorphism of D and takes 0 to 0. By the Schwarz Lemma, $h \circ h_w^{-1}(z) = uz$ for some $|u| = 1$. Replacing z by $h_w(z)$, we conclude that $h(z) = uh_w(z)$. \square

Recall that U is a proper, simply connected, non-empty open subset of \mathbb{C} . Fix $z_0 \in U$ and let \mathcal{F} be the set of one-to-one conformal maps f of U into D such that $f(z_0) = 0$.

Lemma 1.57. *The set \mathcal{F} is non-empty.*

Proof. Since U is proper in \mathbb{C} , there is some $\lambda \notin U$. Then $f(z) = z - \lambda$ is non-vanishing on U , hence has a holomorphic square root g . Since f is one-to-one and non-vanishing, so is g , and hence g is a one-to-one conformal map such that $0 \notin g(U)$. By the Open Mapping Theorem, $g(U)$ is open, so we can find a closed disk $\overline{D}(w_0, r) \subset g(U)$ with $0 < r < \infty$. Moreover, the set $g(U)$ cannot contain both u and $-u$ for any $u \neq 0$, as $f = g^2$ is one-to-one. Thus the reflection $\overline{D}(-w_0, r) \cap g(U) = \emptyset$. It follows that

$$|g(z) + w_0| > r$$

for all $z \in U$. Hence the function

$$p(z) = \frac{r}{g(z) + w_0}$$

is a one-to-one conformal map of U into the unit disk D . For the fixed point $z_0 \in U$, suppose that $p(z_0) = w$. Compose p with the conformal automorphism h_w of D . We obtain a one-to-one conformal map $h_w \circ p$ of U into D which takes z_0 to $h_w(p(z_0)) = h_w(w) = 0$. Hence \mathcal{F} contains $h_w \circ p$. \square

Lemma 1.58. *Let U , z_0 , and \mathcal{F} be as above. Given any $f \in \mathcal{F}$, if f does not map U onto D , then there exists some $g \in \mathcal{F}$ such that $|g'(z_0)| > |f'(z_0)|$.*

Proof. Suppose $w \in D$ and $w \notin f(U)$. Recall that h_w is a conformal automorphism of D with $h_w(w) = 0$. Then $h_w \circ f(z) \neq 0$ for all $z \in U$. Hence $h_w \circ f$ has a holomorphic square root q on U , i.e. $q^2 = h_w \circ f$. If $q(z_0) = \lambda$, then $\lambda^2 = h_w \circ f(z_0) = h_w(0) = -w$, and

$$q'(z_0) = \frac{h'_w(0)}{2q(z_0)} f'(z_0) = \frac{1 - |w|^2}{2\lambda} f'(z_0) = \frac{1 - |\lambda|^4}{2\lambda} f'(z_0).$$

As before q is a one-to-one conformal map of U into D and $q(z_0) = \lambda \neq 0$. So $g = h_\lambda \circ q$ maps z_0 to 0, hence $g \in \mathcal{F}$ and satisfies

$$g'(z_0) = h'_\lambda(\lambda) q'(z_0) = \frac{1 - |\lambda|^4}{2\lambda(1 - |\lambda|^2)} f'(z_0) = \frac{1 + |\lambda|^2}{2\lambda} f'(z_0).$$

Since $0 < (1 - |\lambda|)^2 = 1 + |\lambda|^2 - 2|\lambda|$ and $f'(z_0) \neq 0$ for f being conformal, we conclude that $|g'(z_0)| > |f'(z_0)|$. \square

Proof of the Riemann Mapping Theorem. Let $z_0 \in U$ and \mathcal{F} be as above. Since \mathcal{F} is non-empty, set

$$m = \sup\{|f'(z_0)| : f \in \mathcal{F}\}.$$

Then m is either ∞ or a positive number, since f' is non-vanishing. Choose a sequence $\{f_n\}$ in \mathcal{F} such that

$$\lim_{n \rightarrow \infty} |f'_n(z_0)| = m.$$

Since \mathcal{F} is a uniformly bounded family, it is a normal family, hence there is a subsequence of $\{f_n\}$ which converges uniformly on compact subsets of U to a function h . It follows from Cauchy's Estimates (leaving as an exercise) that f'_n in this subsequence converges uniformly on compact subsets of U to h' , and hence $h'(z_0) = m$.

Since $m \neq 0$, h is not constant. Since each f_n is one-to-one, h is also one-to-one (Hurwitz's Theorem, another exercise). Since $f_n(z_0) = 0$ for all n , we get $h(z_0) = 0$. Hence $h \in \mathcal{F}$ and $h'(z_0) = m$ is attained as the maximum. By the previous lemma, h must map U onto D , hence h is a conformal equivalence of U to D . \square

Remark 1.59. In the proof the only property of simply connected sets we used is that every non-vanishing holomorphic function has a holomorphic square root. Since this property is preserved under conformal equivalence, being simply connected is also preserved under conformal equivalence, and the unit disk is simply connected, we conclude that a connected open subset U in \mathbb{C} is simply connected iff every non-vanishing holomorphic function on U has a holomorphic square root.

Remark 1.60. If U is a simply connected open set with a simple closed curve as boundary, and if $h : U \rightarrow D$ is a conformal equivalence, then h extends to a continuous function from \bar{U} to \bar{D} with a continuous inverse. For a proof, see e.g. Rudin's book.

1.15. Schwarz Reflection. Given a function analytic on a region, one can ask if it can be analytically continued outside of the domain. One way to extend is to use some kind of symmetry. First we introduce a notation. For a subset V of \mathbb{C} , let $V^- = \{\bar{z} : z \in V\}$.

Theorem 1.61 (Schwarz Reflection Principle). *Let U be an open set symmetric about the x -axis. Let $A = U \cap \mathbb{R}$ and V be the part of U in the open upper-half plane (draw a picture). If f is analytic on V , continuous on $V \cup A$, and real-valued on A , then f can extend to an analytic function on all of U by using $\overline{f(\bar{z})}$ on V^- .*

Proof. First, using Cauchy-Riemann Equations one checks that $g(z) = \overline{f(\bar{z})}$ is analytic on V^- (exercise). By assumption, f and g agree on A , hence they define a single continuous function h on $U = V \cup A \cup V^-$ such that h is analytic on U , except possibly at points of A .

Take a point $z_0 \in A$ and let D be a disk neighborhood of z_0 in W . One can show that for any closed path γ in D , $\int_{\gamma} h(z) dz = 0$ (exercise). It follows from Morera's Theorem that h is analytic on D . We thus conclude that h as an extension of f is analytic on U . \square

Identify the set of all lines and circles in \mathbb{C} as the set of all circles in the Riemann sphere S^2 .

Definition 1.62. Suppose $h : W \rightarrow V$ is a conformal equivalence between domains in S^2 . Let $I \subset W$ be an arc on a circle in S^2 . Then the image $h(I)$ is called an *analytic curve*.

By using a Möbius transformation, we may assume that the circle in the above definition is $\mathbb{R} \cup \{\infty\}$ and I is either an open interval on the real line or $\mathbb{R} \cup \{\infty\}$ itself. In the latter the curve will be a simple closed curve.

Let us describe explicitly reflection through an analytic curve. First, a function f on an open set U is called *conjugate analytic* if \bar{f} is analytic on U .

Definition 1.63. Suppose C is an analytic curve in U such that $U \setminus C$ has two connected components V and W . If $\rho : U \rightarrow U$ is a conjugate analytic function which fixes each point of C and interchanges V and W , and if $\rho \circ \rho(z) = z$ for all $z \in U$, then ρ is called a *reflection through C on U* .

Theorem 1.64. *If C is an analytic curve in \mathbb{C} , then there is a reflection ρ through C on some domain V containing C . This reflection is unique in the sense that another reflection through C on some neighborhood V_1 of C must be equal to ρ on the connected component of $V \cap V_1$ which contains C .*

Proof. The idea is to reduce to the case when C lies in \mathbb{R} and use the standard conjugate \bar{z} . Let $h : W \rightarrow V$ be a conformal equivalence that maps L onto C , where L is either a line segment in \mathbb{R} or $\mathbb{R} \cup \{\infty\}$. Replacing W by a connected component of $W \cap W^-$ which contains L , we may assume that $W = W^-$. Let $\kappa(z) = \bar{z}$. Then we get a reflection ρ through C on V by setting

$$\rho = h \circ \kappa \circ h^{-1}.$$

Since h is analytic and κ is conjugate analytic, one checks that ρ is conjugate analytic (exercise). Moreover,

$$\rho \circ \rho = \kappa \circ \kappa = \text{Id},$$

and $\rho(z) = z$ on C because $\kappa(z) = z$ on L and h^{-1} maps C to L . Thus ρ is a reflection through C on V .

If ρ_1 is another reflection through C on V_1 , then $\rho_1 \circ \rho$ is analytic on $V \cap V_1$, since the composition of two conjugate analytic functions is analytic (exercise). Moreover, $\rho_1 \circ \rho(z) = z$ on C , hence it must be equal to z on all connected components of $V \cap V_1$ containing C , by the Uniqueness Theorem of analytic functions. It implies that $\rho_1 = \rho^{-1} = \rho$ on this component. \square

Now we can strengthen the Schwarz Reflection Principle as follows.

Theorem 1.65. *If C is an analytic curve in a domain U such that $U \setminus C$ has two components V and W , and ρ is a reflection through C on U , then any function f which is continuous on $V \cup C$, analytic on V , and real-valued on C can extend to an analytic function on all of U by using $\overline{f(\rho(z))}$ on W .*

Proof. The proof is the same as the proof of the Schwarz Reflection Principle, where ρ plays the role of complex conjugate on U . Let

$$g(z) = \overline{f(\rho(z))}$$

on $W \cup C$. Then g is continuous on $W \cup C$ and analytic on W because it is the composition of two conjugate analytic functions \bar{f} and ρ . Since ρ fixes points on C and f is real-valued on C , g and f agree on C , hence they define a single function h on U such that h is continuous on U and analytic on $U \setminus C$. Arguing as before by Morera's Theorem, h as an extension of f is analytic on all of U . \square

Example 1.66. Let us consider the reflection through an arc on the unit circle.

The map $\rho(z) = \frac{1}{\bar{z}}$ or $\rho(re^{i\theta}) = \frac{e^{i\theta}}{r}$ is defined and conjugate analytic on $\mathbb{C} \setminus \{0\}$. It satisfies $\rho \circ \rho(\bar{z}) = z$. It also fixes each point on the unit circle. Suppose a domain U meets the unit circle in an arc C and is taken to itself by ρ . Then the unique reflection through C on U is the map ρ . For such U and C , the previous theorem says that any function analytic on the part V of U lying on one side of C , continuous on $V \cup C$, and real-valued on C can extend analytically to all of U .

1.16. Analytic Continuation. Here we discuss analytic continuation along curves. For an analytic function f defined on an open disk D , we say that (f, D) is an *analytic function element*, and is an analytic function element at w if $w \in D$. Two analytic function elements (f_1, D_1) and (f_2, D_2) at w are said to be equivalent at w if $f_1 = f_2$ on $D_1 \cap D_2$.

Definition 1.67. Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a curve and (f_0, D_0) an analytic function element at $z_0 = \gamma(0)$. Suppose there exists a partition $0 = t_0 < t_1 < \dots < t_{n+1} = 1$ of $[0, 1]$ and a sequence of function elements $(f_1, D_1), \dots, (f_n, D_n)$ such that $\gamma([t_j, t_{j+1}]) \subset D_j$ and $f_j = f_{j+1}$ on $D_j \cap D_{j+1}$ for all j . Then we say that (f_n, D_n) is an *analytic continuation* of (f_0, D_0) along γ (draw a picture).

Remark 1.68. By the Uniqueness Theorem (after refining the sequence of disks if necessary), it is clear that the analytic continuation of f along γ , up to equivalence, only depends on f and γ , not on the choice of the chain of disks.

Remark 1.69. Suppose (f_0, D_0) is a function element at z_0 and γ is a curve joining z_0 to w . If there is an open set U containing D_0 and γ , and an analytic function f on U such that $f = f_0$ on D_0 , then it is obvious that (f_0, D_0) can be analytically continued along γ using the restriction of f .

Suppose $f \in \mathcal{A}(U)$. Locally around a point $z_0 \in U$, there exists an antiderivative F of f . Now for any smooth path $\gamma : [0, 1] \rightarrow U$ beginning with z_0 , one can analytically continue F along γ as an antiderivative by

$$F(z_t) = F(z_0) + \int_{\gamma} f(w)dw$$

where $\gamma(0) = z_0$ and $\gamma(t) = z_t$. However, this procedure does not necessarily lead to a globally defined antiderivative $F \in \mathcal{A}(U)$. For example, take $f(z) = 1/z$ and $U = \mathbb{C}^*$. If γ is a closed path traversing around 0 one time, then the starting and ending log branches will differ by $2\pi i$. On the other hand, if U is simply connected, then by Cauchy's Integration Theorem such a construction yields a globally defined F . This holds in general for analytic continuation as follows.

Theorem 1.70 (Monodromy Theorem). *Suppose γ_0 and γ_1 are homotopic curves in an open set U joining z_0 to w , and (f_0, D_0) an analytic function element at z_0 , with $D_0 \subset U$. If (f_0, D_0) can be analytically continued along every curve in U , then the continuations of f_0 along γ_0 and γ_1 are equivalent function elements at w .*

Proof. Let $\{\gamma_s\}$ be a one-parameter family of curves from z_0 to w as a homotopy from γ_0 to γ_1 . Then given any $\epsilon > 0$, there is a $\delta > 0$ such that $\|\gamma_s - \gamma_r\| < \epsilon$ whenever $|s - r| < \delta$. Denote by ϕ_s the terminal function element for the continuation along γ_s . We will show that for each $r \in [0, 1]$, there is a $\delta > 0$ such that ϕ_s is equivalent to ϕ_r whenever $|s - r| < \delta$.

Let $0 = t_0 < t_1 < \dots < t_{n+1} = 1$ be a partition and $(f_1, D_1), \dots, (f_n, D_n)$ a sequence of function elements defining $\phi_r = (f_n, D_n)$ as an analytic continuation of (f_0, D_0) along γ_r . Then $\gamma_r([t_j, t_{j+1}]) \subset D_j$ for $j = 0, \dots, n$. For each j , let ϵ_j be the distance from $\gamma_r([t_j, t_{j+1}])$ to the boundary of D_j . If $\|\gamma_s - \gamma_r\| < \epsilon_j$, then $\gamma_s([t_j, t_{j+1}]) \subset D_j$. Thus let $\epsilon = \min\{\epsilon_0, \dots, \epsilon_n\}$, and choose $\delta > 0$ such that $\|\gamma_s - \gamma_r\| < \epsilon$ whenever $|s - r| < \delta$. Then for each s with $|s - r| < \delta$, the same partition and sequence of function elements $(f_1, D_1), \dots, (f_n, D_n)$ also defines

(f_n, D_0) as an analytic continuation of (f_0, D_0) along γ_s . By the uniqueness remark, it implies that ϕ_s is equivalent to ϕ_r whenever $|r - s| < \delta$.

Now we define a function h on $[0, 1]$ by setting $h(s) = 0$ if ϕ_s is equivalent to ϕ_0 and $h(s) = 1$ otherwise. By the previous paragraph h is a continuous function and $h(0) = 0$. Hence $h(s) = 0$ for all s and in particular $h(1) = 0$, which means ϕ_1 is equivalent to ϕ_0 . \square

Corollary 1.71. *Analytic continuations (if they exist) are unique in a simply connected open set.*

Corollary 1.72. *Suppose U is a simply connected open set and (f_0, D_0) is an analytic function element at $z_0 \in U$ with $D_0 \subset U$. If (f_0, D_0) can be analytically continued along every curve in U , then there is a function f which is analytic on U and equal to f_0 on D_0 .*

Proof. If $z \in U$, since any two curves from z_0 to z in U are homotopic, the Monodromy Theorem implies that the terminal function elements of analytic continuations of (f_0, D_0) along them are equivalent in some neighborhood of z . Hence all such analytic continuations determine the same function value at z , and we define $f(z)$ to be this value.

Clearly $f_0(z) = f(z)$ for all $z \in D_0$. It remains to show that $f \in \mathcal{A}(U)$. For $w \in U$, let γ be a curve in U joining z_0 to w , and let (D_n, f_n) be the terminal function element of an analytic continuation of f along γ . If $z \in D_n$, then (D_n, f_n) is also the terminal element of a continuation of (f_0, D_0) along a curve γ_1 from z_0 to z in U , where we extend γ to γ_1 ending at z by joining γ with the line segment from w to z . It follows by definition of f that $f(z) = f_n(z)$ for all $z \in D_n$ (not only for $z = w$). Since f_n is analytic on D_n and w is an arbitrary point in U , we conclude that f is analytic on all of U . \square

1.17. Analytic Covering Maps. Suppose V and W are open subsets of \mathbb{C} .

Definition 1.73. An analytic map $h : V \rightarrow W$ is called an (unramified) *analytic covering map* if for each $w_0 \in W$, there is a neighborhood A of w_0 , contained in W , such that $h^{-1}(A)$ is the disjoint union of a collection $\{B_j\}$ of open subsets of V satisfying that h restricted to B_j is a conformal equivalence onto A for every j .

Example 1.74. The map $\exp : z \rightarrow e^z$ is an analytic covering map from \mathbb{C} to $\mathbb{C} \setminus \{0\}$ (exercise).

Example 1.75. The map $z \mapsto z^n$ is an analytic covering map from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$ (exercise).

Theorem 1.76 (Lifting Through an Analytic Covering Map). *Let $h : V \rightarrow W$ be an analytic covering map and U a simply connected domain in \mathbb{C} . Then any analytic function $f : U \rightarrow W$ can be lifted through h , i.e. there is an analytic map $g : U \rightarrow V$ (not necessarily unique) such that $f = h \circ g$ (draw a diagram).*

Proof. Fix $z_0 \in U$ and let $w_0 = f(z_0)$. Choose a neighborhood A_0 of w_0 such that $h^{-1}(A_0)$ is a disjoint union of open sets, each mapping onto A_0 by h as a conformal equivalence. Denote by B_0 one of these open sets, and let h_0^{-1} be the inverse of $h : B_0 \rightarrow A_0$. Choose an open disk D_0 , centered at z_0 and contained in $f^{-1}(A_0)$. On D_0 we define a function

$$g_0 = h_0^{-1} \circ f.$$

Then g_0 lifts f through h , but only on D_0 . The rest of the proof is to show that the analytic function element (g_0, D_0) can be analytically continued along any path in U beginning at z_0 . If we can do this, then the Monodromy Theorem implies that (g_0, D_0) can be continued to an analytic function g on all of U (since U is simply connected). Moreover, since f and $h \circ g$ agree on D_0 , then they are the same on all of U by the Uniqueness Theorem.

Let $\gamma : [0, 1] \rightarrow U$ be a path beginning at z_0 . Let S be the subset of $I = [0, 1]$ such that (g_0, D_0) can be analytically continued along γ as far as to s for $s \in S$. Clearly if $s \in S$, then $t \in S$ for all $0 \leq t < s$. If $S = [0, 1]$, then we are done. If $S = [0, r]$ for some $r < 1$, then the terminal element is defined on a disk that intersects $(r, 1]$, hence S should be larger, a contradiction. The only remaining case is $S = [0, r)$ with $r < 1$, which cannot happen as we will prove below.

If $S = [0, r)$, choose a neighborhood A of $f(\gamma(r))$ in W such that $h^{-1}(A)$ is a disjoint union of open sets B_j , each mapping onto A conformally under h . Since f is continuous, we can choose an open disk $D \subset U$ such that $f(D) \subset A$ and a point $s \in [0, r)$ such that $w = f(\gamma(s)) \in f(D)$ (draw a picture). Since $s \in S$, the function element (g_0, D_0) can be analytically continued along γ to some (g_n, D_n) with $\gamma(s) \in D_n$. Since $f = h \circ g_0$ on D_0 , again by the Uniqueness Theorem we have $f = h \circ g_n$ on D_n . In particular, $w = f(\gamma(s)) = h(g_n(\gamma(s)))$, hence $g_n(\gamma(s)) \in h^{-1}(A)$, say, $g_n(\gamma(s)) \in B_k$. Now define a new function element (g_{n+1}, D_{n+1}) by setting $D_{n+1} = D$ and $g_{n+1} = f \circ h_k^{-1}$, where $h_k : B_k \rightarrow A$. Then $f = h \circ g_{n+1}$ on D_{n+1} . Note that $g_n(D_n \cap D_{n+1}) \subset B_k$, for otherwise the connected open set $D_n \cap D_{n+1}$ would be separated by $g_n^{-1}(B_k)$ and the union of the sets $g_n^{-1}(B_j)$ for $j \neq k$. It follows that the two inverse functions of h used in the definitions of g_n and g_{n+1} agree on $f(D_n \cap D_{n+1})$, which implies that $g_n = g_{n+1}$ on $D_n \cap D_{n+1}$. Since $\gamma(r) \in D_{n+1}$, it means that (g_0, D_0) can be analytically continued to $\gamma(r)$, hence $r \in S$, contradicting that $S = [0, r)$. \square

1.18. The Picard Theorems. We will apply the previously developed techniques to prove the following two amazing theorems.

Theorem 1.77 (Little Picard Theorem). *A non-constant entire function takes on every value in \mathbb{C} except possibly one.*

Theorem 1.78 (Great Picard Theorem). *An analytic function with an essential singularity at z_0 takes on every complex value, except possibly one, infinitely often in every neighborhood of z_0 .*

The upshot for proving the Picard theorems is to construct an analytic covering map from the unit disk to \mathbb{C} with two points removed. We begin with some facts about reflection through a circle.

Suppose C_1 and C_2 are two circles in the plane that intersect at two points. If the tangents to the two circles are perpendicular at the two points, then we say that the circles *meet at right angles* (draw a picture). Indeed if the tangents are perpendicular at one intersection point, then they are perpendicular at the other point (by symmetry to the line connecting their centers).

Theorem 1.79. *If two circles C_1 and C_2 meet at right angles, and A is the arc of C_1 lying inside C_2 , then reflection through A maps C_2 onto itself (draw a picture).*

Proof. There is a Möbius transformation h that maps C_1 onto $\mathbb{R} \cup \infty$ and maps the inside of C_1 onto the upper-half plane H . We can further choose which point

on C_1 is sent to ∞ , and assume that it is not on \overline{A} . Since Möbius transformations are conformal and take circles to circles (or lines), the image of C_2 under h meets the x -axis at two points with right angles, hence $h(C_2)$ must be a circle such that a diameter of $h(C_2)$ is contained in the x -axis. Then the reflection ρ through the x -axis ($z \mapsto \bar{z}$) maps the inside of $h(C_2)$ onto itself. Now a reflection through the arc A can be described as $h^{-1} \circ \rho \circ h$, and by the uniqueness result, this is the only reflection through A . Since ρ takes the inside of $h(C_2)$ onto itself, reflection through A takes the inside of C_2 onto itself. \square

Theorem 1.80. *With C_1, C_2 as above, the reflection through C_1 takes any other circle C , which meets C_2 at right angles, to another circle which meets C_2 at right angles or a line through the center of C_2 .*

Proof. As discussed before a reflection through a circle is a Möbius transformation followed by conjugation, hence it takes a circle C to either a circle or a line. The previous theorem says that the reflection through C_1 takes the inside of C_2 onto itself, hence the image of C under this reflection still meets C_2 in two points. Since a conformal map preserves angles and conjugation reverses angles, their composition is also angle reversing. It follows that if two curves meet perpendicularly, then the same holds for their image under a reflection. Therefore, reflection through C_1 takes C to a circle or line which also meets C_2 at right angles, and the only way the image of C can be a line is when it passed through the center of C_2 . \square

Now we construct an analytic covering map from the open unit disk D onto $\mathbb{C} \setminus \{0, 1\}$. First note that the three points -1 , $e^{i\pi/3}$, and $e^{-i\pi/3}$ are equidistant points on the unit circle C , hence we can join each pair of these points with an arc of a circle that meets C at right angles. These three arcs bound an open curvilinear triangle V_0 (draw a picture), which is simply connected. By the Riemann Mapping Theorem, there is a conformal equivalence $h : V_0 \rightarrow H$ where H is the upper-half plane. As remarked before, h extends to a continuous map $\overline{V_0} \rightarrow \overline{H} \subset S^2$ which takes the boundary of V_0 to the boundary of H in S^2 . By composing with a suitable Möbius transformation, we may assume that h takes -1 , $e^{i\pi/3}$, and $e^{-i\pi/3}$ to ∞ , 1 , and 0 , respectively.

Since h is real-valued on the three edges of V_0 , by the circle version of the Schwarz Reflection Principle, h has an analytic continuation by reflection across each of these edges. This makes h being defined and analytic in the set V_1 (draw a picture). By the previous two theorems, each of the three new cells is contained in D and bounded by three circles that meet C at right angles. Moreover, as h maps V_0 onto the upper-half plane, it maps each of the new cells of V_1 onto the lower half-plane.

We can now analytically continue h to larger and larger domains $\{V_n\}$ by repeating the above process (draw a picture). One checks that the union of these V_n is D , so we obtain an analytic function h which maps D onto a subset of \mathbb{C} containing the upper and lower open half-planes and the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ on \mathbb{R} . The points ∞ , 1 , and 0 are not in the image of h , because every triangular cell has all of its vertices on the unit circle coming from reflections of the vertices of V_0 that map to ∞ , 1 , and 0 .

Theorem 1.81. *The map $h : D \rightarrow \mathbb{C} \setminus \{0, 1\}$ described above is an analytic covering map.*

Proof. Use the famous “Angels and Demons” illustration (print out the picture and distribute). \square

Theorem 1.82 (Little Picard Theorem). *If f is an entire function and if there are two distinct points in \mathbb{C} that are not in the image of f , then f is constant.*

Proof. Suppose the image of f misses two values, say 0 and 1. Then f is an analytic function from \mathbb{C} to $\mathbb{C} \setminus \{0, 1\}$. Since the map $h : D \rightarrow \{0, 1\}$ constructed above is an analytic covering map and \mathbb{C} is simply connected, we know that f can be lifted through h to an analytic function $g : \mathbb{C} \rightarrow D$ such that $f = h \circ g$. But g becomes a bounded entire function, hence a constant, which implies that f is constant. \square

The proof of the Great Picard Theorem relies on the following technical result.

Theorem 1.83. *Let U be a simply connected subset of \mathbb{C} . Then a sequence of analytic functions on U with values in $\mathbb{C} \setminus \{0, 1\}$ is either a normal family or they converge uniformly to ∞ .*

Proof. See [T, Theorem 7.4.5] for a proof. \square

Theorem 1.84 (Great Picard Theorem). *Let f be a function which is analytic in a neighborhood U of z_0 except at z_0 itself, where it has an essential singularity. Then f takes on every value, except possibly one, infinitely often in every neighborhood of z_0 .*

Proof. Without loss of generality, we may assume that $z_0 = 0$. Prove by contradiction. If the conclusion is false, then there is a disk $D(0, r)$ in which f fails to take on at least two values, say 0 and 1. Then f is an analytic function from $D(0, r) \setminus \{0\}$ to $\mathbb{C} \setminus \{0, 1\}$ with an essential singularity at 0. Define a sequence of functions

$$f_n(z) = f(z/n)$$

for all positive integers n . By the previous theorem, $\{f_n\}$ is a normal family, i.e. it has a subsequence converging uniformly on compact subsets of $D(0, r)$ or they converge uniformly to ∞ . The former implies that f is bounded, hence has a removable singularity at 0. The latter implies that $1/f$ is bounded, and hence has a removable singularity at 0, which further implies that f has at worst a pole at 0. Both cases contradict that f has an essential singularity at 0. \square

1.19. Infinite Products. First, recall that for a half-open half-closed interval I of length 2π , the log function associated to I is

$$\log_I z = \log |z| + i \arg_I z,$$

where $\arg_I z$ is the unique argument of z lying in I for a nonzero complex number z . If $I = (-\pi, \pi]$, we call it the principal branch of the log function and simply denote it by \log . In particular, if u and v both have positive real part, then $\operatorname{im} \log u, \operatorname{im} \log v \in (-\pi/2, \pi/2)$, and $\operatorname{im} \log uv \in (-\pi, \pi)$, hence in this case $\log u + \log v = \log uv$, since they cannot differ by a nonzero multiple of $2\pi i$.

Definition 1.85. If $\{u_k\}$ is a sequence of complex numbers and $p_n = \prod_{k=1}^n u_k$, we say that the infinite product $\prod_{k=1}^{\infty} u_k$ converges to a complex number p if $\lim_{n \rightarrow \infty} p_n = p$.

Theorem 1.86. *The infinite product converges to a non-zero number p if and only if the infinite sums $\sum_{k=1}^{\infty} \log u_k$ converges to a number λ , where $p = e^\lambda$. Moreover, if the infinite series converges absolutely, then the infinite product is unchanged by any rearrangement of the factors.*

Proof. Define p_n as above. First suppose $p = \lim_{n \rightarrow \infty} p_n$ exists and is nonzero. Since the principal branch of \log is continuous at 1, we have

$$\lim_{n \rightarrow \infty} \log(p_n/p) = 0.$$

Then there exists an N such that

$$-\pi/4 < \text{im}(\log(p_n/p)) < \pi/4$$

for $n \geq N$. It follows that $p_n/p_m = (p_n/p)(p_m/p)^{-1}$ is in the right half-plane, i.e. has positive real part for $n, m \geq N$. In particular, $u_{n+1} = p_{n+1}/p_n$ has positive real part for $n \geq N$. Thus we have

$$\log(p_{n+1}/p_N) = \log((p_n/p_N)u_{n+1}) = \log(p_n/p_N) + \log u_{n+1}.$$

Using this equation and an inductive argument beginning with $n = N$, we obtain that

$$\log(p_n/p_N) = \sum_{k=N+1}^n \log u_k$$

for all $n > N$. Since the left side converges as $n \rightarrow \infty$, so does the right side, which implies the convergence of the series.

Conversely if the infinite series converges, let

$$\lambda_n = \sum_{k=1}^n \log u_k$$

be its n th partial sum. Then $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. Since $p_n = e^{\lambda_n}$ and the exponential function is continuous, the sequence $\{p_n\}$ converges to e^λ .

Finally if the series converges absolutely, then each of its rearrangement converges to the same number. It follows that each rearrangement of the infinite product also converges to the same number. \square

Now consider a sequence of functions $\{u_k(z)\}$ defined on a set S . We say that the infinite product of $\{u_k(z)\}$ converges uniformly on S if the sequence $\{p_n(z)\}$ of the partial products converges uniformly on S .

Theorem 1.87. *Let $\{u_k(z)\}$ be a sequence of complex-valued functions defined and bounded uniformly on a set S . If the series*

$$\sum_{k=1}^{\infty} \log u_k(z)$$

converges uniformly to $\lambda(z)$ on S , then the infinite product

$$\prod_{k=1}^{\infty} u_k(z)$$

converges uniformly to $e^{\lambda(z)}$ on S .

Proof. As before, let $\lambda_n(z)$ be the n th partial sum of $\log u_k(z)$, and $p_n(z)$ the n th partial product of $u_k(z)$. By assumption, $\lambda_n(z) - \lambda(z)$ converges uniformly to 0 on S . Since the exponential function is continuous, it implies that

$$\frac{p_n(z)}{p(z)} = e^{\lambda_n(z) - \lambda(z)}$$

converges uniformly to 1.

Since the $p_n(z)$ are bounded uniformly on S , then $\Re(\lambda_n(z)) = \log |p_n(z)|$ are bounded uniformly on S , and hence $\Re(\lambda)$ is bounded on S . It follows that $p(z) = e^{\lambda(z)}$ is bounded on S . Hence $p_n = (p_n/p)p$ converges uniformly to p on S . \square

Theorem 1.88. *Let $\{a_k(z)\}$ be a sequence of complex-valued functions defined on a set S . If the series*

$$\sum_{k=1}^{\infty} |a_k(z)|$$

converges uniformly on S , then the infinite product

$$\prod_{k=1}^{\infty} (1 + a_k(z))$$

converges uniformly on S , and each rearrangement of the product converges to the same function. If the infinite product converges to $p(z)$, then each zero of $p(z)$ is a zero, with the same order, of some finite product of the factors $(1 + a_k(z))$.

Proof. For small w , say $|w| < 1/2$, one checks that

$$\frac{1}{2}|w| \leq |\log(1 + w)| \leq 2|w|.$$

If the series converges uniformly on S , then there is a K such that $|a_k(z)| \leq 1/2$ for all $k \geq K$ and all $z \in S$. It follows that one of the two series

$$\sum_{k=K}^{\infty} |\log(1 + a_k(z))| \quad \text{and} \quad \sum_{k=K}^{\infty} |a_k(z)|$$

converges uniformly on S if and only if the other one does.

Therefore, if $\sum_{k=1}^{\infty} |a_k(z)|$ converges uniformly, then $\sum_{k=K}^{\infty} \log(1 + a_k(z))$ converges uniformly and absolutely. By the previous theorems, this implies the uniform convergence of $\prod_{k=K}^{\infty} (1 + a_k(z))$ to a function on S with no zeros. It follows that $\prod_{k=1}^{\infty} (1 + a_k(z))$ converges uniformly on S , the limit is unaffected by rearrangement of the factors, and each of its zeros is a zero, with the same order, of the product of the factors $1 + a_k(z)$ for $k < K$. \square

Example 1.89. Consider the infinite product

$$\prod_{k=1}^{\infty} (1 + z^2/k^2)$$

on $D(0, R)$. Since $|z^2/k^2| \leq R^2/k^2$ for all $z \in D(0, R)$ and $\sum_{k=1}^{\infty} R^2/k^2$ converges, it follows that the series

$$\sum_{k=1}^{\infty} \frac{|z^2|}{k^2}$$

converges uniformly on $D(0, R)$. Hence by the above theorem, the infinite product also converges uniformly on $D(0, R)$ for each R , and hence on each bounded subset of \mathbb{C} .

Definition 1.90. Let $f \in \mathcal{A}(U)$, not identically zero. Then the meromorphic function f'/f is called the *log derivative* of f on U .

This definition arises formally from $(\log f)' = f'/f$.

Theorem 1.91. Let $\{f_n\}$ be a sequence of analytic functions on a connected open set U . If $\{f_n\}$ converges uniformly to f (not identically zero) on U , then the sequence $\{f'_n/f_n\}$ converges uniformly to f'/f on compact subsets of $U \setminus S$, where S is the set of zeros of f .

Proof. Similar to one of the homework problems. \square

Corollary 1.92. Let $\{u_k\}$ be a sequence of analytic functions on a connected open set U . If the infinite product

$$f(z) = \prod_{k=1}^{\infty} u_k(z)$$

converges uniformly on compact subsets of U to a function f which is not identically zero, then the infinite sum

$$\sum_{k=1}^{\infty} \frac{u'_k(z)}{u_k(z)}$$

converges uniformly to f'/f on compact subsets of $U \setminus S$, where S is the set of zeros of f .

Example 1.93. Consider the function

$$f(z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

where the infinite product converges uniformly on each bounded subset of \mathbb{C} . Its log derivative can be written as

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

Since

$$\frac{2z}{z^2 - k^2} = \frac{1}{z - k} + \frac{1}{z + k},$$

we obtain that

$$\frac{f'(z)}{f(z)} = \lim_{n \rightarrow \infty} \sum_{k=-n}^n \frac{1}{z - k}.$$

Indeed with some more work one can show that

$$f(z) = \sin(\pi z).$$

Given a sequence of points in U such that they do not have any accumulation point in U , can one construct an analytic function on U such that it has exactly the points of this sequence as its zeros, with each zero order equal to the number of times it appears in the sequence? In order to answer this question, we first introduce the following functions.

Denote by $E_0(z) = 1 - z$ and

$$E_p(z) = (1 - z)e^{z+z^2/2+\dots+z^p/p}$$

for $p > 0$. Note that $E_p(1) = 0$ as a simple zero (by checking the derivative nonzero at 1), the only zero of $E_p(z)$ occurs at $z = 1$, and $z + z^2/2 + \dots + z^p/p$ is the p th partial sum of the power series expansion of $-\log(1 - z)$ at $z = 0$.

Theorem 1.94. $E_p(z)$ is an entire function and if $|z| \leq 1$, then $|E_p(z) - 1| \leq |z|^{p+1}$.

Proof. Note that $E_p(0) = 1$ and

$$(1 - E_p(z))' = -E_p'(z) = z^p e^{z+z^2/2+\dots+z^p/p}.$$

Since the derivative has a zero of order p at $z = 0$, the function $1 - E_p(z)$ has a zero of order $p + 1$ at $z = 0$. Note that the expansion of e^z at $z = 0$ and $z + z^2/2 + \dots + z^p/p$ have all coefficients as non-negative real numbers. It follows that the expansion of

$$h(z) = \frac{1 - E_p(z)}{z^{p+1}}$$

at $z = 0$ also has non-negative real numbers as coefficients. It implies that the maximal modulus of $h(z)$ for $|z| \leq 1$ is at $h(1) = 1$. \square

Given $f \in \mathcal{A}(U)$ and a sequence $\{z_k\} \subset U$, we say that $\{z_k\}$ is a list of zeros of f (counting with multiplicity) if each z_k is a zero of f and each zero w of f occurs $\text{ord}_w(f)$ times in this sequence. Now we can define an infinite product called *Weierstrass product* to answer the previous question.

Theorem 1.95. If $\{z_k\}$ is a sequence of non-zero complex numbers and $\{p_k\}$ is a sequence of nonnegative integers such that

$$\sum_{k=1}^{\infty} \left| \frac{z}{z_k} \right|^{p_k+1} < \infty$$

for all z , then the Weierstrass product

$$f(z) = \prod_{k=1}^{\infty} E_{p_k}(z/z_k)$$

converges uniformly on compact subsets of \mathbb{C} to an entire function that has $\{z_k\}$ as a list of zeroes (counting with multiplicity).

Proof. By the previous theorem, we have

$$|E_{p_k}(z/z_k) - 1| \leq \left| \frac{z}{z_k} \right|^{p_k+1}$$

if $|z| \leq |z_k|$. The assumption implies that for given z , $|z| \leq |z_k|$ holds for all sufficiently large k . Now we can apply Theorem 1.88 with $a_k(z) = E_{p_k}(z/z_k) - 1$, since $\sum_{k=1}^{\infty} |a_k(z)|$ converges uniformly on compact subsets of \mathbb{C} . \square

Theorem 1.96 (The Weierstrass Theorem). *If $\{z_k\}$ is a sequence of complex numbers converging to infinity, then there exists an entire function with $\{z_k\}$ as a list of its zeros (counting with multiplicity).*

Proof. Without loss of generality, suppose exactly m terms in the sequence are 0. We may arrange them to be the first m terms. Then $\{z_k\}_{k=m+1}^\infty$ is a sequence of nonzero complex numbers. Given any $R > 0$, since $z_k \rightarrow \infty$, there is $K > m$ such that $|z_k| > 2R$ for all $k \geq K$. Then the series

$$\sum_{k=m+1}^{\infty} \left| \frac{z}{z_k} \right|^k$$

converges uniformly on $|z| \leq R$ by comparing with the geometric series. Hence the hypotheses of the previous theorem hold if we set $p_k = k - 1$ for each k . The resulting Weierstrass product

$$\prod_{k=m+1}^{\infty} E_{p_k}(z/z_k)$$

converges uniformly on compact subsets of \mathbb{C} to an entire function which has $\{z_k\}_{k=m+1}^\infty$ as a list of its zeros. Finally we can set

$$f(z) = z^m \prod_{k=m+1}^{\infty} E_{p_k}(z/z_k)$$

which has $\{z_k\}_{k=1}^\infty$ as a list of its zeros. \square

Example 1.97. Let us construct an entire function which has a zero of order k at each positive integer k . The corresponding sequence of zeros is

$$\{z_k\} = 1, 2, 2, 3, 3, 3, \dots$$

Then we have

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^{p+1}} = \sum_{n=1}^{\infty} \frac{n}{n^{p+1}} = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

which converges say for $p = 2$. Hence the Weierstrass product for the sequence $\{z_k\}$ and $\{p_k = 2\}$ gives

$$\prod_{k=1}^{\infty} E_2(z/z_k) = \prod_{n=1}^{\infty} \left((1 - z/n) e^{z/n + z^2/(2n^2)} \right)^n.$$

By the previous theorem, this infinite product converges to an entire function with the desired zeros.

In the opposite direction, we have the following factorization result.

Theorem 1.98 (Weierstrass Factorization). *Let f be an entire function, not identically zero. Let m be the zero order of f at 0, and let $\{z_k\}$ be a list of the non-zero zeros of f (counting with multiplicity). Then there exist non-negative integers $\{p_k\}$ and an entire function h such that*

$$f(z) = e^{h(z)} z^m \prod_{k=1}^{\infty} E_{p_k}(z/z_k).$$

Proof. Choose $\{p_k\}$ such that the hypotheses of the previous theorem hold (say, taking $p_k = k - 1$ always works). Then the resulting function

$$g(z) = z^m \prod_{k=1}^{\infty} E_{p_k}(z/z_k)$$

converges uniformly on compact sets and has the same zeros as f with the same multiplicities. Therefore, fg^{-1} is an entire function with no zeros, hence $fg^{-1} = e^h$ for some entire function h . \square

Example 1.99. Consider $f(z) = \sin(\pi z)$. It has a simple zero at each integer and no other zeros. Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

we can take $p_k = 1$ for all k . Then the above theorem implies that

$$\sin(\pi z) = e^{h(z)} z \prod_{k \neq 0} E_1(z/k) = e^{h(z)} z \prod_{k \neq 0} (1 - z/k) e^{z/k}.$$

Note that

$$(1 - z/k) e^{z/k} (1 + z/k) e^{-z/k} = 1 - z^2/k^2.$$

Hence we can write

$$\sin(\pi z) = e^{h(z)} z \prod_{k=1}^{\infty} (1 - z^2/k^2).$$

Indeed with some more work one can show that

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} (1 - z^2/k^2).$$

Below we introduce some variations of the Weierstrass Theorem. First we replace \mathbb{C} by an arbitrary non-empty, proper open subset of S^2 .

Theorem 1.100 (General Weierstrass Theorem). *Let U be a non-empty, proper open subset of S^2 . If $\{z_k\}$ is any sequence of points of U with no limit points in U , then there is an analytic function f on U such that f has $\{z_k\}$ as a list of its zeros (counting with multiplicity).*

Proof. Using Möbius transformation, we may assume that $\infty \in U$ (and not equal to any z_k), hence $S^2 \setminus U$ is a compact subset of \mathbb{C} . Since $\{z_k\}$ has no limit point in U , the distance between z_k and $S^2 \setminus U$ must approach 0 as $k \rightarrow \infty$. Hence we can choose a sequence $\{w_k\}$ of points in $S^2 \setminus U$ such that $\lim_{k \rightarrow \infty} |z_k - w_k| = 0$. Define

$$f(z) = \prod_{k=1}^{\infty} E_k \left(\frac{z_k - w_k}{z - w_k} \right).$$

Since $\lim_{k \rightarrow \infty} |z_k - w_k| = 0$, the series

$$\sum_{k=1}^{\infty} \left| \frac{z_k - w_k}{z - w_k} \right|^{k+1}$$

converges uniformly on compact subsets of U , hence f is analytic on U and has $\{z_k\}$ as a list of its zeros. \square

Next we consider poles as well.

Theorem 1.101. *If U is an open subset of \mathbb{C} , then every meromorphic function on U has the form f/g , where f and g are analytic on U and g is not identically zero.*

Proof. Let $h \in \mathcal{M}(U)$ and let $\{z_k\}$ be a sequence of the poles of h , where each z_k appears as many times as its pole order. By the previous theorem, there exists $g \in \mathcal{A}(U)$ such that g has $\{z_k\}$ as a list of zeros. It implies that $f = gh \in \mathcal{A}(U)$, and $h = f/g$ as desired. \square

Theorem 1.102 (Mittag-Leffler Theorem). *Let $R > 0$ or $R = \infty$, let S be a discrete set of points of $D(0, R)$ with no limit points in $D(0, R)$, and let $\{h_w : w \in S\}$ be a set of polynomials with no constant terms. Then there exists a meromorphic function f whose poles are exact at w for each $w \in S$ with principal part given by $h_w((z - w)^{-1})$.*

Proof. Choose an increasing sequence of radii $\{r_n\}$ with $r_n \rightarrow R$. Denote by S_1 the subset of S contained in $\overline{D}(0, r_1)$, and for $n > 1$, let

$$S_n = \{w \in S : r_{n-1} < |w| \leq r_n\}.$$

Then for each n ,

$$g_n(z) = \sum_{w \in S_n} h_w((z - w)^{-1})$$

is a meromorphic function on \mathbb{C} with a pole at w and with the required principal part for each $w \in S_n$. We might hope to take the infinite sum of g_n for all n , but there is no reason why the sum converges on $D(0, R)$. Nevertheless, we can modify each g_n without changing its poles and principal parts such that the series converges.

More precisely, g_n is analytic on an open set containing $\overline{D}(0, r_{n-1})$. Hence g_n is the uniform limit of its power series at 0 on $\overline{D}(0, r_{n-1})$. It follows that there is a polynomial p_n such that

$$|g_n(z) - p_n(z)| < 2^{-n}$$

for all $|z| \leq r_{n-1}$. Set $f_1 = g_1$ and $f_n = g_n - p_n$. Then for each m , the series

$$\sum_{n=m+1}^{\infty} f_n(z)$$

converges uniformly to an analytic function on $D(0, r_m)$. It implies that

$$f(z) = \sum_{n=1}^{\infty} f_n(z)$$

is defined as a meromorphic function on $D(0, r_m)$ and has the required poles and principal parts at the points of S contained in $D(0, r_m)$. Since $\lim_{m \rightarrow \infty} r_m = R$, we thus conclude that $f(z)$ is meromorphic on the entire $D(0, R)$ and has the desired poles and principal parts. \square

2. RIEMANN SURFACES

In this section we carry out complex analysis on Riemann surfaces, introduce fundamental results about Riemann surfaces, and interpret them from a modern viewpoint.

2.1. Preliminaries. We go over some of the previous concepts and results in the context of Riemann surfaces.

Definition 2.1 (Riemann surface). A *Riemann surface* X is a one-dimensional complex manifold, i.e., X is a real surface with a complex atlas of charts

$$\{\phi_i : U_i \rightarrow V_i \subset \mathbb{C}\},$$

where $\{U_i\}$ is an open covering of X , ϕ_i is a homeomorphism, and

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$$

is a conformal equivalence. We say that such a complex atlas of charts is a *complex structure* on the underlying real surface.

Example 2.2. (a) Recall that $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, homeomorphic to the real sphere. Take

$$U_1 = \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C},$$

$$U_2 = \mathbb{P}^1 \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}.$$

Define $\phi_1(z) = z$, $\phi_2(z) = 1/z$ for $z \neq \infty$ and $\phi_2(\infty) = 0$. Then $\phi_2 \circ \phi_1^{-1} : \mathbb{C}^* \rightarrow \mathbb{C}^*$ is given by $z \mapsto 1/z$, which is a conformal equivalence. Therefore, \mathbb{P}^1 is a Riemann surface, called the *Riemann sphere* or the (complex) *projective line*.

(b) Suppose $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over \mathbb{R} . Define

$$\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\},$$

called the lattice spanned by ω_1 and ω_2 . Define an equivalence relation $z_1 \sim z_2$ on \mathbb{C} if $z_1 - z_2 \in \Gamma$. Denote by E the set of equivalence classes, i.e., $E = \mathbb{C}/\Gamma$. Equip E with the quotient topology, i.e., $U \subset E$ is open if and only if $\pi^{-1}(U)$ is open for the projection $\pi : \mathbb{C} \rightarrow E$. Then E is homeomorphic to a torus.

One can enable E a complex structure as follows. Let $V \subset \mathbb{C}$ be an open subset such that no two points in V are equivalent with respect to Γ . Then $U = \pi(V)$ is open in E , and let $\phi : U \rightarrow V$ be the inverse of π , which forms a complex chart on E .

Exercise 2.3. Prove that the above complex charts form a complex structure on E .

Definition 2.4 (Holomorphic function). Let X be a Riemann surface. A function $f : X \rightarrow \mathbb{C}$ is called *holomorphic*, if for every chart $\phi : U \rightarrow V \subset \mathbb{C}$ on X the function

$$f \circ \phi^{-1} : V \rightarrow \mathbb{C}$$

is holomorphic. The set of all holomorphic functions on X is denoted by $\mathcal{O}(X)$ (new notation, previously we used $\mathcal{H}(X)$).

Definition 2.5 (Holomorphic map). Let X and Y be Riemann surfaces. A continuous map $f : X \rightarrow Y$ is called *holomorphic*, if for every chart $\phi : U_1 \rightarrow V_1 \subset \mathbb{C}$ on X and $\psi : U_2 \rightarrow V_2 \subset \mathbb{C}$ on Y with $f(U_1) \subset U_2$ the function

$$\psi \circ f \circ \phi^{-1} : V_1 \rightarrow V_2$$

is holomorphic. If f is biholomorphic, then X and Y are called *isomorphic*.

Definition 2.6 (Meromorphic function). Let X be a Riemann surface and $\Sigma \subset X$ a discrete set of points. Suppose $f : X \setminus \Sigma \rightarrow \mathbb{C}$ is a holomorphic function such that

$$\lim_{x \rightarrow p} |f(x)| = \infty$$

for every $p \in \Sigma$, then f is called *meromorphic*, and the points in Σ are *poles* of f . The set of all meromorphic functions on X is denoted by $\mathcal{M}(X)$.

Remark 2.7. In a neighborhood of a pole p , let z be a suitable coordinate with $z(p) = 0$. Then as we saw before the Laurent series expansion of f is

$$f = \sum_{i=-k}^{\infty} a_i z^i$$

for some $k \in \mathbb{Z}^+$ with $a_{-k} \neq 0$. We say that k is the *pole order* of f at p .

Example 2.8. Let $f = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial on $\mathbb{C} \subset \mathbb{P}^1$ with $n \geq 1$. Then $\lim_{z \rightarrow \infty} |f(z)| = \infty$, hence $f \in \mathcal{M}(\mathbb{P}^1)$.

Theorem 2.9. Suppose X is a Riemann surface and $f \in \mathcal{M}(X)$. For each pole p of f , define $f(p) = \infty$. Then $f : X \rightarrow \mathbb{P}^1$ is a holomorphic map.

Proof. Take the Laurent series expansion

$$f(z) = \sum_{i=-k}^{\infty} a_i z^i = z^{-k} g(z)$$

locally at a pole p with a suitable coordinate $z(p) = 0$, where k is the pole order of p and $g(p) \neq 0$. Then $1/f = z^k(1/g)$ is holomorphic at p . \square

Exercise 2.10. Prove that every holomorphic function on a compact Riemann surface is constant. *Hint: a non-constant holomorphic function is open.*

Definition 2.11 (Covering, branch and ramification). Suppose X and Y are Riemann surfaces and $\pi : Y \rightarrow X$ is a non-constant holomorphic map. Then π is called a (*possibly branched*) *covering*.

A point $y \in Y$ is called a *ramification point* of π , if there is no neighborhood V of y such that $\pi|_V$ is injective. A point $x \in X$ is called a *branch point* of π , if $\pi^{-1}(x)$ contains a ramification point. The map π is called *unramified* if it has no branch point (i.e., no ramification point).

For $p \in Y$ and $q = \pi(p) \in X$, by properties of non-constant holomorphic functions, there exist neighborhoods U of p and V of q as well as suitable coordinates x and y such that $x(p) = y(q) = 0$ and $f|_U(x) = x^k$ for some $k \in \mathbb{Z}^+$. If $k = 1$, then p is not a ramification point of π . If $k > 1$, then p is a ramification point. We say that the *multiplicity* of π at p is k , denoted by $\text{mult}_p(\pi)$, and the *ramification order* of π at p is $k - 1$, denoted by $\text{ord}_p(\pi)$. In particular, $\text{mult}_p(\pi) - 1 = \text{ord}_p(\pi)$.

Example 2.12. (a) Let $\pi : \mathbb{C} \rightarrow \mathbb{C}$ be $\pi(z) = z^k$ for an integer $k \geq 2$. Then 0 is the only ramification point.

(b) Consider the *exponential map* $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ by $z \mapsto e^z$. Then \exp is unramified. In particular, if $V \subset \mathbb{C}$ does not contain two points that differ by an integer multiple of $2\pi i$, then $\exp|_V$ is injective.

(c) Let Γ be a lattice in \mathbb{C} , and $E = \mathbb{C}/\Gamma$ the associated torus. Then the projection $\mathbb{C} \rightarrow E$ is unramified.

Theorem 2.13 (Degree of covering maps). *Let $\pi : Y \rightarrow X$ be a covering map of two connected compact Riemann surfaces. Then for $q \in X$, the fiber cardinality over q , counting with multiplicity, is independent of q . Namely,*

$$d = \sum_{p \in \pi^{-1}(q)} \text{mult}_p(\pi)$$

is constant for all $q \in X$, called the degree of π .

Proof. Since π is non-constant, π is open, $\pi(Y)$ is both open and closed in X , hence π is onto. For $q \in X$, suppose $\pi^{-1}(q) = \{p_1, \dots, p_n\}$ as a set. At each preimage p_i , suppose π is of type $z \mapsto z^{k_i}$ under a suitable local coordinate. Then $d = \sum_{i=1}^n k_i$ for all q' in a neighborhood of q , and hence d is a local constant. Since X is connected, d is a global constant. \square

Exercise 2.14. Suppose X is a compact Riemann surface and f is a non-constant meromorphic function on X . Then f has as many zeros as poles, counting with multiplicity.

2.2. Sheaves. We introduce an important concept “sheaf” which has been used extensively in modern mathematics.

Let X be a topological space. A *sheaf* \mathcal{F} on X associates to each open set U an abelian group $\mathcal{F}(U)$, called the *sections* of \mathcal{F} over U , along with a *restriction map* $r_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ for any open sets $U \subset V$ (for a section $\sigma \in \mathcal{F}(V)$, we often write $\sigma|_U$ to denote $r_{V,U}(\sigma)$), satisfying the following conditions:

- (1) For any open sets $U \subset V \subset W$, $r_{V,U} \circ r_{W,V} = r_{W,U}$;
- (2) For a collection of open sets $\{U_i\}_{i \in I}$ and sections $\alpha_i \in \mathcal{F}(U_i)$, if $\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$ for any $i, j \in I$, then there exists a unique $\alpha \in \mathcal{F}(\cup_i U_i)$ such that $\alpha|_{U_i} = \alpha_i$ for any i .

Remark 2.15. If \mathcal{F} satisfies (1) only, we call it a *presheaf*. One can perform *sheafification* for a presheaf to make it become a sheaf. For many sheaves we consider, the restriction maps are the obvious ones by restricting functions from bigger subsets to smaller ones.

Exercise 2.16. Show that $\mathcal{F}(\emptyset)$ consists of exactly one element.

Example 2.17. Let G be an abelian group. We have the *sheaf of locally constant functions* \mathbb{G} on a topological space X , where $\mathbb{G}(U)$ is the group of locally constant maps $f : U \rightarrow G$ on a non-empty open set $U \subset X$ and $\mathbb{G}(\emptyset) = 0$.

Exercise 2.18. Show that for the sheaf \mathbb{G} of locally constant functions, we have $\mathbb{G}(U) = G$ for any non-empty connected open set U .

Exercise 2.19. Suppose we define $\mathbb{G}(U) = G$ as the set of constant functions on a non-empty open set U with the natural restriction maps. If G contains at least two elements and if X has two disjoint non-empty open subsets, show that \mathbb{G} is a sheaf.

Example 2.20. Let X be a complex manifold and $U \subset X$ an open set.

- (1) *Sheaf \mathcal{O} of holomorphic functions:*

$$\mathcal{O}(U) = \{\text{holomorphic functions on } U\}.$$

The group law is given by addition.

(2) *Sheaf \mathcal{O}^* of nowhere vanishing holomorphic functions:*

$$\mathcal{O}^*(U) = \{\text{holomorphic functions } f \text{ on } U : f(p) \neq 0 \text{ for any } p \in U\}.$$

The group law is given by multiplication.

(3) *Sheaf \mathcal{M} of meromorphic functions:* strictly speaking, a meromorphic function is not a function, even we take ∞ into account. If X is compact, we cannot take a meromorphic function as a quotient of two holomorphic functions, since any globally defined holomorphic function is a constant on X . Instead, we define $f \in \mathcal{M}(U)$ as local quotients of holomorphic functions compatible with each other. Namely, there exists an open covering $\{U_i\}$ of U such that on each U_i , f is given by g_i/h_i for some $g_i, h_i \in \mathcal{O}(U_i)$ satisfying $g_i/h_i = g_j/h_j$, i.e. $g_i h_j = g_j h_i \in \mathcal{O}(U_i \cap U_j)$, hence these local quotients can be glued together over U .

(4) *Sheaf \mathcal{M}^* of meromorphic functions not identically zero:* this is defined similarly as above and the group law is given by multiplication.

2.3. Maps between sheaves. Let \mathcal{E} and \mathcal{F} be two sheaves on a topological space X . A *map* $f : \mathcal{E} \rightarrow \mathcal{F}$ is a collection of group homomorphisms f_U for each open subset U

$$\{f_U : \mathcal{E}(U) \rightarrow \mathcal{F}(U)\}$$

such that they commute with the restriction maps, i.e., for any open sets $U \subset V$ and $\sigma \in \mathcal{E}(V)$ we have

$$f_V(\sigma)|_U = f_U(\sigma|_U).$$

Define the *sheaf of kernel* $\ker(f)$ as

$$\ker(f)(U) = \{\ker(f_U : \mathcal{E}(U) \rightarrow \mathcal{F}(U))\}.$$

Exercise 2.21. Prove that in the above definition $\ker(f)$ is a sheaf.

Example 2.22. Let X be a complex manifold. Define the *exponential map*

$$\exp : \mathcal{O} \rightarrow \mathcal{O}^*$$

by $\exp(h) = e^{2\pi i h}$ for any open set $U \subset X$ and section $h \in \mathcal{O}(U)$. It is easy to see that $\ker(\exp)$ is the locally constant sheaf \mathbb{Z} .

The *sheaf of cokernel* is harder to define. Naively, one would like to define $\text{coker}(f)(U) = \text{coker}(f_U : \mathcal{E}(U) \rightarrow \mathcal{F}(U))$, but this is problematic. For instance, consider the exponential map $\exp : \mathcal{O} \rightarrow \mathcal{O}^*$ on the punctured plane $\mathbb{C} \setminus \{0\}$. The section $z \in \mathcal{O}^*(\mathbb{C} \setminus \{0\})$ is not in the image of f , hence it would define a section in the cokernel. Nevertheless, restricted to any simply connected open subset $U \subset \mathbb{C} \setminus \{0\}$, z lies in the image of f . Now cover $\mathbb{C} \setminus \{0\}$ by simply connected open subsets. By the gluing property of sheaves, z would be zero everywhere, leading to a contradiction.

Instead, we define a section of $\text{coker}(f)(U)$ to be a collection of sections $\sigma_\alpha \in \mathcal{F}(U_\alpha)$ for an open covering $\{U_\alpha\}$ of U such that for all α, β we have

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in f_{U_\alpha \cap U_\beta}(\mathcal{E}(U_\alpha \cap U_\beta)).$$

Here the definition still depends on the choice of an open covering. To get rid of this, we pass to the direct limit. More precisely, we further identify two collections $\{(U_\alpha, \sigma_\alpha)\}$ and $\{(V_\beta, \sigma_\beta)\}$ if for all $p \in U_\alpha \cap V_\beta$, there exists an open set W satisfying $p \in W \subset U_\alpha \cap V_\beta$ such that

$$\sigma_\alpha|_W - \sigma_\beta|_W \in f_W(\mathcal{E}(W)).$$

This identification yields an equivalence relation and correspondingly we define $\text{coker}(f)(U)$ as the group of equivalence classes of the above sections.

Exercise 2.23. Prove that in the above definition $\text{coker}(f)$ is a sheaf.

If $\ker(f)$ (resp. $\text{coker}(f)$) is the zero sheaf, we say that f is *injective* (resp. *surjective*).

Consider the following sequence of maps between sheaves:

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0.$$

We say that it is a *short exact sequence* if $\mathcal{E} = \ker(\beta)$ and $\mathcal{G} = \text{coker}(\alpha)$. In this case we also say that \mathcal{E} is a *subsheaf* of \mathcal{F} and \mathcal{G} is the *quotient sheaf* \mathcal{F}/\mathcal{E} .

Example 2.24. Let X be a complex manifold. We have the exact *exponential sequence*:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0,$$

where i is the natural inclusion and $\exp(f) = e^{2\pi\sqrt{-1}f}$ for $f \in \mathcal{O}(U)$.

Exercise 2.25. Prove that the exponential sequence is exact.

Example 2.26. Let X be a complex manifold and $Y \subset X$ a submanifold. Define the *ideal sheaf* $\mathcal{I}_{Y/X}$ of Y in X (or simply \mathcal{I}_Y if there is no confusion) by

$$\mathcal{I}_Y(U) = \{\text{holomorphic functions in } U \text{ vanishing on } Y \cap U\}.$$

We have the exact sequence:

$$0 \longrightarrow \mathcal{I}_Y \xrightarrow{i} \mathcal{O}_X \xrightarrow{r} i_*\mathcal{O}_Y \longrightarrow 0,$$

where i is the natural inclusion and r is defined by the natural restriction map. Here $i_*\mathcal{O}_Y$ is the extension of \mathcal{O}_Y by zero outside Y , as a sheaf defined on X .

Exercise 2.27. Prove that the above sequence is exact.

2.4. Stalks and germs. Let \mathcal{F} be a sheaf on a topological space X and $p \in X$ a point. Suppose U and V are two open subsets, both containing p , with two sections $\alpha \in \mathcal{F}(U)$ and $\beta \in \mathcal{F}(V)$. Define an equivalence relation $\alpha \sim \beta$, if there exists an open subset W satisfying $p \in W \subset U \cap V$ such that $\alpha|_W = \beta|_W$. Define the *stalk* \mathcal{F}_p as the union of all sections in open neighborhoods of p modulo this equivalence relation. Namely, \mathcal{F}_p is the direct limit

$$\mathcal{F}_p := \varinjlim_{U \ni p} \mathcal{F}(U) = \left(\bigsqcup_{U \ni p} \mathcal{F}(U) \right) / \sim.$$

Note that \mathcal{F}_p is also a group, by adding representatives of two equivalence classes. There is a group homomorphism $r_U : \mathcal{F}(U) \rightarrow \mathcal{F}_p$ mapping a section $\alpha \in \mathcal{F}(U)$ to its equivalence class. The image is called the *germ* of α .

Example 2.28 (Skyscraper sheaf). Let $p \in X$ be a point on a topological space X . Define the *skyscraper sheaf* \mathcal{F} at p by $\mathcal{F}(U) = \{0\}$ for $p \notin U$ and $\mathcal{F}(U) = A$ for $p \in U$, where A is an abelian group. The restriction maps are either the identity map $A \rightarrow A$ or the zero map. For $q \neq p$, the stalk $\mathcal{F}_q = \{0\}$. At p , we have $\mathcal{F}_p = A$. Note that \mathcal{F} can also be obtained by extending the constant sheaf A at p by zero to $X \setminus \{p\}$.

Exercise 2.29. Let X be a Riemann surface and $p \in X$ a point. Let \mathcal{I}_p be the ideal sheaf of p in X parameterizing holomorphic functions vanishing at p . We have the exact sequence

$$0 \longrightarrow \mathcal{I}_p \xrightarrow{i} \mathcal{O}_X \xrightarrow{r} \mathcal{O}_p \longrightarrow 0.$$

Show that the quotient sheaf \mathcal{O}_p is isomorphic to the skyscraper sheaf with stalk \mathbb{C} at p .

It is more convenient to verify injectivity and surjectivity for maps of sheaves by using stalks.

Proposition 2.30. Let $\phi : \mathcal{E} \rightarrow \mathcal{F}$ be a map for sheaves \mathcal{E} and \mathcal{F} on a topological space X .

(1) ϕ is injective if and only if the induced map $\phi_p : \mathcal{E}_p \rightarrow \mathcal{F}_p$ is injective for the stalks at every point p .

(2) ϕ is surjective if and only if the induced map $\phi_p : \mathcal{E}_p \rightarrow \mathcal{F}_p$ is surjective for the stalks at every point p .

(3) ϕ is an isomorphism if and only if the induced map $\phi_p : \mathcal{E}_p \rightarrow \mathcal{F}_p$ is an isomorphism for the stalks at every point p .

Proof. The claim (3) follows from (1) and (2). Let us prove (1) only, and one can find the proof of (2) in many standard textbooks.

Suppose ϕ is injective. Take a section $\sigma \in \mathcal{E}(U)$ on an open subset U . If $\phi([\sigma]) = 0 \in \mathcal{F}_p$, there exists a smaller open subset $V \subset U$ such that $\phi_V(\sigma) = 0 \in \mathcal{F}(V)$, hence $\sigma|_V = 0 \in \mathcal{E}(V)$. Consequently the equivalence class $[\sigma] = 0 \in \mathcal{E}_p$ and we conclude that ϕ_p is injective.

Conversely, suppose ϕ_p is injective for every point p . Take a section $\sigma \in \mathcal{E}(U)$. If $\phi(\sigma) = 0 \in \mathcal{F}(U)$, then for every point $p \in U$, $[\phi(\sigma)] = 0 \in \mathcal{F}_p$. Since ϕ_p is injective, it implies that $[\sigma] = 0 \in \mathcal{E}_p$, i.e. there exists an open subset $U_p \ni p$ such that $\sigma|_{U_p} = 0 \in \mathcal{E}(U_p)$. Applying the gluing property to the open covering $\{U_p\}$ of U , we conclude that $\sigma = 0 \in \mathcal{E}(U)$. \square

Remark 2.31. The image of ϕ does not automatically form a sheaf. In general, it is only a *presheaf*. If the sheafification of $\text{Im}(\phi)$ equals \mathcal{F} , we say that ϕ is surjective. In particular, it does *not* mean $\mathcal{E}(U) \rightarrow \mathcal{F}(U)$ is surjective for every open set U . Sometimes one has to pass to a refined open covering in order to obtain a surjection between sections.

Example 2.32. Consider the exponential map $\exp : \mathcal{O} \rightarrow \mathcal{O}^*$ on the punctured plane $\mathbb{C} \setminus \{0\}$. As a map of sheaves it is surjective, but the section z over $\mathbb{C} \setminus \{0\}$ does not have an inverse. Nevertheless, it does have an inverse over any simply connected open subset.

2.5. Sheaf cohomology. Let \mathcal{F} be a sheaf on a topological space X . Take an open covering $\underline{U} = \{U_\alpha\}$ of X . Define the k -th *cochain group*

$$C^k(\underline{U}, \mathcal{F}) := \prod_{\alpha_0, \dots, \alpha_k} \mathcal{F}(U_{\alpha_0} \cap \dots \cap U_{\alpha_k}).$$

An element σ of $C^k(\underline{U}, \mathcal{F})$ consists of a section $\sigma_{\alpha_0, \dots, \alpha_k} \in \mathcal{F}(U_{\alpha_0} \cap \dots \cap U_{\alpha_k})$ for every $U_{\alpha_0} \cap \dots \cap U_{\alpha_k}$, where $\alpha_0, \dots, \alpha_k$ are ordered.

Define a *coboundary map* $\delta : C^k(\underline{U}, \mathcal{F}) \rightarrow C^{k+1}(\underline{U}, \mathcal{F})$ by

$$(\delta\sigma)_{\alpha_0, \dots, \alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \sigma_{\alpha_0, \dots, \hat{\alpha}_j, \dots, \alpha_{k+1}}|_{U_{\alpha_0} \cap \dots \cap U_{\alpha_{k+1}}}.$$

Example 2.33. Consider $\underline{U} = \{U_1, U_2, U_3\}$ as an open covering of X . Take a cochain element $\sigma \in C^0(\underline{U}, \mathcal{F})$, i.e. σ is a collection of a section $\sigma_i \in \mathcal{F}(U_i)$ for every i . Then we have

$$(\delta\sigma)_{ij} = (\sigma_j - \sigma_i)|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_j).$$

Now take $\tau \in C^1(\underline{U}, \mathcal{F})$, i.e. τ is a collection of a section $\tau_{ij} \in \mathcal{F}(U_i \cap U_j)$ for every pair i, j . Then we have

$$(\delta\tau)_{123} = (\tau_{23} - \tau_{13} + \tau_{12})|_{U_1 \cap U_2 \cap U_3} \in \mathcal{F}(U_1 \cap U_2 \cap U_3).$$

A cochain $\sigma \in C^k(\underline{U}, \mathcal{F})$ is called a *cocycle* if $\delta\sigma = 0$. We say that σ is a *coboundary* if there exists $\tau \in C^{k-1}(\underline{U}, \mathcal{F})$ such that $\delta\tau = \sigma$.

Lemma 2.34. *A coboundary is a cocycle, i.e. $\delta \circ \delta = 0$.*

Proof. Let us prove it for the above example. The same idea can be applied to general cases. Under the above setting, we have

$$\begin{aligned} ((\delta \circ \delta)\sigma)_{123} &= (\delta\sigma)_{23} - (\delta\sigma)_{13} + (\delta\sigma)_{12} \\ &= (\sigma_3 - \sigma_2) - (\sigma_3 - \sigma_1) + (\sigma_2 - \sigma_1) \\ &= 0 \in \mathcal{F}(U_1 \cap U_2 \cap U_3). \end{aligned}$$

Here we omit the restriction notation, since it is obvious. \square

Exercise 2.35. Prove in full generality that $\delta \circ \delta = 0$.

For the coboundary map $\delta_k : C^k(\underline{U}, \mathcal{F}) \rightarrow C^{k+1}(\underline{U}, \mathcal{F})$, define the *k-th cohomology group* (respect to \underline{U}) by

$$H^k(\underline{U}, \mathcal{F}) := \frac{\ker(\delta_k)}{\text{Im}(\delta_{k-1})}.$$

This is well-defined due to the above lemma.

Example 2.36. For $k = 0$, we have $H^0(\underline{U}, \mathcal{F}) = \ker(\delta_0)$. Take an element $\{\sigma_i \in \mathcal{F}(U_i)\}$ in this group. Because it is a cocycle, it satisfies

$$\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_j).$$

By the gluing property of sheaves, there exists a global section $\sigma \in \mathcal{F}(X)$ such that $\sigma|_{U_i} = \sigma_i$. Conversely, if σ is a global section, then define $\sigma_i = \sigma|_{U_i} \in \mathcal{F}(U_i)$. In this way we obtain a cocycle in $C^0(\underline{U}, \mathcal{F})$. From the discussion we see that $H^0(\underline{U}, \mathcal{F}) = \mathcal{F}(X)$, which is independent of the choice of an open covering. Hence $H^0(\underline{U}, \mathcal{F})$ is called the *group of global sections* of \mathcal{F} and we often denote it by $H^0(X, \mathcal{F})$ or simply $H^0(\mathcal{F})$.

In general, we would like to define sheaf cohomology independent of open coverings. Take two open coverings $\underline{U} = \{U_\alpha\}_{\alpha \in I}$ and $\underline{V} = \{V_\beta\}_{\beta \in J}$. We say that \underline{U} is a refinement of \underline{V} if for every U_α there exists a V_β such that $U_\alpha \subset V_\beta$ and we write it as $\underline{U} < \underline{V}$. Then we also have an index map $\phi : I \rightarrow J$ sending α to β . It induces a map

$$\rho_\phi : C^k(\underline{V}, \mathcal{F}) \rightarrow C^k(\underline{U}, \mathcal{F})$$

given by

$$\rho_\phi(\sigma)_{\alpha_0, \dots, \alpha_k} = \sigma_{\phi(\alpha_0), \dots, \phi(\alpha_k)}|_{U_{\alpha_0} \cap \dots \cap U_{\alpha_k}}.$$

One checks that it commutes with the coboundary map δ , i.e. $\delta \circ \rho_\phi = \rho_\phi \circ \delta$. Moreover, it induces a map

$$\rho : H^k(\underline{V}, \mathcal{F}) \rightarrow H^k(\underline{U}, \mathcal{F}),$$

which is independent of the choice of ϕ . Finally, we define the k -th (Čech) cohomology group by passing to the direct limit:

$$H^k(X, \mathcal{F}) := \varinjlim H^k(\underline{U}, \mathcal{F}).$$

The definition involves direct limit, which is inconvenient to use in practice. Nevertheless, we can simplify the situation once the open covering \underline{U} is fine enough. We say that $\underline{U} = \{U_i\}_{i \in I}$ is *acyclic* respect to \mathcal{F} , if for any $k > 0$ and $i_1, \dots, i_l \in I$ we have

$$H^k(U_{i_1} \cap \dots \cap U_{i_l}, \mathcal{F}) = 0.$$

Theorem 2.37 (Leray's Theorem). *If the open covering \underline{U} is acyclic respect to \mathcal{F} , then $H^*(\underline{U}, \mathcal{F}) \cong H^*(X, \mathcal{F})$.*

Remark 2.38. In the context of complex manifolds, if U_i 's are contractible, then \underline{U} is acyclic respect to the sheaves we will consider. While for algebraic varieties, if U_i 's are affine, then \underline{U} is acyclic.

Example 2.39. Let us compute the cohomology of the structure sheaf \mathcal{O} on \mathbb{P}^1 . It is clear that $H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}$, since any global holomorphic function on \mathbb{P}^1 is constant. For higher cohomology, use $[X, Y]$ to denote the homogeneous coordinates of \mathbb{P}^1 . Take the standard open covering $U = \{[X, Y] : X \neq 0\}$ and $V = \{[X, Y] : Y \neq 0\}$. It is acyclic respect to the structure sheaf \mathcal{O} (morally because $U, V \cong \mathbb{C}$ is contractible). Let $s = Y/X$ and $t = X/Y$ as affine coordinates of U and V , respectively. Suppose h is an element in $C^1(\{U, V\}, \mathcal{O})$, i.e. $h \in \mathcal{O}(U \cap V)$. We can write

$$h = \sum_{i=-\infty}^{\infty} a_i s^i.$$

Now take

$$f = - \sum_{i=0}^{\infty} a_i s^i \in \mathcal{O}(U),$$

$$g = \sum_{i=-\infty}^{-1} a_i s^i = \sum_{i=-\infty}^{-1} a_i t^{-i} \in \mathcal{O}(V).$$

Then we have $(f, g) \in C^0(\{U, V\}, \mathcal{O})$ and $\delta((f, g)) = g - f = h$. It implies that $H^1(\mathbb{P}^1, \mathcal{O}) = 0$. All the other $H^k(\mathbb{P}^1, \mathcal{O}) = 0$ for $k > 1$, since there are only two open subsets in the covering.

Example 2.40. Let Ω denote the *sheaf of holomorphic one-forms* on a Riemann surface, i.e. locally a section of Ω can be expressed as $f(z)dz$, where z is local coordinate and $f(z)$ a holomorphic function, satisfying the obvious change of coordinates. Let us compute the cohomology of Ω on \mathbb{P}^1 . Take the above open covering.

Suppose ω is a global holomorphic one-form. Then on the open chart U , it can be written as

$$\left(\sum_{i=0}^{\infty} a_i s^i \right) ds.$$

Using the relation $s = 1/t$ and $ds = -dt/t^2$, on V it can be expressed as

$$-\left(\sum_{i=0}^{\infty} a_i t^{-i-2} \right) dt,$$

which is holomorphic if and only if $a_i = 0$ for all i . Hence w is the zero one-form and $H^0(\mathbb{P}^1, \Omega) = 0$. Now take $\omega \in C^1(\{U, V\}, \Omega)$, i.e. $\omega \in \Omega(U \cap V) = \Omega(\mathbb{C}^*)$, we express it as

$$\omega = \left(\sum_{i=-\infty}^{\infty} a_i t^i \right) dt.$$

Note that any $\alpha \in \Omega(U)$ and $\beta \in \Omega(V)$ can be written as

$$\alpha = \left(\sum_{i=0}^{\infty} b_i s^i \right) ds,$$

$$\beta = \left(\sum_{i=0}^{\infty} c_i t^i \right) dt.$$

Hence on $U \cap V$ we have

$$\delta((\alpha, \beta)) = \beta - \alpha = -\left(\sum_{i=0}^{\infty} b_i t^{-i-2} \right) dt + \left(\sum_{i=0}^{\infty} c_i t^i \right) dt.$$

Note that only the term t^{-1} is missing from the expression. We conclude that $H^1(\mathbb{P}^1, \Omega) = \{a_{-1} t^{-1} dt\} \cong \mathbb{C}$.

Remark 2.41. In general, the rank of $H^1(X, \mathcal{O}) \cong H^0(X, \Omega)$ (by Serre Duality) is called the *genus* of a Riemann surface (or a complex algebraic curve) X .

Exercise 2.42. Let $D = p_1 + \cdots + p_n$ be a collection of n points in \mathbb{P}^1 . We say that D is an *effective divisor of degree n* . Define the sheaf $\mathcal{O}(D)$ on \mathbb{P}^1 by

$$\mathcal{O}(D)(U) = \{f \in \mathcal{M}(U) : f \in \mathcal{O}(U \setminus \{p_1, \dots, p_n\}) \text{ with at worst a simple pole at each } p_i\}.$$

Assume that the standard covering of \mathbb{P}^1 is acyclic respect to $\mathcal{O}(D)$. Use it to calculate the cohomology groups $H^*(\mathbb{P}^1, \mathcal{O}(D))$.

As many other homology/cohomology theories, one can associate a *long exact sequence of cohomology* to a short exact sequence. Suppose we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0.$$

Then α and β induce maps

$$\alpha : C^k(\underline{U}, \mathcal{E}) \rightarrow C^k(\underline{U}, \mathcal{F}), \quad \beta : C^k(\underline{U}, \mathcal{F}) \rightarrow C^k(\underline{U}, \mathcal{G}).$$

Since the coboundary map δ is given by alternating sums of restrictions, α and β commute with δ , hence they send a cocycle to cocycle and a coboundary to coboundary. Consequently they induce maps for cohomology

$$\alpha_* : H^k(X, \mathcal{E}) \rightarrow H^k(X, \mathcal{F}), \quad \beta_* : H^k(X, \mathcal{F}) \rightarrow H^k(X, \mathcal{G}).$$

Next we define the coboundary map

$$\delta_* : H^k(X, \mathcal{G}) \rightarrow H^{k+1}(X, \mathcal{E}).$$

For $\sigma \in C^k(\underline{U}, \mathcal{G})$ satisfying $\delta\sigma = 0$, after refining \underline{U} (still denoted by \underline{U}) such that there exists $\tau \in C^k(\underline{U}, \mathcal{F})$ satisfying $\beta(\tau) = \sigma$, because β is surjective. Then $\beta(\delta\tau) = \delta(\beta(\tau)) = \delta\sigma = 0$, hence after refining further there exists $\mu \in C^{k+1}(\underline{U}, \mathcal{E})$ satisfying $\alpha(\mu) = \delta\tau$. Note that μ is a cocycle. It is because $\alpha(\delta\mu) = \delta(\alpha(\mu)) = \delta\delta(\tau) = 0$ and α is injective, hence $\delta\mu = 0$ and $\mu \in \ker(\delta)$. We thus take $\delta_*\sigma := [\mu] \in H^{k+1}(X, \mathcal{E})$. One checks that this is independent of the choice of τ and μ .

We say that a sequence of maps

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \longrightarrow \cdots$$

is *exact* if $\text{Im}(\alpha_{n-1}) = \ker(\alpha_n)$.

Proposition 2.43. *The long sequence of cohomology associated to a short exact sequence of sheaves is exact.*

Proof. We prove it under an extra assumption that there exists an acyclic open covering \underline{U} such that for any $U = U_{i_1} \cap \cdots \cap U_{i_k}$ we have the short exact sequence:

$$0 \rightarrow \mathcal{E}(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow 0.$$

At least for sheaves considered in this course, this assumption always holds. It further implies the following sequence is exact:

$$0 \rightarrow C^k(\underline{U}, \mathcal{E}) \rightarrow C^k(\underline{U}, \mathcal{F}) \rightarrow C^k(\underline{U}, \mathcal{G}) \rightarrow 0.$$

Let us prove that

$$H^k(\underline{U}, \mathcal{F}) \xrightarrow{\beta_*} H^k(\underline{U}, \mathcal{G}) \xrightarrow{\delta_*} H^{k+1}(\underline{U}, \mathcal{E})$$

is exact. The other cases are easier.

Consider $\tau \in Z^k(\underline{U}, \mathcal{F})$. In the definition of δ_* , take $\sigma = \beta(\tau)$. Then there exists $\mu \in C^k(\underline{U}, \mathcal{E})$ such that $\alpha(\mu) = \delta\tau = 0$. Then we have $\mu = 0$ since α is injective. Consequently $\delta_*\beta_*(\tau) = \delta_*(\sigma) = \mu = 0$, hence $\delta_*\beta_* = 0$ and $\text{Im}(\beta_*) \subset \ker(\delta_*)$.

Conversely, suppose $\delta_*\sigma = 0$ for $\sigma \in Z^k(\underline{U}, \mathcal{G})$. In the definition of δ_* , it implies that $\mu = 0 \in H^{k+1}(\underline{U}, \mathcal{E})$, hence there exists $\gamma \in C^k(\underline{U}, \mathcal{E})$ such that $\delta\gamma = \mu$. Since $\alpha(\mu) = \delta\tau$, we have $\delta\tau = \delta\alpha(\gamma)$ and $\tau - \alpha(\gamma) \in Z^k(\underline{U}, \mathcal{F})$ is a cocycle. Moreover, $\beta(\tau - \alpha(\gamma)) = \beta(\tau) = \sigma$, hence $\beta_*(\tau - \alpha(\gamma)) = \sigma$. We conclude that $\ker(\delta_*) \subset \text{Im}(\beta_*)$. \square

Exercise 2.44. Prove in general the cohomology sequence is exact.

Example 2.45. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_p \xrightarrow{i} \mathcal{O}_{\mathbb{P}^1} \xrightarrow{r} \mathcal{O}_p \longrightarrow 0.$$

Its long exact sequence of cohomology is as follows:

$$0 \rightarrow H^0(\mathcal{I}_p) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}) \rightarrow H^0(\mathcal{O}_p) \rightarrow H^1(\mathcal{I}_p) \rightarrow H^1(\mathcal{O}_{\mathbb{P}^1}) \rightarrow 0.$$

The last term is zero because p is a point so it does not have higher cohomology. We have $H^0(\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$ because any global regular function on \mathbb{P}^1 is constant. Note that $H^0(\mathcal{I}_p) = 0$, because vanishing at p forces such a constant function to be zero. Moreover we have seen that $H^1(\mathcal{O}_{\mathbb{P}^1}) = 0$. Altogether it implies $H^1(\mathcal{I}_p) = 0$, because $H^0(\mathcal{O}_{\mathbb{P}^1}) \rightarrow H^0(\mathcal{O}_p)$ is an isomorphism by evaluating at p .

Exercise 2.46. Let D be an effective divisor of degree n on \mathbb{P}^1 . We have the short exact sequence

$$0 \longrightarrow \mathcal{I}_D \xrightarrow{i} \mathcal{O}_{\mathbb{P}^1} \xrightarrow{r} \mathcal{O}_D \longrightarrow 0.$$

Use the associated long exact sequence to calculate the cohomology $H^*(\mathbb{P}^1, \mathcal{I}(D))$.

2.6. Holomorphic vector bundles. Let k be a positive integer. Consider $\pi : E \rightarrow X$ a holomorphic map between complex manifolds, such that for every $x \in X$ the fiber $E_x = \pi^{-1}(x)$ is isomorphic to \mathbb{C}^k and there exists an open neighborhood U of x along with an isomorphism

$$\phi_U : E_U = \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^k$$

mapping E_x to $\{x\} \times \mathbb{C}^k$ which is a linear isomorphism between vector spaces. Then E is called a *holomorphic vector bundle of rank k* on X and has a *trivialization* $\{(U, \phi_U)\}$. If E is of rank one, we say that E is a *line bundle*.

We give another characterization of vector bundles based on transition functions. Suppose $\underline{U} = \{U_\alpha\}$ is an open covering of X . Given holomorphic functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(\mathbb{C}^k),$$

we can construct a vector bundle E by gluing $U_\alpha \times \mathbb{C}^k$ together. More precisely,

$$E = \sqcup (U_\alpha \times \mathbb{C}^k) / \sim$$

as a complex manifold is defined by identifying (x, v) with $(x, g_{\alpha\beta}(v))$ for $x \in U_\alpha \cap U_\beta$ and $v \in \mathbb{C}^k$ and $E \rightarrow X$ is given by projection to the bases U_α . Call $\{g_{\alpha\beta}\}$ the transition functions of E . They have to satisfy the following compatibility conditions:

$$\begin{aligned} g_{\alpha\beta}(x) \cdot g_{\beta\alpha}(x) &= I, & \text{for all } x \in U_\alpha \cap U_\beta, \\ g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) &= I, & \text{for all } x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}$$

Exercise 2.47. Let E and F be two vector bundles on X of rank k and l , respectively. Define the *direct sum* $E \oplus F$, the *tensor product* $E \otimes F$, the *dual* E^* , and the *wedge product* $\wedge^r E$ for $r \leq k$. Calculate the ranks of these bundles and represent their transition functions in terms of the transition functions of E and F .

A *map* between two vector bundles E and F on X is given by a holomorphic map $f : E \rightarrow F$ such that $f(E_x) \subset F_x$ and $f_x = f|_{E_x} : E_x \rightarrow F_x$ is linear. Note that if $f(E_x)$ has the same rank for every x , then $\ker(f)$ and $\text{Im}(f)$ are naturally subbundles of E and F , respectively. We say that E and F are *isomorphic* if f_x is a linear isomorphism for every x . A vector bundle is called *trivial* if it is isomorphic to $X \times \mathbb{C}^k$.

Exercise 2.48. Give an example of a map between vector bundles $f : E \rightarrow F$ on X such that the image of f is not a vector bundle.

Exercise 2.49. Let L be a line bundle on X . Prove that $L \otimes L^*$ is a trivial line bundle.

Define a *section* σ as a holomorphic map $\sigma : X \rightarrow E$ such that $\sigma(x) \in E_x$ for every $x \in X$, i.e. $\pi \circ \sigma$ is identity. If $\sigma(x) = 0 \in E_x$, we say that σ is vanishing on x .

Exercise 2.50. Let L be a line bundle on X . Prove that L is trivial if and only if it possesses a nowhere vanishing section.

Example 2.51 (Holomorphic tangent bundles). Let X be an n -dimensional complex manifold. Suppose $\phi_U : U \rightarrow \mathbb{C}^n$ are coordinate charts of X . Define the (holomorphic) *tangent bundle* T_X by setting $T_X = \sqcup T_x$ with

$$T_x = \mathbb{C}\left\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right\} \cong \mathbb{C}^n$$

as well as transition functions $g_{UV} = J(\phi_V \phi_U^{-1})$, where J denotes the Jacobian matrix $(\frac{\partial z_i}{\partial w_j})$ for $1 \leq i, j \leq n$. The dual bundle T_X^* is called the *cotangent bundle* of X . The determinant $\det(T_X^*)$ is called the *canonical line bundle* of X .

Remark 2.52. Alternatively, one can define vector bundles on a topological space, a differential manifold and an algebraic variety. The above definitions and properties go through word by word once replacing “holomorphic map” by “homomorphism”, “smooth map” or “regular map”.

2.7. Vector bundles and locally free sheaves. There is a *one-to-one correspondence* between isomorphism classes of vector bundles of rank n and isomorphism classes of locally free sheaves of rank n on a complex variety X . Here we briefly explain the idea. The reader can refer to Hartshorne II 5, especially Ex. 5.18 for more details.

Let \mathcal{O}_X be the structure sheaf of a complex manifold X . Note that $\mathcal{O}_X(U)$ has a *ring* structure (not only a group) for any open set U . A *sheaf of \mathcal{O}_X -modules* is a sheaf \mathcal{F} on X such that for each open set U , the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module. An \mathcal{O}_X -module \mathcal{F} is called *free* if it is isomorphic to a direct sum of \mathcal{O}_X . It is called *locally free* if there is an open covering $\underline{U} = \{U_\alpha\}$ such that for each open subset U_α , $\mathcal{F}|_{U_\alpha}$ is a free $\mathcal{O}_X(U_\alpha)$ -module. The *rank* of \mathcal{F} on U is the number of copies of \mathcal{O} in the summation. If X is connected, the rank of \mathcal{F} does not vary with the open subsets. In particular, a locally free sheaf of rank 1 is also called an *invertible sheaf*.

Roughly speaking, if \mathcal{F} is locally free of rank n , we can choose a set of n generators x_1, \dots, x_n for the $\mathcal{O}_X(U)$ -module $\mathcal{F}(U)$. They span an n -dimensional vector space $\mathbb{C}[x_1, \dots, x_n]$ over U . By changing to a different set of generators over another open subset, one can write down the transition functions, hence it associates to \mathcal{F} a vector bundle structure. Conversely if F is a vector bundle on X , locally we have $F|_U \cong U \times \mathbb{C}^n$ with x_1, \dots, x_n a basis (i.e. n linearly independent sections) of \mathbb{C}^n over U . Then we can associate to $F|_U$ an $\mathcal{O}_X(U)$ -module of rank n using x_1, \dots, x_n as generators.

Example 2.53. Let $X \subset \mathbb{P}^n$ be a submanifold and $Y \subset X$ a hypersurface, i.e. Y is cut out (transversely) by a hypersurface F in \mathbb{P}^n with X . We have the short exact sequence

$$0 \rightarrow \mathcal{I}_{Y/X} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0.$$

The ideal sheaf $\mathcal{I}_{Y/X}$ is an invertible sheaf. Indeed, for an open subset $U \subset X$, $\mathcal{I}_{Y/X}(U)$ can be expressed as $(F|_U) \cdot \mathcal{O}_X(U)$, hence is locally free of rank 1. The sheaf \mathcal{O}_Y (extended to X by zero outside Y) is *not* locally free. For $U \cap Y = \emptyset$, $\mathcal{O}_Y(U) = 0$ and for $U \cap Y \neq \emptyset$, $\mathcal{O}_Y(U)$ is non-zero. Later we will see how to explicitly construct a line bundle corresponding to $\mathcal{I}_{Y/X}$.

2.8. Divisors. Let X be a complex manifold. Suppose $Y \subset X$ is an irreducible subspace of codimension one satisfying that for every $p \in Y$ there exists an open neighborhood $U \subset X$ of p such that $U \cap Y$ is cut out by a holomorphic function

f . Then we say that Y is an *irreducible divisor* of X . We call f a local defining equation for Y at p . A *divisor* D on X is a formal linear combination of irreducible divisors:

$$D = \sum_{i=1}^n a_i Y_i,$$

where $a_i \in \mathbb{Z}$ (or \mathbb{Q}, \mathbb{R} depending on the context). If $a_i \geq 0$ for all i , we say that D is *effective* and denote it by $D \geq 0$. The divisors on X form an additive group $\text{Div}(X)$.

Suppose f is a local defining equation of an irreducible divisor $Y \subset X$ on an open subset $U \subset X$. For another holomorphic function g on X , locally we can write

$$g = f^a \cdot h$$

for some nonnegative integer a such that the holomorphic function h is coprime with f in $\mathcal{O}_X(U)$. We say that a is the *vanishing order* of g along $Y \cap U$. Note that the vanishing order is locally a constant, hence is independent of U . We use

$$\text{ord}_Y(g) = a$$

to denote the vanishing order of g along Y .

For two holomorphic functions g, h on X , we have

$$\text{ord}_Y(gh) = \text{ord}_Y(g) + \text{ord}_Y(h).$$

For a function $f = g/h$, we define

$$\text{ord}_Y(f) = \text{ord}_Y(g) - \text{ord}_Y(h).$$

If $\text{ord}_Y(f) > 0$, we say that f has a *zero* along Y . If $\text{ord}_Y(f) < 0$, we say that f has a *pole* along Y . We also define the *divisor associated to f* by

$$(f) = \sum_Y \text{ord}_Y(f) \cdot Y,$$

as well as the *divisor of zeros*

$$(f)_0 = \sum_Y \text{ord}_Y(g) \cdot Y$$

and the *divisor of poles*

$$(f)_\infty = \sum_Y \text{ord}_Y(h) \cdot Y.$$

They satisfy

$$(f) = (f)_0 - (f)_\infty.$$

If $D = (f)$ is the associated divisor of a meromorphic function f on X , then D is called a *principal divisor*.

Example 2.54. Suppose $D = x_1 + \cdots + x_n - y_1 - \cdots - y_n$ for $x_i, y_j \in \mathbb{C}$, not necessarily distinct. Consider

$$f = \frac{(z - x_1) \cdots (z - x_n)}{(z - y_1) \cdots (z - y_n)}$$

regarded as a meromorphic function on \mathbb{P}^1 . Then D is the associated divisor of f , hence it is a principal divisor. Conversely, any meromorphic function on \mathbb{P}^1 has the same number of zeros as poles, counting with multiplicity. Hence any principal divisor on \mathbb{P}^1 must be of this type (up to a Möbius transformation that sends ∞ to an ordinary point).

Recall that \mathcal{M}^* is the multiplicative sheaf of (not identically zero) meromorphic functions and \mathcal{O}^* the multiplicative sheaf of nowhere vanishing holomorphic functions, which is a subsheaf of \mathcal{M}^* .

Proposition 2.55. $\text{Div}(X) \cong H^0(X, \mathcal{M}^*/\mathcal{O}^*)$.

Proof. Suppose $\{f_\alpha\}$ represents a global section of $\mathcal{M}^*/\mathcal{O}^*$ with respect to an open covering $\underline{U} = \{U_\alpha\}$. Associate to it a divisor $D_\alpha = (f_\alpha)$ in U_α . We claim that $D_\alpha = D_\beta$ in $U_\alpha \cap U_\beta$. This is due to

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta),$$

hence f_α and f_β define the same divisor. Consequently $\{D_\alpha\}$ defines a global divisor. Moreover, if $\{f_\alpha\}$ and $\{g_\alpha\}$ define the same divisor, then $f_\alpha/g_\alpha \in \mathcal{O}^*(U_\alpha)$, hence $\{f_\alpha\}$ and $\{g_\alpha\}$ represent the same section of $\mathcal{M}^*/\mathcal{O}^*$. This shows an injection

$$H^0(X, \mathcal{M}^*/\mathcal{O}^*) \hookrightarrow \text{Div}(X).$$

Conversely, suppose $D = \sum a_i Y_i$ is a divisor on X with $a_i \in \mathbb{Z}$ and Y_i being actual codimension-one in X . We can choose an open covering $\underline{U} = \{U_\alpha\}$ such that Y_i is locally defined by $g_{i\alpha} \in \mathcal{O}(U_\alpha)$. Consider

$$f_\alpha = \prod_i (g_{i\alpha})^{a_i} \in \mathcal{M}^*(U_\alpha).$$

Then we have

$$\frac{f_\alpha}{f_\beta} = \prod_i \left(\frac{g_{i\alpha}}{g_{i\beta}} \right)^{a_i}.$$

Both $g_{i\alpha}$ and $g_{i\beta}$ define the same divisor $Y_i|_{U_\alpha \cap U_\beta}$ in $U_\alpha \cap U_\beta$, hence we see that

$$\frac{g_{i\alpha}}{g_{i\beta}} \in \mathcal{O}^*(U_\alpha \cap U_\beta), \quad \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

Then $\{f_\alpha\}$ defines a global section of $\mathcal{M}^*/\mathcal{O}^*$. Finally if D determines the zero section of $\mathcal{M}^*/\mathcal{O}^*$ (which is 1 since the group structure is multiplicative), it means locally $f_\alpha \in \mathcal{O}^*(U_\alpha)$ (after refining the open covering). Then it does not have zeros or poles, hence $D|_{U_\alpha} = 0$ for each U_α and D is globally zero. This shows the other injection

$$\text{Div}(X) \hookrightarrow H^0(X, \mathcal{M}^*/\mathcal{O}^*).$$

□

2.9. Line bundles. Recall that a line bundle L on X is a vector bundle of rank 1. Equivalently, it is a locally free sheaf of rank 1. Define the *Picard group* $\text{Pic}(X)$ parameterizing isomorphism classes of line bundles on X . The group law is given by tensor product. We can interpret $\text{Pic}(X)$ as a cohomology group.

Proposition 2.56. *There is a one-to-one correspondence between the isomorphism classes of line bundles on X and $H^1(X, \mathcal{O}^*)$, i.e.*

$$\text{Pic}(X) \cong H^1(X, \mathcal{O}^*).$$

Proof. Take an open covering $\underline{U} = \{U_\alpha\}$ of X with respect to the trivialization of a line bundle L . The transition functions $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfy

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1.$$

Therefore, $\{g_{\alpha\beta}\}$ is a cocycle in $C^1(\underline{U}, \mathcal{O}^*)$, hence it represents a cohomology class in $H^1(X, \mathcal{O}^*)$.

Suppose M is another line bundle with transition functions $\{h_{\alpha\beta}\}$. If M and L are isomorphic, then $L \otimes M^*$ is trivial, i.e. $\{g_{\alpha\beta}/h_{\alpha\beta}\}$ are transition functions of $L \otimes M^*$, which has a nowhere vanishing section σ . Suppose on U_α we have $\sigma_\alpha : U_\alpha \rightarrow \mathbb{C}^*$ as the restriction of σ . Then on $U_\alpha \cap U_\beta$ we have

$$\frac{g_{\alpha\beta}}{h_{\alpha\beta}} \cdot \sigma_\alpha = \sigma_\beta.$$

Therefore we conclude that

$$\frac{g_{\alpha\beta}}{h_{\alpha\beta}} = \frac{\sigma_\beta}{\sigma_\alpha} \in \delta C^0(\underline{U}, \mathcal{O}^*).$$

Hence L and M (via transition functions) represent the same class in $H^1(X, \mathcal{O}^*)$.

Reversing the above argument, we thus obtain the desired isomorphism. \square

Now we describe another important correspondence between line bundles and divisors. Suppose D is a divisor in X with local defining equations $\{f_\alpha\}$ such that $f_\alpha \in \mathcal{M}^*(U_\alpha)$. Define

$$g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

Then we have $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Moreover, $\{g_{\alpha\beta}\}$ satisfy the assumptions imposed to transition functions, hence they define a line bundle, denoted by $L = [D]$ or $L = \mathcal{O}_X(D)$. We have a group homomorphism

$$\text{Div}(X) \rightarrow \text{Pic}(X)$$

induced by

$$D + D' \mapsto [D] \otimes [D'].$$

We say that D and D' are *linearly equivalent*, if $[D]$ and $[D']$ are isomorphic line bundles. We denote linear equivalence by

$$D \sim D'.$$

The following result says that the kernel of the above map consists of principal divisors. In other words, two divisors $D \sim D'$ if and only if $D - D'$ is a principal divisor.

Proposition 2.57. *The associated line bundle $[D]$ is trivial if and only if D is a principal divisor, i.e. $D = (f)$ for some $f \in \mathcal{M}^*(X)$.*

Proof. Suppose $D = (f)$ is the associated divisor of a meromorphic function f on X . Then D has local defining equations $\{f_\alpha = f|_{U_\alpha}\}$. The transition functions associated to $[D]$ are all equal to 1, hence $[D]$ is a trivial line bundle. Conversely, suppose $[D]$ is trivial. Then it has a nowhere vanishing section σ whose restriction to U_α is denoted by σ_α . The transition functions $g_{\alpha\beta} = f_\beta/f_\alpha$ defined above satisfy

$$g_{\alpha\beta} \cdot \sigma_\alpha = \sigma_\beta,$$

hence we have

$$\frac{f_\alpha}{\sigma_\alpha} = \frac{f_\beta}{\sigma_\beta} \in \mathcal{M}^*(U_\alpha \cap U_\beta).$$

We can glue $\{f_\alpha/\sigma_\alpha\}$ to form a global function $f \in \mathcal{M}^*(X)$. Since σ is nowhere vanishing, we obtain that $(f) = D$. \square

Let us summarize, using the short exact sequence

$$0 \rightarrow \mathcal{O}^* \rightarrow \mathcal{M}^* \rightarrow \mathcal{M}^*/\mathcal{O}^* \rightarrow 0.$$

Recall that

$$\mathrm{Div}(X) \cong H^0(X, \mathcal{M}^*/\mathcal{O}^*), \quad \mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*).$$

Then we have the long exact sequence

$$0 \rightarrow H^0(X, \mathcal{O}^*) \rightarrow H^0(X, \mathcal{M}^*) \xrightarrow{(\cdot)} \mathrm{Div}(X) \xrightarrow{[\cdot]} \mathrm{Pic}(X) \rightarrow \dots$$

which encodes all the information in the above discussion.

2.10. Sections of a line bundle. Let L be a line bundle on X with transition functions $\{g_{\alpha\beta}\}$. A *holomorphic section* s of L has restriction $s_\alpha \in \mathcal{O}(U_\alpha)$, satisfying

$$g_{\alpha\beta}s_\alpha = s_\beta.$$

Conversely, a collection $\{s_\alpha \in \mathcal{O}(U_\alpha)\}$ such that $s_\beta/s_\alpha = g_{\alpha\beta}$ determines a holomorphic section of L . Similarly, we define a *meromorphic section* s to be a collection

$$\{s_\alpha \in \mathcal{M}(U_\alpha)\}$$

such that $g_{\alpha\beta}s_\alpha = s_\beta$.

Proposition 2.58. *For any nontrivial section s of L , we have $L \cong [(s)]$, where (\cdot) is the associated divisor and $[\cdot]$ is the associated line bundle. A line bundle L is associated to a divisor D if and only if it has a meromorphic section s such that $(s) = D$. In particular, L has a holomorphic section if and only if it is associated to an effective divisor.*

Proof. For a meromorphic section $s \neq 0$, consider the divisor $D_\alpha = (s_\alpha)$ associated to the local section s_α in U_α . Since

$$\frac{s_\beta}{s_\alpha} = g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta),$$

$D_\alpha = (s_\alpha)$ form a *global* divisor $D = (s)$ on X . Conversely, suppose D is an arbitrary divisor with local defining equations given by $\{s_\alpha \in \mathcal{M}^*(U_\alpha)\}$. Then the transition functions of the line bundle $[D]$ are $\{g_{\alpha\beta} = s_\beta/s_\alpha\}$ and consequently the collection $\{s_\alpha\}$ gives rise to a section of $[D]$. It is clear that the divisor (s) is effective if and only if s is a holomorphic section. \square

Now we treat a line bundle as a locally free sheaf of rank 1 and reinterpret the above correspondence. Let D be a divisor on X . Define a sheaf $\mathcal{O}_X(D)$ or simply $\mathcal{O}(D)$ by

$$\mathcal{O}(D)(U) = \{f \in \mathcal{M}(U) : (f) + D|_U \geq 0\}.$$

It has a vector space structure since $(f) + D|_U \geq 0$ and $(g) + D|_U \geq 0$ implies that $(af + bg) + D|_U \geq 0$ for any $a, b \in \mathbb{C}$.

Proposition 2.59. *The space of holomorphic sections of the line bundle $[D]$ can be identified with $H^0(X, \mathcal{O}(D))$.*

Proof. A global section $s \in H^0(X, \mathcal{O}(D))$ is a meromorphic function satisfying

$$(s) + D \geq 0.$$

Suppose D is locally defined by $\{f_\alpha\}$. The line bundle $[D]$ has transition functions

$$\{g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}\}.$$

Then the collection $\{s \cdot f_\alpha\}$ defines a section σ of $[D]$. Since

$$(s) + (f_\alpha) \geq 0$$

for every U_α , σ is a holomorphic section of $[D]$.

Conversely, given a holomorphic section σ of $[D]$, i.e. σ is a collection $\{h_\alpha \in \mathcal{O}(U_\alpha)\}$ such that

$$\frac{h_\beta}{h_\alpha} = g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

Then $\{h_\alpha/f_\alpha\}$ defines a global meromorphic function g . Since $(h_\alpha) \geq 0$ in every U_α , we have

$$(g|_{U_\alpha}) + (f_\alpha) = (h_\alpha) \geq 0,$$

hence $(g) + D \geq 0$ globally on X and $g \in H^0(X, \mathcal{O}(D))$. \square

Remark 2.60. Replacing X by any open subset U , the above proposition implies that sections of $\mathcal{O}(D)(U)$ can be identified with holomorphic sections of the line bundle $[D]$ over U . If $D \sim D'$, i.e. $D' - D = (f)$ for a meromorphic function f on X , then for any $g \in \mathcal{O}(D')(U)$, we have

$$0 \leq (g) + D' = (g) + (f) + (D) = (fg) + (D)$$

restricted to U . So we obtain an isomorphism

$$\mathcal{O}(D')(U) \xrightarrow{\cdot f} \mathcal{O}(D)(U)$$

for any open subset U , compatible with the sheaf restriction maps. In this sense, the sheaf $\mathcal{O}(D)$ and the line bundle $[D]$ have a one-to-one correspondence up to isomorphism and linear equivalence, respectively, assuming that every line bundle can be associated to a divisor (i.e. having a nontrivial meromorphic section).

Let $|D|$ be the set of effective divisors that are linearly equivalent to D . We call $|D|$ the *linear system* associated to D .

Proposition 2.61. *Let X be compact and D a divisor on X . Then we have*

$$\mathbb{P}H^0(X, \mathcal{O}(D)) = |D|,$$

i.e. an effective divisor in $|D|$ and a holomorphic section of $[D]$ (up to scale) determine each other.

Proof. For any $D' \in |D|$, by definition $D' - D = (f)$ is principal for some $f \in \mathcal{M}(X)$, hence $(f) + D = D' \geq 0$ and $f \in H^0(X, \mathcal{O}(D))$. If g is another function such that $D' - D = (g)$, then $(f/g) = 0$, i.e. f/g is holomorphic, hence it is a constant as X is compact.

Conversely, any $f \in H^0(X, \mathcal{O}(D))$ defines an effective divisor $D' = (f) + D$. If $(f) + D = (g) + D$, then $(f/g) = 0$ and f/g is a constant since X is compact. \square

Exercise 2.62. Let $D = \sum_{i=1}^n a_i p_i$ be a divisor on \mathbb{P}^1 with $a_i \in \mathbb{Z}$ and $p_i \in \mathbb{P}^1$. Define the *degree* of D by $\deg(D) = \sum_{i=1}^n a_i$.

- (1) Prove that $D \sim D'$ if and only if $\deg(D) = \deg(D')$.
- (2) Calculate the cohomology $H^*(\mathbb{P}^1, \mathcal{O}(D))$ in terms of $\deg(D)$.

2.11. The Riemann-Roch formula. Let X be a compact and connected Riemann surface (i.e. a smooth complex algebraic curve). Define its *arithmetic genus*

$$g := h^1(\mathcal{O}_X)$$

where $h^i(\mathcal{F})$ denotes the rank of $H^i(\mathcal{F})$ for a sheaf \mathcal{F} .

Theorem 2.63 (Riemann-Roch Formula). *Let D be a divisor on X and $\mathcal{O}(D)$ the associated line bundle (or invertible sheaf). Then we have*

$$h^0(\mathcal{O}(D)) - h^1(\mathcal{O}(D)) = 1 - g + \deg(D).$$

Remark 2.64. Define the (holomorphic) *Euler characteristic* of a sheaf \mathcal{F} by

$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i h^i(\mathcal{F}).$$

Then the Riemann-Roch formula can be written as

$$\chi(\mathcal{O}(D)) - \chi(\mathcal{O}_X) = \deg(D).$$

Proof. Let us first prove it for effective divisors of degree ≥ 0 . Do induction on n . The formula obviously holds for \mathcal{O}_X . Suppose it is true for $\deg(D) < n$. Consider $D = p + D'$ with D' an effective divisor of degree $n - 1$. We have the short exact sequence

$$0 \rightarrow \mathcal{O}(D') \rightarrow \mathcal{O}(D) \rightarrow \mathbb{C}_p \rightarrow 0,$$

where \mathbb{C}_p is the skyscraper sheaf with one-dimensional stalk supported at p . The exactness can be easily checked. The map $\mathcal{O}(D') \rightarrow \mathcal{O}(D)$ is an inclusion, since

$$(f) + D' \geq 0$$

implies that

$$(f) + D = (f) + D' + p \geq 0.$$

The quotient corresponds to germs of functions f at p such that

$$(f)|_U + D'|_U = -p$$

in arbitrarily small neighborhoods U of p . In other words, if $\text{ord}_p(D') = m \geq 0$, we can write

$$f = z^{-m-1}h(z),$$

where $h \in \mathcal{O}^*(U)$. So the quotient sheaf is given by $\mathbb{C} \cdot \{z^{-m-1}\} \cong \mathbb{C}$ supported at p . Since the associated cohomology sequence is long exact, we have

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D')) + 1 = 1 - g + (n - 1) + 1 = 1 - g + n.$$

In general, write a divisor $D = D_1 - D_2$, where D_1 and D_2 are both effective divisors of degree d_1 and d_2 , respectively, and $d_1 - d_2 = \deg(D)$. By the same token, we have the short exact sequence

$$0 \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(D_1) \rightarrow \mathbb{C}^{d_2} \rightarrow 0.$$

Then we obtain that

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D_1)) - d_2 = 1 - g + d_1 - d_2 = 1 - g + \deg(D).$$

□

Remark 2.65. Assuming the Serre duality

$$H^1(\mathcal{O}(D)) \cong H^0(K \otimes \mathcal{O}(-D)),$$

where K is the canonical line bundle of X , then we can rewrite the Riemann-Roch formula as

$$h^0(L) - h^0(K \otimes L^*) = 1 - g + \deg(L),$$

where L is a line bundle on X . Note that K is a degree $2g - 2$ line bundle (to be discussed later). We conclude that

$$h^0(K) = g, \quad h^1(K) = h^0(\mathcal{O}) = 1.$$

It implies that the space of holomorphic one-forms on a genus g Riemann surface is g -dimensional.

2.12. The Riemann-Hurwitz formula. Recall that a *branched cover* $\pi : X \rightarrow Y$ between two (compact, connected) Riemann surfaces is a (surjective) holomorphic map. For a general point $q \in Y$, $\pi^{-1}(q)$ consists of d distinct points. Call d the *degree* of π . Locally around $p \mapsto q$, if the map is given by

$$x \mapsto y = x^m,$$

where x, y are local coordinates of p, q , respectively, call m the *vanishing order* of π at p and denote it by

$$\text{ord}_p(\pi) = m.$$

If $\text{ord}_p(\pi) > 1$, we say that p is a *ramification point*. If $\pi^{-1}(q)$ contains a ramification point, then q is called a *branch point*. Define the *pullback*

$$\pi^*(q) = \sum_{p \in \pi^{-1}(q)} \text{ord}_p(\pi) \cdot p \in \text{Div}(X).$$

Note that $\pi^*(q)$ is a degree d effective divisor on X .

Theorem 2.66 (Riemann-Hurwitz Formula). *Let $\pi : X \rightarrow Y$ be a branched cover of Riemann surfaces. Then we have*

$$K_X \sim \pi^* K_Y + \sum_{p \in X} (\text{ord}_p(\pi) - 1) \cdot p,$$

where K_X and K_Y are canonical divisor classes of X and Y , respectively.

Proof. Take a one-form ω on Y locally expressed as $f(w)dw$ around a point $q = \pi(p)$. Suppose the covering at p is given by

$$z \mapsto w = z^m,$$

then we have

$$\pi^*(f(w)dw) = mf(z^m)z^{m-1}dz.$$

Namely, the associated divisors satisfy the relation

$$(\pi^*\omega)|_U = (\pi^*(\omega))|_U + (\text{ord}_p(\pi) - 1) \cdot p$$

in a local neighborhood U of p . So globally it implies that

$$(\pi^*\omega) = \pi^*(\omega) + \sum_{p \in X} (\text{ord}_p(\pi) - 1) \cdot p.$$

Since $\pi^*\omega$ is a one-form on X , $(\pi^*\omega)$ is a canonical divisor of X and the claimed formula follows. \square

We can interpret the (numerical) Riemann-Hurwitz formula from a topological viewpoint. Let $\chi(X)$ denote the *topological Euler characteristic* of X . If X is a Riemann surface of genus g , take a triangulation of X and suppose the number of k -dimensional edges is c_k for $k = 0, 1, 2$. Then we have

$$\chi(X) = c_0 - c_1 + c_2 = 2 - 2g.$$

Proposition 2.67. *Let $\pi : X \rightarrow Y$ be a degree d branched cover of two Riemann surfaces. Then we have*

$$\chi(X) = d \cdot \chi(Y) - \sum_{p \in X} (\text{ord}_p(\pi) - 1).$$

Proof. Take a triangulation of Y such that every branch point is a vertex. Pull it back as a triangulation of X . Note that it pulls back a face to d faces, an edge to d edges and a vertex v to $|\pi^{-1}(v)|$ vertices. Note that if

$$\pi^{-1}(v) = \sum_{i=1}^k m_i p_i$$

for distinct points p_i , then $|\pi^{-1}(v)| = k$. In other words, we have

$$|\pi^{-1}(v)| = d - \sum_{p \in \pi^{-1}(v)} (\text{ord}_p(\pi) - 1).$$

Then the claimed formula follows right away. \square

Corollary 2.68 (Numerical Riemann-Hurwitz). *Let $\pi : X \rightarrow Y$ be a degree d branched cover of two compact Riemann surfaces of genus g and h , respectively. Then we have*

$$2g - 2 = d(2h - 2) + \sum_{p \in X} (\text{ord}_p(\pi) - 1).$$

In particular, if $g < h$, such branched covers do not exist.

Corollary 2.69. *The canonical line bundle of a genus g Riemann surface X has degree equal to $2g - 2$.*

Proof. Every Riemann surface X possesses a nontrivial meromorphic function, say by the Riemann-Roch formula. It induces a branched cover $\pi : X \rightarrow \mathbb{P}^1$ of some degree d . By the Riemann-Hurwitz Formula we know

$$\deg(K_X) = d(-2) + \sum_{p \in X} (\text{ord}_p(\pi) - 1),$$

since we have seen that $\deg(K_{\mathbb{P}^1}) = -2$. By the Numerical Riemann-Hurwitz we have

$$2 - 2g = 2d - \sum_{p \in X} (\text{ord}_p(\pi) - 1).$$

Then the claim follows immediately. \square

Exercise 2.70. Let X be a compact Riemann surface of genus g . If X admits a branched cover of degree 2 to \mathbb{P}^1 , we say that X is *hyperelliptic*. Prove that every Riemann surface of $g \leq 2$ is hyperelliptic.

2.13. Genus formula of plane curves. In this section we consider a Riemann surface as a complex (one-dimensional) curve. Suppose $F(Z_0, Z_1, Z_2)$ is a general degree d homogeneous polynomial whose vanishing locus is a complex curve $C \subset \mathbb{P}^2$ in the complex projective plane. Since F is general, C is a (smooth) Riemann surface. In other words, the singularities of C locate at the common zeros of $F = 0$ and $\partial F / \partial Z_i = 0$ for all i , which are empty for a general F . On the other hand, if F is special, its vanishing locus C may be singular. For example, let $F = Z_0^2 - Z_1^2$. Then C is a union of two lines, hence has a (nodal) singularity at the intersection of the two lines. If $F = Z_0 Z_1^2 - Z_2^3$. Then C has a (cuspidal) singularity at $[1, 0, 0]$.

Theorem 2.71. *In the above setting, the genus g of C is given by*

$$g = \frac{(d-1)(d-2)}{2}.$$

Proof. We give two proofs. The first one is more algebraic. Suppose C_1 and C_2 are two plane curves of degree d , defined by F_1 and F_2 . Then F_1/F_2 is a meromorphic function on \mathbb{P}^2 , hence C_1 and C_2 are linearly equivalent. It follows that all degree d plane curves are linearly equivalent. Hence it makes sense to use $\mathcal{O}_{\mathbb{P}^2}(d)$ to denote the line bundle associated to a plane curve of degree d . In particular, $\mathcal{O}(1)$ is the line bundle associated to a line L in \mathbb{P}^2 . The ideal sheaf of L has sections given by holomorphic functions vanishing along L , hence it can be identified with $\mathcal{O}(-1)$, the dual of $\mathcal{O}(1)$. Then we have the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_L \rightarrow 0.$$

Tensor it with $\mathcal{O}_{\mathbb{P}^2}(1-m)$. We obtain that

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-m) \rightarrow \mathcal{O}_{\mathbb{P}^2}(-(m-1)) \rightarrow \mathcal{O}_{\mathbb{P}^2}(1-m)|_L \rightarrow 0.$$

Since $\mathcal{O}_{\mathbb{P}^2}(1-m)|_L$ is the line bundle associated to a degree $1-m$ divisor on $L \cong \mathbb{P}^1$, we conclude that

$$\chi(\mathcal{O}_{\mathbb{P}^2}(-(m-1))) - \chi(\mathcal{O}_{\mathbb{P}^2}(-m)) = \chi(\mathcal{O}_{\mathbb{P}^1}(1-m)) = 2-m,$$

where we apply the Riemann-Roch formula to \mathbb{P}^1 in the last equality. Then we obtain that

$$\chi(\mathcal{O}_{\mathbb{P}^2}) - \chi(\mathcal{O}_{\mathbb{P}^2}(-d)) = \sum_{m=1}^d (2-m) = -\frac{d(d-3)}{2}.$$

Now by the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_C \rightarrow 0,$$

we have

$$1-g = \chi(\mathcal{O}_C) = -\frac{d(d-3)}{2},$$

hence the genus formula follows. Here we implicitly assumed that $h^i(\mathcal{O}_C) = 0$ for $i \geq 2$. In general, for any line bundle L on a Riemann surface X , we have $h^i(X, L) = 0$ for $i \geq 2$.

The other proof is an application of the Riemann-Hurwitz formula. Without loss of generality, suppose $o = [0, 0, 1] \notin C$. Let L be the line $Z_2 = 0$ and project C to L from o , i.e.,

$$[Z_0, Z_1, Z_2] \mapsto [Z_0, Z_1].$$

In affine coordinates $x = Z_1/Z_0$ and $y = Z_2/Z_0$, this map is given by vertical projection

$$(x, y) \mapsto x,$$

i.e., we project C vertically to the x -axis. This yields a degree d branched cover

$$\pi : C \rightarrow L \cong \mathbb{P}^1.$$

A point p is a ramification point of π if and only if there exists a vertical line tangent to C at p , i.e., p is a common zero of F and $\partial F/\partial Z_2$. Since F and $\partial F/\partial Z_2$ have degree d and $d - 1$, respectively, they intersect at $d(d - 1)$ points. By Riemann-Hurwitz, we have

$$2g - 2 = d(-2) + d(d - 1),$$

hence the genus formula follows. In order to make sure that all the ramification points are simple, we can choose a general projection direction such that it is different from those given by the (finitely many) lines with higher tangency order to C . \square

Remark 2.72. In the first proof, indeed we did not use the smoothness of C . So the (arithmetic) genus formula holds for an arbitrary plane curve, even if it is singular. Similarly in the second proof, even if the projection has higher ramification points, a detailed local study plus Riemann-Hurwitz can provide the same formula.

2.14. Base point free and very ample line bundles. Let L be a line bundle on a compact complex manifold X . We say that L has a *base point* at $p \in X$ if p belongs to the vanishing locus of every holomorphic section of L . If the base locus of L is empty, then L is called *base point free*.

For a base point free line bundle L , let $\sigma_0, \dots, \sigma_n$ be a basis of the space $H^0(X, L)$ of holomorphic sections. Locally around a point $p \in X$, consider σ_i as a holomorphic function and associate to p the point

$$[\sigma_0(p), \dots, \sigma_n(p)] \in \mathbb{P}^n.$$

This is well-defined, since if we take a different chart, then we get the same point

$$[g_{\alpha\beta}\sigma_0(p), \dots, g_{\alpha\beta}\sigma_n(p)] \in \mathbb{P}^n$$

where $\{g_{\alpha\beta}\}$ are transition functions of L . Therefore, we obtain a holomorphic map

$$\phi_L : X \rightarrow \mathbb{P}^n.$$

Remark 2.73. We can give a more conceptual and coordinate free description of ϕ_L . Since L is base point free, the space of holomorphic sections σ vanishing at p forms a hyperplane $H_p \subset H^0(X, L) \cong \mathbb{C}^{n+1}$. Then one can define $\phi_L(p) = [H_p] \in (\mathbb{P}^n)^*$ in the dual projective space parameterizing hyperplanes.

Proposition 2.74. *In the above setting, there is a one-to-one correspondence between (the pullback of) hyperplane sections of X and effective divisors in the linear system $|L|$.*

Proof. This is just a reformulation of the one-to-one correspondence

$$|L| = \mathbb{P}H^0(X, L),$$

which we proved before. In other words, an effective divisor in $|L|$ uniquely determines a holomorphic section $\sigma = \sum_{i=0}^n a_i \sigma_i$ up to scaling, which defines a hyperplane in $\mathbb{P}H^0(X, L)$. \square

Example 2.75. If $L = \mathcal{O}$, then $H^0(\mathcal{O}) = \mathbb{C}$, hence $\phi_{\mathcal{O}}$ maps X to a single point.

Example 2.76. Let $X = \mathbb{P}^1$ and $L = \mathcal{O}(2p)$ where $p = [0, 1]$. Then $H^0(\mathbb{P}^1, L)$ is 3-dimensional and we can choose a basis by

$$1, \quad \frac{Y}{X}, \quad \frac{Y^2}{X^2}.$$

Recall that around p the sections of L are given by $f \cdot (X/Y)^2$. Hence we obtain that

$$\phi_L([X, Y]) = [X^2, XY, Y^2],$$

which is a smooth conic in \mathbb{P}^2 . The genus formula for a plane curve of degree two also implies that the image has $g = 0$.

Exercise 2.77. A submanifold $X \subset \mathbb{P}^n$ is called *non-degenerate* if it is not contained in any hyperplane. If X is isomorphic to \mathbb{P}^1 , we call it a smooth *rational curve*. For any complex curve X in \mathbb{P}^n , the intersection number of X with a general hyperplane is called the degree of X .

(1) Show that any non-degenerate smooth rational curve in \mathbb{P}^n has degree $\geq n$.

(2) For $d \geq n \geq 3$, show that there exist non-degenerate smooth degree d rational curves in \mathbb{P}^n .

Example 2.78. Let E be a complex curve of genus one and $L = \mathcal{O}(2p)$ for a point $p \in E$. By Riemann-Roch, $h^0(E, L) = 2$. Moreover, L is base point free. Otherwise if q is a base point, then q has to be p and there exists another effective divisor $p + r \in |2p|$ such that $p + r \sim 2p$. But this implies that $r - p$ is principal and $E \cong \mathbb{P}^1$, leading to a contradiction. Now $\phi_L : E \rightarrow \mathbb{P}^1$ is a branched cover of degree two. Two points s and t lie in the same fiber of ϕ_L if and only if $s + t \sim 2p$.

The above example indicates that ϕ_L is *not* always an embedding. We say that L is *very ample* if ϕ_L is an embedding and that L is *ample* if $L^{\otimes m}$ is very ample for some $m > 0$.

Example 2.79. The line bundle $\mathcal{O}(d)$ is very ample on \mathbb{P}^1 if and only if $d > 0$. The induced map ϕ embeds \mathbb{P}^1 into \mathbb{P}^d as a degree d smooth rational curve, which is called a *rational normal curve*.

Let us give a criterion for base point free and very ample line bundles.

Proposition 2.80. *Let L be a line bundle on a Riemann surface X .*

(1) *L is base point free if and only if*

$$h^0(X, L \otimes \mathcal{O}(-p)) = h^0(X, L) - 1$$

for any $p \in X$.

(2) *L is very ample if and only if L is base point free and for any $p, q \in X$ (not necessarily distinct)*

$$h^0(X, L \otimes \mathcal{O}(-p - q)) = h^0(X, L \otimes \mathcal{O}(-p)) - 1 = h^0(X, L \otimes \mathcal{O}(-q)) - 1.$$

Proof. Treat L as a locally free sheaf of rank one. By the short exact sequence

$$0 \rightarrow L \otimes \mathcal{O}(-p) \rightarrow L \rightarrow \mathbb{C}_p \rightarrow 0,$$

we have

$$h^0(X, L) - 1 \leq h^0(X, L \otimes \mathcal{O}(-p)) \leq h^0(X, L).$$

Then L has a base point at p if and only if all holomorphic sections of L vanish at p , i.e., $H^0(X, L \otimes \mathcal{O}(-p)) = H^0(X, L)$. This proves (1).

For (2), a very ample line bundle is necessarily base point free by definition. If $p \neq q \in X$ have the same image under ϕ_L , it is equivalent to saying that the subspace of sections vanishing at p is the same as the subspace of sections vanishing at q , which is further equivalent to

$$h^0(X, L \otimes \mathcal{O}(-p)) = h^0(X, L \otimes \mathcal{O}(-p - q)) = h^0(X, L \otimes \mathcal{O}(-q)).$$

Moreover, ϕ_L induces an injection restricted to the tangent space $T_p(X)$ if and only if there exists a hyperplane such that it cuts out X locally a simple point at p , namely, if and only if there is a section vanishing at p with multiplicity one, i.e.,

$$h^0(X, L \otimes \mathcal{O}(-2p)) < h^0(X, L \otimes \mathcal{O}(-p)).$$

But we have

$$h^0(X, L \otimes \mathcal{O}(-2p)) \geq h^0(X, L \otimes \mathcal{O}(-p)) - 1.$$

Hence (2) follows from combining the two cases. \square

Remark 2.81. In (2), for $p \neq q$ the condition geometrically means that the sections of L separate any two points. When $p = q$, it says that the sections of L separate tangent vectors at p .

Example 2.82. Let E be an elliptic curve, i.e., a Riemann surface of genus one. Fix a point $p \in E$. Consider the morphism

$$\tau : E \rightarrow \text{Pic}^0(E)$$

by $\tau(q) = [q - p]$, where $\text{Pic}^0(E)$ is the Picard group of isomorphism classes of line bundles of degree zero. For $q_1 \neq q_2$, we have $q_1 - p \not\sim q_2 - p$, hence τ is injective. For any line bundle L of degree zero, by Riemann-Roch $h^0(L \otimes \mathcal{O}(p)) \geq 1$, hence $L \otimes \mathcal{O}(p)$ has a section vanishing at a single point q . It implies that $L = [q - p]$, hence τ is surjective. Therefore, we thus conclude that τ is an isomorphism. This defines a group law on E with respect to p , i.e., $q + r = s$, where $s \in E$ is the unique point satisfying that

$$(q - p) + (r - p) \sim s - p.$$

Now consider the linear system $|3p|$ on E . Since

$$h^0(E, \mathcal{O}(3p)) = 3, \quad h^0(E, \mathcal{O}(2p)) = 2, \quad h^0(E, \mathcal{O}(p)) = 1,$$

$\mathcal{O}(3p)$ is very ample. It induces an embedding of E into \mathbb{P}^2 as a plane cubic curve. A line cuts out a divisor of degree three in E , say, $q + r + s$ (not necessarily distinct) if and only if

$$q + r + s \sim 3p.$$

Note that the tangent line L of E at p is a *flex line*, i.e., the tangency multiplicity $(L \cdot E)_p = 3$. Such p is called a *flex point*.

Exercise 2.83. Show that there are in total nine flex points in a smooth plane cubic curve.

Let $V \subset |L|$ be a linear subspace. We say that V is a *linear series* of L . The linear system $|L|$ is also called a *complete linear series*. The above definitions and properties go through similarly for the induced map ϕ_V .

Exercise 2.84. Write down a linear series of $|\mathcal{O}(3)|$ on \mathbb{P}^1 such that it maps \mathbb{P}^1 into \mathbb{P}^2 as a *singular* plane cubic curve. How many different types of such singular plane cubics can you describe?

2.15. Canonical maps. Let K be the canonical line bundle on a Riemann surface X . If X is \mathbb{P}^1 , $\deg(K) = -2$ and K is not effective. If X is an elliptic curve, then $K \cong \mathcal{O}$ and the induced map ϕ_K is onto a point. From now on, assume that the genus of X satisfies $g \geq 2$. We say that X is *hyperelliptic* if it admits a degree 2 branched cover of \mathbb{P}^1 . Two points $p, q \in X$ are called *conjugate* if they have the same image in \mathbb{P}^1 . A ramification point of the double cover is called a *Weierstrass point* of X , i.e., it is self conjugate. By Riemann-Hurwitz, a genus $g \geq 2$ hyperelliptic curve possesses $2g + 2$ Weierstrass points.

Lemma 2.85. *If X is a hyperelliptic curve of genus ≥ 2 , then X admits a unique double cover of \mathbb{P}^1 .*

Proof. Otherwise suppose $h^0(X, \mathcal{O}(p+q)) = 2$ and $h^0(X, \mathcal{O}(p+r)) = 2$ for $q \neq r$. Let $L = \mathcal{O}_X(p+q+r)$. If $h^0(X, L) = 3$, since $h^0(X, L(-x-y)) \leq 1$ and $h^0(X, L(-x)) \leq 2$ by degree reason, ϕ_L would map X as a plane cubic of genus one, leading to a contradiction. Then we conclude that

$$H^0(X, \mathcal{O}(p+q)) = H^0(X, \mathcal{O}(p+r)) = H^0(X, \mathcal{O}(p+q+r)),$$

which implies that both q, r are base points of $|p+q+r|$ and $h^0(X, \mathcal{O}(p)) = 2$, $X \cong \mathbb{P}^1$, leading to a contradiction. \square

Proposition 2.86. *Let X be a curve of genus $g \geq 2$. Then the canonical line bundle K is base point free. The induced map*

$$\phi_K : X \rightarrow \mathbb{P}^{g-1}$$

is an embedding if and only if X is not hyperelliptic. If X is hyperelliptic, ϕ_K is a double cover of a rational normal curve in \mathbb{P}^{g-1} .

Proof. First, let us show that K is base point free. For any point $p \in X$, by Riemann-Roch we have

$$h^0(X, K \otimes \mathcal{O}(-p)) - h^0(X, \mathcal{O}(p)) = 1 - g + (2g - 3),$$

$$h^0(X, K \otimes \mathcal{O}(-p)) = g - 1 = h^0(X, K) - 1.$$

Hence K satisfies the criterion of base point freeness.

Next, K fails to separate p, q (not necessarily distinct) if and only if

$$h^0(X, K \otimes \mathcal{O}(-p-q)) = h^0(X, K \otimes \mathcal{O}(-p)) = g - 1,$$

which is equivalent to, by Riemann-Roch again, that

$$h^0(X, \mathcal{O}(p+q)) = 2.$$

In other words, the linear system $|p+q|$ induces a double cover $X \rightarrow \mathbb{P}^1$.

Finally, if X is hyperelliptic of genus ≥ 2 , it admits a unique double cover of \mathbb{P}^1 . By the above analysis, two points p, q have the same image under the canonical map if and only if $h^0(X, \mathcal{O}(p+q)) = 2$, i.e., p, q are conjugate. Then the canonical map is a double cover of a rational curve of degree $\deg(K)/2 = g - 1$ in \mathbb{P}^{g-1} , i.e., a rational normal curve. A hyperplane section of $\phi_K(X)$ pulls back to X a divisor

$$\sum_{i=1}^{g-1} (p_i + q_i),$$

where p_i, q_i are conjugate or $p_i = q_i$ a Weierstrass point. \square

Remark 2.87. For a non-hyperelliptic curve X , ϕ_K is called the *canonical embedding* of X and its image is called a *canonical curve*.

Example 2.88. Let X be a curve of genus two. Then $h^0(X, K) = 2$, hence X is hyperelliptic and the double cover of \mathbb{P}^1 is induced by the canonical line bundle, as we have seen.

Example 2.89. A non-hyperelliptic curve of genus three admits a canonical embedding to \mathbb{P}^2 as a plane quartic. An effective canonical divisor corresponds to a line section of the quartic. By the genus formula, any smooth plane quartic also has genus equal to 3. Moreover, a smooth plane quartic X gives rise to a line bundle L of degree 4 on X by restricting $\mathcal{O}_{\mathbb{P}^2}(1)$. By Riemann-Roch, $h^0(X, K \otimes L^*) \geq 1$, but $\deg(K \otimes L^*) = 0$, hence $L \cong K$. So any plane quartic is a canonical embedding of a non-hyperelliptic curve of genus three.

2.16. Dimension of linear systems. Let $D = p_1 + \cdots + p_d$ be an effective divisor of degree d on a genus g complex curve X . Recall that the linear system $|D|$ can be identified with $\mathbb{P}H^0(X, \mathcal{O}(D))$ parameterizing effective divisors linearly equivalent to D . Suppose as a projective space

$$r = \dim |D| = h^0(X, \mathcal{O}(D)) - 1.$$

By Riemann-Roch and Serre Duality, we have

$$\dim |K \otimes \mathcal{O}(-D)| = r + g - d - 1.$$

Note that $|K \otimes \mathcal{O}(-D)|$ can be identified with the linear system of effective canonical divisors that contain D . By the canonical map

$$\phi_K : X \rightarrow \mathbb{P}^{g-1},$$

it says that the space of hyperplanes of \mathbb{P}^{g-1} that contain the points $\phi_K(D) = \{\phi_K(p_1), \dots, \phi_K(p_d)\}$ is $(r + g - d - 1)$ -dimensional. In other words, the linear subspace in \mathbb{P}^{g-1} spanned of $\phi_K(p_1), \dots, \phi_K(p_d)$ has dimension

$$(g - 2) - (r + g - d - 1) = (d - 1) - r.$$

Since we expect d points to span a $(d-1)$ -dimensional linear subspace, geometrically it says that $\phi_K(D)$ fails to impose

$$r = \dim |D|$$

independent conditions. We summarize the discussion as the following geometric version of the Riemann-Roch formula.

Theorem 2.90 (Geometric Riemann-Roch). *In the above setting, let $\overline{\phi_K(D)}$ be the linear subspace in \mathbb{P}^{g-1} spanned by the image of D under the canonical map. Then we have*

$$\dim |D| = \deg(D) - 1 - \dim \overline{\phi_K(D)}.$$

Remark 2.91. Even if D contains points with multiplicity, the above formulation still holds. Say, if D contains $2p$, then $2p$ spans the tangent line at p . If D contains $3p$, then $3p$ spans an osculating 2-plane at p etc.

Example 2.92. Let us revisit the canonical embedding of a non-hyperelliptic curve X of genus three in \mathbb{P}^2 . Consider $D = p+q+r$. Then $h^0(\mathcal{O}(D)) \leq 2$, i.e. $\dim |D| = 1$ or 0. Note that $\dim |D| = 1$ if and only if

$$\dim \overline{\phi_K(D)} = 3 - 1 - 1 = 1$$

by Geometric Riemann-Roch, i.e., if and only if p , q and r are collinear in \mathbb{P}^2 .

Let us study in detail the dimension of a linear system.

Lemma 2.93. *Let D be a divisor on a complex curve X . Then $\dim |D| \geq k$ if and only if for every k points $p_1, \dots, p_k \in X$ there exists an effective divisor in $|D|$ containing all of them.*

Proof. First, suppose for every k points $p_1, \dots, p_k \in X$ there exists an effective divisor in $|D|$ containing all of them. Since $\sum_{i=1}^k p_i$ varies in a complex k -dimensional family, then $\dim |D| \geq k$ is obvious. Alternatively, we can prove it by induction. Suppose it holds for $\leq k$. Assume for every p_1, \dots, p_{k+1} , there exists $D' \in |D|$ containing all of them. Then we conclude that $\dim |D - p| \geq k$ for any $p \in X$. Choose a point p not in the base locus of $|D|$. Consequently we have

$$\dim |D| = \dim |D - p| + 1 \geq k + 1.$$

Conversely, suppose $\dim |D| \geq k$. Then we have

$$h^0 \left(X, \mathcal{O} \left(D - \sum_{i=1}^k p_i \right) \right) \geq h^0(X, \mathcal{O}(D)) - k \geq 1.$$

It implies that there exists a meromorphic function f such that

$$(f) + D - \sum_{i=1}^k p_i \geq 0,$$

hence $(f) + D = D'$ is an effective divisor in $|D|$ containing p_1, \dots, p_k . \square

Corollary 2.94. *For any two effective divisors D_1 and D_2 on X , we have*

$$\dim |D_1| + \dim |D_2| \leq \dim |D_1 + D_2|.$$

Proof. Suppose $\dim |D_i| = k_i$ for $i = 1, 2$. Take any $k_1 + k_2$ points

$$p_1, \dots, p_{k_1}, q_1, \dots, q_{k_2}$$

in X . By the above lemma, there exist $D'_1 \in |D_1|$ and $D'_2 \in |D_2|$ such that D'_1 contains all the p_i and D'_2 contains all the q_j . Then $D'_1 + D'_2 \in |D_1 + D_2|$ contains all the p_i, q_j , hence we obtain that

$$\dim |D_1 + D_2| \geq k_1 + k_2$$

by using the lemma again. \square

Note that if $h^0(X, K \otimes (-D)) = 0$, then Riemann-Roch implies that that

$$h^0(X, \mathcal{O}(D)) = 1 - g + \deg(D).$$

Some subtlety may occur if

$$h^0(X, K \otimes (-D)) > 0$$

and we call such a divisor D a *special divisor* and the associated linear system $|D|$ a *special linear system*. By Riemann-Roch, any divisor D with $\deg(D) > 2g - 2$ is

non-special. By Geometric Riemann-Roch, D is non-special if and only if the linear span of $\phi_K(D)$ is the entire space \mathbb{P}^{g-1} .

Theorem 2.95 (Clifford's Theorem). *Let D be an effective divisor such that $\deg(D) \leq 2g - 2$ on a genus g complex curve X . Then we have*

$$\dim |D| \leq \frac{1}{2} \cdot \deg(D).$$

Proof. If D is non-special, we have

$$\dim |D| = \deg(D) - g < \frac{1}{2} \deg(D).$$

If D is special, then there exists an effective divisor D' such that $D + D' \sim K$. By the above lemma we have

$$\dim |D| + \dim |D'| \leq \dim |K| = g - 1.$$

By Riemann-Roch and Serre Duality, we have

$$\dim |D| - \dim |D'| = 1 - g + \deg(D).$$

The desired inequality follows by combining the two relations. \square

Remark 2.96. Indeed, the above equality holds only if $D = 0$, $D = K$ or X is hyperelliptic. If $D = 0$ or $D = K$, one easily checks that the equality holds. If X is hyperelliptic, we can take $D = p + q$, where p, q are conjugate and $\dim |p + q| = 1$. To prove that these are the only possibilities, we need the *uniform position theorem* regarding a general hyperplane section of a non-degenerate space curve, see [GH, p. 249].

Exercise 2.97. Let X be a hyperelliptic curve of genus ≥ 2 . For $0 < 2k \leq g$, find an effective divisor D of degree $2k$ on X such that $\dim |D| = k$. Classify all such divisors up to linear equivalence.

REFERENCES

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