1. Sheaves and cohomology

1.1. Sheaves. Let $X$ be a topological space. A sheaf $\mathcal{F}$ on $X$ associates to each open set $U$ an abelian group $\mathcal{F}(U)$, called the sections of $\mathcal{F}$ over $U$, along with a restriction map $r_{V,U}: \mathcal{F}(V) \to \mathcal{F}(U)$ for any open sets $U \subset V$ (for a section $\sigma \in \mathcal{F}(V)$, we often write $\sigma|_U$ to denote $r_{V,U}(\sigma)$), satisfying the following conditions:

1. For any open sets $U \subset V \subset W$, $r_{V,U} \circ r_{W,V} = r_{W,U}$;

2. For a collection of open sets $\{U_i\}_{i \in I}$ and sections $\alpha_i \in \mathcal{F}(U_i)$, if $\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$ for any $i,j \in I$, then there exists a unique $\alpha \in \mathcal{F}(\bigcup U_i)$ such that $\alpha|_{U_i} = \alpha_i$ for any $i$.

Remark 1.1. If $\mathcal{F}$ satisfies (1) only, we call it a presheaf. One can perform sheafification for a presheaf to make it become a sheaf. For many sheaves we consider, the restriction maps are the obvious ones by restricting functions from bigger subsets to smaller ones.

Exercise 1.2. Show that $\mathcal{F}(\emptyset)$ consists of exactly one element.

Example 1.3. Let $G$ be an abelian group. We have the sheaf of locally constant functions $\mathcal{G}$ on a topological space $X$, where $\mathcal{G}(U)$ is the group of locally constant maps $f: U \to G$ on a non-empty open set $U \subset X$ and $\mathcal{G}(\emptyset) = 0$.

Exercise 1.4. Show that for the sheaf $\mathcal{G}$ of locally constant functions, we have $\mathcal{G}(U) = G$ for any non-empty connected open set $U$.

Exercise 1.5. Suppose we define $\mathcal{G}(U) = G$ as the set of constant functions on a non-empty open set $U$ with the natural restriction maps. If $G$ contains at least two elements and if $X$ consists of two disjoint non-empty open subsets, show that $\mathcal{G}$ is not a sheaf.

Example 1.6. Let $X$ be a complex manifold and $U \subset X$ an open set.

1. Sheaf $\mathcal{O}$ of holomorphic functions:

$$\mathcal{O}(U) = \{\text{holomorphic functions on } U\}.$$ 

The group law is given by addition.

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(2) Sheaf $\mathcal{O}^*$ of nowhere zero holomorphic functions:

$$\mathcal{O}^*(U) = \{\text{holomorphic functions } f \text{ on } U : f(p) \neq 0 \text{ for every } p \in U\}.$$  

The group law is given by multiplication.

(3) Sheaf $\mathcal{M}$ of meromorphic functions: strictly speaking, a meromorphic function is not a function, even we take $\infty$ into account. If $X$ is compact, we cannot take a meromorphic function as a quotient of two holomorphic functions, since any globally defined holomorphic function is a constant on $X$. Instead, we define $f \in \mathcal{M}(U)$ as local quotients of holomorphic functions compatible with each other. Namely, there exists an open covering $\{U_i\}$ of $U$ such that on each $U_i$, $f$ is given by $g_i/h_i$ for some $g_i,h_i \in \mathcal{O}(U_i)$ satisfying $g_i/h_i = g_j/h_j$, i.e. $g_i h_j = g_j h_i \in \mathcal{O}(U_i \cap U_j)$, hence these local quotients can be glued together over $U$.

(4) Sheaf $\mathcal{M}^*$ of meromorphic functions not identically zero: this is defined similarly as above and the group law is given by multiplication.

1.2. Maps between sheaves. Let $\mathcal{E}$ and $\mathcal{F}$ be two sheaves on a topological space $X$. A map $f: \mathcal{E} \to \mathcal{F}$ is a collection of group homomorphisms

$$\{f_U: \mathcal{E}(U) \to \mathcal{F}(U)\}$$

such that they commute with the restriction maps, i.e., for any open sets $U \subset V$ and $\sigma \in \mathcal{E}(V)$ we have

$$f_V(\sigma)|_U = f_U(\sigma|_U).$$

Define the sheaf of kernel $\ker(f)$ as

$$\ker(f)(U) = \{\ker(f_U: \mathcal{E}(U) \to \mathcal{F}(U))\}.$$  

Exercise 1.7. Prove that in the above definition $\ker(f)$ is a sheaf.

Example 1.8. Let $X$ be a complex manifold. Define the exponential map

$$\exp: \mathcal{O} \to \mathcal{O}^*$$

by $\exp(h) = e^{2\pi i h}$ for any open set $U \subset X$ and section $h \in \mathcal{O}(U)$. It is easy to see that $\ker(\exp)$ is the locally constant sheaf $\mathbb{Z}$.

The sheaf of cokernel is harder to define. Naively, one would like to define $\text{coker}(f)(U) = \ker(f_U: \mathcal{E}(U) \to \mathcal{F}(U))$, but this is problematic. For instance, consider the exponential map $\exp: \mathcal{O} \to \mathcal{O}^*$ on the punctured plane $\mathbb{C}\setminus\{0\}$. The section $z \in \mathcal{O}^*(\mathbb{C}\setminus\{0\})$ is not in the image of $f$, hence it would define a section in the cokernel. Nevertheless, restricted to any contractible open set $U \subset \mathbb{C}\setminus\{0\}$, $z$ lies in the image of $f$. Now cover $\mathbb{C}\setminus\{0\}$ by contractible open sets. By the gluing property of sheaves, $z$ would be zero everywhere, leading to a contradiction.

Instead, we define a section of $\text{coker}(f)(U)$ to be a collection of sections $\{\sigma_\alpha \in \mathcal{F}(U_\alpha)\}$ for an open covering $\{U_\alpha\}$ of $U$ such that for all $\alpha,\beta$ we have

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in f_{U_\alpha \cap U_\beta}(\mathcal{E}(U_\alpha \cap U_\beta)).$$

Here the definition still depends on the choice of an open covering. To get rid of this, we pass to the direct limit. More precisely, we further identify two collections $\{(U_\alpha, \sigma_\alpha)\}$ and $\{(V_\beta, \sigma_\beta)\}$ if for all $p \in U_\alpha \cap V_\beta$, there exists an open set $W$ satisfying $p \in W \subset U_\alpha \cap V_\beta$ such that

$$\sigma_\alpha|_W - \sigma_\beta|_W \in f_W(\mathcal{E}(W)).$$

This identification yields an equivalence relation and correspondingly we define $\text{coker}(f)(U)$ as the group of equivalence classes of the above sections.
Exercise 1.9. Prove that in the above definition coker(f) is a sheaf.

If ker(f) (resp. coker(f)) is the zero sheaf, we say that f is injective (resp. surjective).

Consider the following sequence of maps between sheaves:

$$0 \longrightarrow \mathcal{E} \overset{\alpha}{\longrightarrow} \mathcal{F} \overset{\beta}{\longrightarrow} \mathcal{G} \longrightarrow 0.$$ 

We say that it is a short exact sequence if $\mathcal{E} = \ker(\beta)$ and $\mathcal{G} = \text{coker}(\alpha)$. In this case we also say that $\mathcal{E}$ is a subsheaf of $\mathcal{F}$ and $\mathcal{G}$ is the quotient sheaf $\mathcal{F}/\mathcal{E}$.

Example 1.10. Let $X$ be a complex manifold. We have the exact exponential sequence:

$$0 \longrightarrow \mathbb{Z} \overset{i}{\longrightarrow} \mathcal{O} \overset{\exp}{\longrightarrow} \mathcal{O}^* \longrightarrow 0,$$

where $i$ is the inclusion and $\exp(f) = e^{2\pi \sqrt{-1}f}$ for $f \in \mathcal{O}(U)$.

Exercise 1.11. Prove that the exponential sequence is exact.

Example 1.12. Let $X$ be a complex manifold and $Y \subset X$ a submanifold. Define the ideal sheaf $I_{Y/X}$ of $Y$ in $X$ (or simply $I_Y$ if there is no confusion) by

$I_Y(U) =$ \{holomorphic functions in $U$ that are vanishing on $Y \cap U$\}.

We have the exact sequence:

$$0 \longrightarrow I_Y \overset{i}{\longrightarrow} \mathcal{O}_X \overset{r}{\longrightarrow} i_* \mathcal{O}_Y \longrightarrow 0,$$

where $i$ is the inclusion and $r$ is defined by restriction. Here we define

$i_* \mathcal{O}_Y(U) =$ \{holomorphic functions on $Y \cap U$\},

which as a sheaf defined on $X$ is the extension of $\mathcal{O}_Y$ by zero outside $Y$. Hence we may also abuse notation to write it as $\mathcal{O}_Y$.

Exercise 1.13. Prove that the above sequence is exact.

1.3. Stalks and germs. Let $\mathcal{F}$ be a sheaf on a topological space $X$ and $p \in X$ a point. Suppose $U$ and $V$ are two open subsets, both containing $p$, with two sections $\alpha \in \mathcal{F}(U)$ and $\beta \in \mathcal{F}(V)$. Define an equivalence relation $\alpha \sim \beta$, if there exists an open subset $W$ satisfying $p \in W \subset U \cap V$ such that $\alpha|_W = \beta|_W$. Define the stalk $\mathcal{F}_p$ as the union of all sections in open neighborhoods of $p$ modulo this equivalence relation. Namely, $\mathcal{F}_p$ is the direct limit

$$\mathcal{F}_p := \lim_{\longrightarrow \substack{U \ni p \to \mathcal{F}(U)}} \left( \bigcup_{U \ni p} \mathcal{F}(U) \right)/\sim.$$

Note that $\mathcal{F}_p$ is also a group, by adding representatives of two equivalence classes. There is a group homomorphism $r_U : \mathcal{F}(U) \rightarrow \mathcal{F}_p$ mapping a section $\alpha \in \mathcal{F}(U)$ to its equivalence class. The image is called the germ of $\alpha$.

Example 1.14 (Skyscraper sheaf). Let $p \in X$ be a point in a Hausdorff space $X$. Define the skyscraper sheaf $\mathcal{F}$ at $p$ by $\mathcal{F}(U) =$ \{0\} for $p \notin U$ and $\mathcal{F}(U) =$ $A$ for $p \in U$, where $A$ is an abelian group. The restriction maps are either the identity map $A \rightarrow A$ or the zero map. For $q \neq p$, the stalk $\mathcal{F}_q =$ \{0\}. At $p$, we have $\mathcal{F}_p =$ $A$. Note that $\mathcal{F}$ can also be obtained by extending the constant sheaf $A$ at $p$ by zero to $X\setminus\{p\}$.
Exercise 1.15. Let \( X \) be a Riemann surface and \( p \in X \) a point. Let \( \mathcal{I}_p \) be the ideal sheaf of \( p \) in \( X \) parameterizing holomorphic functions vanishing at \( p \). We have the exact sequence

\[
0 \longrightarrow \mathcal{I}_p \overset{i}{\longrightarrow} \mathcal{O}_X \overset{r}{\longrightarrow} \mathcal{O}_p \longrightarrow 0.
\]

Show that the quotient sheaf \( \mathcal{O}_p \) is isomorphic to the skyscraper sheaf with stalk \( \mathbb{C} \) at \( p \).

It is more convenient to verify injectivity and surjectivity for maps of sheaves by using stalks. First note that given a sheaf map \( \phi: \mathcal{E} \to \mathcal{F} \) and a point \( p \in X \), one can define the induced map \( \phi_p: \mathcal{E}_p \to \mathcal{F}_p \) on the stalks of \( p \) by \( \phi_p([\sigma]) = [\phi_U(\sigma)] \) for a neighborhood \( U \) of \( p \) and a section \( \sigma \in \mathcal{E}(U) \). One can check that the definition does not depend on the choice of the representative \( \sigma \) and the neighborhood \( U \).

Proposition 1.16. Let \( \phi: \mathcal{E} \to \mathcal{F} \) be a map for sheaves \( \mathcal{E} \) and \( \mathcal{F} \) on a topological space \( X \).

1. \( \phi \) is injective if and only if the induced map \( \phi_p: \mathcal{E}_p \to \mathcal{F}_p \) is injective for the stalks at every point \( p \).
2. \( \phi \) is surjective if and only if the induced map \( \phi_p: \mathcal{E}_p \to \mathcal{F}_p \) is surjective for the stalks at every point \( p \).
3. \( \phi \) is an isomorphism if and only if the induced map \( \phi_p: \mathcal{E}_p \to \mathcal{F}_p \) is an isomorphism for the stalks at every point \( p \).

Proof. The claim (3) follows from (1) and (2). Let us prove (1) only, and one can find the proof of (2) in standard textbooks.

Suppose \( \phi \) is injective. Take a section \( \sigma \in \mathcal{E}(U) \) in a neighborhood \( U \) of \( p \). If \( \phi_p([\sigma]) = 0 \in \mathcal{F}_p \), i.e. \( [\phi_U(\sigma)] = 0 \), then there exists a smaller open subset \( V \subset U \) such that \( 0 = \phi_U(\sigma)|_V = \phi_V(\sigma|_V) \in \mathcal{F}(V) \), hence \( \sigma|_V = 0 \in \mathcal{E}(V) \). Consequently the equivalence class \([\sigma] = 0 \in \mathcal{E}_p \) and we conclude that \( \phi_p \) is injective.

Conversely, suppose \( \phi_p \) is injective for every point \( p \). Take a section \( \sigma \in \mathcal{E}(U) \) for an open subset \( U \). If \( \phi_U(\sigma) = 0 \in \mathcal{F}(U) \), then for every point \( p \in U \), \([\phi_U(\sigma)] = 0 \in \mathcal{F}_p \). Since \( \phi_p \) is injective, it implies that \([\sigma] = 0 \in \mathcal{E}_p \), i.e. there exists an open subset \( U_p \ni p \) such that \( \sigma|_{U_p} = 0 \in \mathcal{E}(U_p) \). Applying the gluing property to the open covering \( \{U_p\} \) of \( U \), we thus conclude that \( \sigma = 0 \in \mathcal{E}(U) \).

Remark 1.17. The image of \( \phi \) does not automatically form a sheaf. In general, it is only a presheaf. If the sheafification of \( \text{Im}(\phi) \) equals \( \mathcal{F} \), we say that \( \phi \) is surjective. In particular, it does not mean \( \mathcal{E}(U) \to \mathcal{F}(U) \) is surjective for every open set \( U \). Sometimes one has to pass to a refined open covering in order to obtain a surjection between sections.

Example 1.18. Consider the exponential map \( \exp: \mathcal{O} \to \mathcal{O}^* \) on the punctured plane \( \mathbb{C}\setminus\{0\} \). As a map of sheaves it is surjective, but the section \( z \) over \( \mathbb{C}\setminus\{0\} \) does not have an inverse. Nevertheless, it does have an inverse over any contractible open subset.

1.4. Sheaf cohomology. Let \( \mathcal{F} \) be a sheaf on a topological space \( X \). Take an open covering \( \mathcal{U} = \{U_\alpha\} \) of \( X \). Define the \( k \)-th cochain group

\[
C^k(\mathcal{U}, \mathcal{F}) := \prod_{\alpha_0, \ldots, \alpha_k} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}).
\]
An element $\sigma$ of $C^k(U, \mathcal{F})$ consists of a section $\sigma_{\alpha_0, \ldots, \alpha_k} \in \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$ for every $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$, where $\alpha_0, \ldots, \alpha_k$ are ordered and distinct indices.

Define a coboundary map $\delta: C^k(U, \mathcal{F}) \to C^{k+1}(U, \mathcal{F})$ by

$$(\delta \sigma)_{\alpha_0, \ldots, \alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \sigma_{\alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}}|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_{k+1}}}.$$  

**Example 1.19.** Consider $U = \{U_1, U_2, U_3\}$ as an open covering of $X$. Take a cochain element $\sigma \in C^0(U, \mathcal{F})$, i.e. $\sigma$ is a collection of a section $\sigma_i \in \mathcal{F}(U_i)$ for every $i$. Then we have

$$(\delta \sigma)_{ij} = (\sigma_j - \sigma_i)|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_j).$$

Now take $\tau \in C^1(U, \mathcal{F})$, i.e. $\tau$ is a collection of a section $\tau_{ij} \in \mathcal{F}(U_i \cap U_j)$ for every pair $i, j$. Then we have

$$(\delta \tau)_{123} = (\tau_{23} - \tau_{13} + \tau_{12})|_{U_1 \cap U_2 \cap U_3} \in \mathcal{F}(U_1 \cap U_2 \cap U_3).$$

A cochain $\sigma \in C^k(U, \mathcal{F})$ is called a cocycle if $\delta \sigma = 0$. We say that $\sigma$ is a coboundary if there exists $\tau \in C^{k-1}(U, \mathcal{F})$ such that $\delta \tau = \sigma$.

**Lemma 1.20.** A coboundary is a cocycle, i.e. $\delta \circ \delta = 0$.

**Proof.** Let us prove it for the above example. The same idea applies in general with heavier notation. Under the above setting, we have

$$(\delta \circ \delta)_{123} = (\delta \sigma)_{23} - (\delta \sigma)_{13} + (\delta \sigma)_{12} = (\sigma_3 - \sigma_2) - (\sigma_3 - \sigma_1) + (\sigma_2 - \sigma_1) = 0 \in \mathcal{F}(U_1 \cap U_2 \cap U_3).$$

Here we omit the restriction notation, since it is obvious. \hfill \Box

**Exercise 1.21.** Prove in full generality that $\delta \circ \delta = 0$.

For the coboundary map $\delta_k: C^k(U, \mathcal{F}) \to C^{k+1}(U, \mathcal{F})$, define the $k$-th cohomology group (respect to $U$) by

$$H^k(U, \mathcal{F}) := \frac{\ker(\delta_k)}{\operatorname{im}(\delta_{k-1})}.$$  

This is well-defined due to the above lemma.

**Example 1.22.** For $k = 0$, we have $H^0(U, \mathcal{F}) = \ker(\delta_0)$. Take an element $\{\sigma_i \in \mathcal{F}(U_i)\}$ in this group. Because it is a cocycle, it satisfies

$$\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_j).$$

By the gluing property of sheaves, there exists a global section $\sigma \in \mathcal{F}(X)$ such that $\sigma|_{U_i} = \sigma_i$. Conversely, if $\sigma$ is a global section, then define $\sigma_i = \sigma|_{U_i} \in \mathcal{F}(U_i)$. In this way we obtain a cocycle in $C^1(U, \mathcal{F})$. From the discussion we see that $H^0(U, \mathcal{F}) = \mathcal{F}(X)$, which is independent of the choice of an open covering. Hence $H^0(U, \mathcal{F})$ is called the group of global sections of $\mathcal{F}$, and we denote it by $H^0(X, \mathcal{F})$ or simply by $H^0(\mathcal{F})$.

In general, we would like to define cohomology independent of open coverings. Take two open coverings $U = \{U_\alpha\}_{\alpha \in I}$ and $V = \{V_\beta\}_{\beta \in J}$. We say that $U$ is a refinement of $V$ if for every $U_\alpha$ there exists a $V_\beta$ such that $U_\alpha \subset V_\beta$ and we write
it as $U < V$. Then we have an index map $\phi: I \to J$ sending $\alpha$ to $\beta$. It induces a map

$$\rho_\phi: C^k(V, \mathcal{F}) \to C^k(U, \mathcal{F})$$

given by

$$\rho_\phi(\sigma)_{\alpha_0, \ldots, \alpha_k} = \sigma_{\phi(\alpha_0), \ldots, \phi(\alpha_p)}|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}}.$$

One checks that it commutes with the coboundary map $\delta$, i.e. $\delta \circ \rho_\phi = \rho_\phi \circ \delta$. Therefore, it induces a map $\rho: H^k(V, \mathcal{F}) \to H^k(U, \mathcal{F})$ (which is independent of the choice of $\phi$). Finally, we define the $k$-th (Čech) cohomology group by passing to the direct limit:

$$H^k(X, \mathcal{F}) := \lim_{\longrightarrow} H^k(U_i, \mathcal{F}).$$

The definition involves direct limit, which is inconvenient to use in practice. Nevertheless, we can simplify the situation if the open covering $U$ is fine enough. We say that $U = \{U_i\}_{i \in I}$ is acyclic respect to $\mathcal{F}$, if for any $k > 0$ and $i_1, \ldots, i_l \in I$ we have

$$H^k(U_{i_1} \cap \cdots \cap U_{i_l}, \mathcal{F}) = 0.$$

**Theorem 1.23** (Leray’s Theorem). If the open covering $U$ is acyclic respect to $\mathcal{F}$, then $H^*(U, \mathcal{F}) \cong H^*(X, \mathcal{F})$.

**Remark 1.24.** In the context of complex manifolds, if $U_i$’s are contractible, then $U$ is acyclic respect to the sheaves we will consider. While for algebraic varieties, if $U_i$’s are affine, then $U$ is acyclic.

**Example 1.25.** Let us compute the cohomology of the structure sheaf $\mathcal{O}$ on the Riemann sphere $\mathbb{P}^1$. It is clear that $H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}$, since any holomorphic function on $\mathbb{P}^1$ is constant. For higher cohomology, use $[X,Y]$ to denote the homogeneous coordinates of $\mathbb{P}^1$. Take the standard open covering $U = \{[X,Y] : X \neq 0\}$ and $V = \{[X,Y] : Y \neq 0\}$. It is acyclic respect to the structure sheaf $\mathcal{O}$ (morally because $U, V \cong \mathbb{C}$ is contractible). Let $s = Y/X$ and $t = X/Y$ be affine coordinates of $U$ and $V$, respectively. Suppose $h$ is an element in $C^1([U, V], \mathcal{O})$, i.e. $h \in \mathcal{O}(U \cap V)$. We can write

$$h = \sum_{i = -\infty}^{\infty} a_i s^i.$$

Now take

$$f = -\sum_{i=0}^{\infty} a_i s^i \in \mathcal{O}(U),$$

$$g = \sum_{i = -\infty}^{-1} a_i s^i = \sum_{i = -\infty}^{-1} a_i t^{-i} \in \mathcal{O}(V).$$

Then we have $(f, g) \in C^0([U, V], \mathcal{O})$ and $\delta((f, g)) = g - f = h$. It implies that $H^1(\mathbb{P}^1, \mathcal{O}) = 0$. All other $H^k(\mathbb{P}^1, \mathcal{O}) = 0$ for $k > 1$, since there are only two open subsets in the covering.
Example 1.26. Let $\Omega$ denote the sheaf of holomorphic one-forms on a Riemann surface, i.e. locally a section of $\Omega$ can be expressed as $f(z)dz$, where $z$ is local coordinate and $f(z)$ a holomorphic function, consistent with change of coordinates. Let us compute the cohomology of $\Omega$ on $\mathbb{P}^1$. Take the above open covering. Suppose $\omega$ is a global holomorphic one-form. Then in the open subset $U$, it can be written as

$$\left(\sum_{i=0}^{\infty} a_i s^i\right)ds.$$  

Using the relation $s = 1/t$ and $ds = -dt/t^2$, in $V$ it can be expressed as

$$-\left(\sum_{i=0}^{\infty} a_i t^{-i-2}\right)dt,$$

which is holomorphic if and only if $a_i = 0$ for all $i$. Hence $\omega$ is the zero one-form and $H^0(\mathbb{P}^1, \Omega) = 0$. Now take $\omega \in C^1(\{U, V\}, \Omega)$, i.e. $\omega \in \Omega(U \cap V) = \Omega(\mathbb{C}^*)$, we express it as

$$\omega = \left(\sum_{i=-\infty}^{\infty} a_i t^i\right)dt.$$  

Note that any $\alpha \in \Omega(U)$ and $\beta \in \Omega(V)$ can be written as

$$\alpha = \left(\sum_{i=0}^{\infty} b_i s^i\right)ds,$$

$$\beta = \left(\sum_{i=0}^{\infty} c_i t^i\right)dt.$$  

Hence on $U \cap V$ we have

$$\delta((\alpha, \beta)) = \beta - \alpha = -\left(\sum_{i=0}^{\infty} b_i t^{-i-2}\right)dt + \left(\sum_{i=0}^{\infty} c_i t^i\right)dt.$$  

Note that only the term $t^{-1}$ is missing from the expression. We conclude that $H^1(\mathbb{P}^1, \Omega) \cong \mathbb{C}$. 

Remark 1.27. In general, the rank of $H^1(X, \mathcal{O}) \cong H^0(X, \Omega)$ (by Serre Duality) is called the genus of a Riemann surface (or a complex algebraic curve) $X$.

Exercise 1.28. Let $D = p_1 + \cdots + p_n$ be a collection of $n$ points in $\mathbb{P}^1$. We say that $D$ is an effective divisor of degree $n$. Define the sheaf $\mathcal{O}(D)$ on $\mathbb{P}^1$ by

$$\mathcal{O}(D)(U) = \{f \in \mathcal{M}(U) : f \in \mathcal{O}(U \setminus \{p_1, \ldots, p_n\}) \text{ with at worst a simple pole at each } p_i\}.$$  

Assume that the standard covering of $\mathbb{P}^1$ is acyclic respect to $\mathcal{O}(D)$. Use it to calculate the cohomology groups $H^*(\mathbb{P}^1, \mathcal{O}(D))$.

As many other homology/cohomology theories, one can associate a long exact sequence of cohomology to a short exact sequence. Suppose we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0.$$  

Then $\alpha$ and $\beta$ induce maps

$$\alpha: C^k(U, \mathcal{E}) \to C^k(U, \mathcal{F}), \quad \beta: C^k(U, \mathcal{F}) \to C^k(U, \mathcal{G}).$$
Since the coboundary map $\delta$ is given by alternating sums of restrictions, $\alpha$ and $\beta$ commute with $\delta$, and hence they send a cocycle to cocycle and a coboundary to coboundary. Consequently they induce maps for cohomology

$$\alpha_*: H^k(X, \mathcal{F}) \to H^k(X, \mathcal{G}), \quad \beta_*: H^k(X, \mathcal{F}) \to H^k(X, \mathcal{G}).$$

Next we define the coboundary map

$$\delta_*: H^k(X, \mathcal{G}) \to H^{k+1}(X, \mathcal{G}).$$

For $\sigma \in C^k(U, \mathcal{G})$ satisfying $\delta \sigma = 0$, because $\beta$ is surjective, after refining $U$ (still denoted by $U$) there exists $\tau \in C^k(U, \mathcal{F})$ satisfying $\beta(\tau) = \sigma$. Then $\beta(\delta \sigma) = \delta(\beta(\tau)) = \delta \sigma = 0$, hence after refining $U$ further there exists $\mu \in C^{k+1}(U, \mathcal{F})$ satisfying $\alpha(\mu) = \delta \sigma$. Note that $\mu$ is a cocycle. This is because $\alpha(\delta \mu) = \delta(\alpha(\mu)) = \delta \delta(\sigma) = 0$ and $\alpha$ is injective, hence $\delta \mu = 0$ and $\mu \in \ker(\delta)$. We thus take $\delta_* \sigma := [\mu] \in H^{k+1}(X, \mathcal{F})$. One checks that this is independent of the choice of $\tau$ and $\mu$.

We say that a sequence of maps

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \longrightarrow \cdots$$

is exact if $\ker(\alpha_{n-1}) = \text{im}(\alpha_n)$.

**Proposition 1.29.** The long sequence of cohomology associated to a short exact sequence of sheaves is exact.

**Proof.** We prove it under an extra assumption that there exists an acyclic open covering $U$ such that for any $U = U_i \cap \cdots \cap U_k$ we have the short exact sequence:

$$0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \to 0.$$

At least for sheaves considered in this course, this assumption always holds. It further implies the following sequence is exact:

$$0 \to C^k(U, \mathcal{E}) \to C^k(U, \mathcal{F}) \to C^k(U, \mathcal{G}) \to 0.$$

Let us prove that

$$H^k(U, \mathcal{F}) \xrightarrow{\beta_*} H^k(U, \mathcal{G}) \xrightarrow{\delta_*} H^{k+1}(U, \mathcal{E})$$

is exact. The other cases are easier.

Consider $\tau \in Z^k(U, \mathcal{F})$. In the definition of $\delta_*$, take $\sigma = \beta(\tau)$. Then there exists $\mu \in C^{k+1}(U, \mathcal{F})$ such that $\alpha(\mu) = \delta \tau = 0$. Then we have $\mu = 0$ since $\alpha$ is injective. Consequently $\delta_* \beta_* (\tau) = \delta_*(\sigma) = \mu = 0$, hence $\delta_* \beta_* = 0$ and $\text{im}(\beta_* \sigma) \subseteq \ker(\delta_*)$.

Conversely, suppose $\delta_\sigma = 0$ for $\sigma \in Z^k(U, \mathcal{G})$. In the definition of $\delta_*$, it implies that $\mu = 0 \in H^{k+1}(U, \mathcal{E})$, hence there exists $\gamma \in C^k(U, \mathcal{E})$ such that $\delta \gamma = \mu$. Since $\alpha(\mu) = \delta \tau$, we have $\delta \gamma = \delta \alpha(\gamma)$ and $\tau - \alpha(\gamma) \in Z^k(U, \mathcal{F})$ is a cocycle. Moreover, $\beta(\tau - \alpha(\gamma)) = \beta(\tau) = \sigma$, hence $\beta_* (\tau - \alpha(\gamma)) = \sigma$. We conclude that $\ker(\delta_*) \subseteq \text{im}(\beta_*)$.

**Exercise 1.30.** Prove in general the cohomology sequence is exact.

**Example 1.31.** Consider the short exact sequence

$$\cdots \longrightarrow \mathcal{F}_p \xrightarrow{i} \mathcal{O}_{\mathbb{P}^1} \xrightarrow{r} \mathcal{O}_p \longrightarrow 0.$$

Its long exact sequence of cohomology is as follows:

$$0 \to H^0(\mathcal{F}_p) \to H^0(\mathcal{O}_{\mathbb{P}^1}) \to H^0(\mathcal{O}_p) \to H^1(\mathcal{F}_p) \to H^1(\mathcal{O}_{\mathbb{P}^1}) \to 0.$$

The last term is zero because $p$ is a point so it does not have higher cohomology. We have $H^0(\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$ because any holomorphic function on $\mathbb{P}^1$ is constant. Note
that $H^0(I_p) = 0$, because vanishing at $p$ forces such a constant function to be zero. Moreover we have seen that $H^1(O_{P^1}) = 0$. Altogether it implies that $H^1(I_p) = 0$, because $H^0(O_{P^1}) \to H^0(O_p)$ is an isomorphism by evaluating at $p$.

**Exercise 1.32.** Let $D$ be an effective divisor of degree $n$ on $P^1$. We have the short exact sequence

$$
0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{O}_{P^1} \longrightarrow \mathcal{O}_D \longrightarrow 0.
$$

Use the associated long exact sequence to calculate the cohomology $H^*(P^1, \mathcal{I}(D))$. 
