1. Sheaves and cohomology

1.1. Sheaves. Let $X$ be a topological space. A sheaf $\mathcal{F}$ on $X$ associates to each open set $U$ an abelian group $\mathcal{F}(U)$, called the sections of $\mathcal{F}$ over $U$, along with a restriction map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ for any open sets $U \subset V$ (for a section $\sigma \in \mathcal{F}(U)$, we often write $\sigma|_U$ to denote $r_{V,U}(\sigma)$), satisfying the following conditions:

1. For any open sets $U \subset V \subset W$, $r_{V,U} \circ r_{W,V} = r_{W,U}$;
2. For a collection of open sets $\{U_i\}_{i \in I}$ and sections $\alpha_i \in \mathcal{F}(U_i)$, if $\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$ for any $i, j \in I$, then there exists a unique $\alpha \in \mathcal{F}(\bigcup U_i)$ such that $\alpha|_{U_i} = \alpha_i$ for any $i$.

Remark 1.1. If $\mathcal{F}$ satisfies (1) only, we call it a presheaf. One can perform sheafification for a presheaf to make it become a sheaf. For many sheaves we consider, the restriction maps are the obvious ones by restricting functions from bigger subsets to smaller ones.

Exercise 1.2. Show that $\mathcal{F}(\emptyset)$ consists of exactly one element.

Example 1.3. Let $G$ be an abelian group. We have the sheaf of locally constant functions $\mathcal{G}$ on a topological space $X$, where $\mathcal{G}(U)$ is the group of locally constant maps $f : U \to G$ on a non-empty open set $U \subset X$ and $\mathcal{G}(\emptyset) = 0$.

Exercise 1.4. Show that for the sheaf $\mathcal{G}$ of locally constant functions, we have $\mathcal{G}(U) = G$ for any non-empty connected open set $U$.

Exercise 1.5. Suppose we define $\mathcal{G}(U) = G$ as the set of constant functions on a non-empty open set $U$ with the natural restriction maps. If $G$ contains at least two elements and if $X$ consists of two disjoint non-empty open subsets, show that $\mathcal{G}$ is not a sheaf.
Example 1.6. Let $X$ be a complex manifold and $U \subset X$ an open set.

1. Sheaf $\mathcal{O}$ of holomorphic functions:
   \[ \mathcal{O}(U) = \{ \text{holomorphic functions on } U \} \]
   The group law is given by addition.

2. Sheaf $\mathcal{O}^*$ of nowhere zero holomorphic functions:
   \[ \mathcal{O}^*(U) = \{ \text{holomorphic functions } f \text{ on } U : f(p) \neq 0 \text{ for every } p \in U \} \]
   The group law is given by multiplication.

3. Sheaf $\mathcal{M}$ of meromorphic functions: strictly speaking, a meromorphic function is not a function, even we take $\infty$ into account. If $X$ is compact, we cannot take a meromorphic function as a quotient of two holomorphic functions, since any globally defined holomorphic function is a constant on $X$. Instead, we define $f \in \mathcal{M}(U)$ as local quotients of holomorphic functions compatible with each other. Namely, there exists an open covering $\{U_i\}$ of $U$ such that on each $U_i$, $f$ is given by $g_i/h_i$ for some $g_i, h_i \in \mathcal{O}(U_i)$ satisfying $g_i/h_i = g_j/h_j$, i.e. $g_i h_j = g_j h_i \in \mathcal{O}(U_i \cap U_j)$, hence these local quotients can be glued together over $U$.

4. Sheaf $\mathcal{M}^*$ of meromorphic functions not identically zero: this is defined similarly as above and the group law is given by multiplication.

1.2. Maps between sheaves. Let $\mathcal{E}$ and $\mathcal{F}$ be two sheaves on a topological space $X$. A map $f : \mathcal{E} \to \mathcal{F}$ is a collection of group homomorphisms
   \[ \{ f_U : \mathcal{E}(U) \to \mathcal{F}(U) \} \]
   such that they commute with the restriction maps, i.e., for any open sets $U \subset V$ and $\sigma \in \mathcal{E}(V)$ we have
   \[ f_V(\sigma)|_U = f_U(\sigma|_U). \]
   Define the sheaf of kernel $\ker(f)$ as
   \[ \ker(f)(U) = \{ \ker(f_U : \mathcal{E}(U) \to \mathcal{F}(U)) \}. \]

Exercise 1.7. Prove that in the above definition $\ker(f)$ is a sheaf.

Example 1.8. Let $X$ be a complex manifold. Define the exponential map
   \[ \exp : \mathcal{O} \to \mathcal{O}^* \]
   by $\exp(h) = e^{2\pi i h}$ for any open set $U \subset X$ and section $h \in \mathcal{O}(U)$. It is easy to see that $\ker(\exp)$ is the locally constant sheaf $\mathbb{Z}$.

The sheaf of cokernel is harder to define. Naively, one would like to define $\text{coker}(f)(U) = \text{coker}(f_U : \mathcal{E}(U) \to \mathcal{F}(U))$, but this is problematic. For instance, consider the exponential map $\exp : \mathcal{O} \to \mathcal{O}^*$ on the punctured plane $\mathbb{C} \setminus \{0\}$. The section $z \in \mathcal{O}^*(\mathbb{C} \setminus \{0\})$ is not in the image of $f$, hence it would define a section in the cokernel. Nevertheless, restricted to any contractible open set $U \subset \mathbb{C} \setminus \{0\}$, $z$ lies in the image of $f$. Now cover $\mathbb{C} \setminus \{0\}$ by contractible open sets. By the gluing property of sheaves, $z$ would be zero everywhere, leading to a contradiction.

Instead, we define a section of $\text{coker}(f)(U)$ to be a collection of sections $\{ \sigma_\alpha \in \mathcal{F}(U_\alpha) \}$ for an open covering $\{U_\alpha\}$ of $U$ such that for all $\alpha, \beta$ we have
   \[ \sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in f_{U_\alpha \cap U_\beta}(\mathcal{E}(U_\alpha \cap U_\beta)). \]
Here the definition still depends on the choice of an open covering. To get rid of this, we pass to the direct limit. More precisely, we further identify two collections
\{ (U_\alpha, \sigma_\alpha) \} and \{ (V_\beta, \sigma_\beta) \} if for all p \in U_\alpha \cap V_\beta, there exists an open set W satisfying
p \in W \subset U_\alpha \cap V_\beta \text{ such that }
\sigma_\alpha|_W - \sigma_\beta|_W \in f_\mathcal{E}(W).
This identification yields an equivalence relation and correspondingly we define \( \operatorname{coker}(f)(U) \) as the group of equivalence classes of the above sections.

**Exercise 1.9.** Prove that in the above definition \( \operatorname{coker}(f) \) is a sheaf.

If \( \ker(f) \) (resp. \( \operatorname{coker}(f) \)) is the zero sheaf, we say that \( f \) is injective (resp. surjective).

Consider the following sequence of maps between sheaves:

\[ 0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0, \]
We say that it is a short exact sequence if \( \mathcal{E} = \ker(\beta) \) and \( \mathcal{G} = \operatorname{coker}(\alpha) \). In this case we also say that \( \mathcal{E} \) is a subsheaf of \( \mathcal{F} \) and \( \mathcal{G} \) is the quotient sheaf \( \mathcal{F}/\mathcal{E} \).

**Example 1.10.** Let \( X \) be a complex manifold. We have the exact exponential sequence:

\[ 0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0, \]
where \( i \) is the inclusion and \( \exp(f) = e^{2\pi \sqrt{-1}f} \) for \( f \in \mathcal{O}(U) \).

**Exercise 1.11.** Prove that the exponential sequence is exact.

**Example 1.12.** Let \( X \) be a complex manifold and \( Y \subset X \) a submanifold. Define the ideal sheaf \( \mathcal{I}_{Y/X} \) of \( Y \) in \( X \) (or simply \( \mathcal{I}_Y \) if there is no confusion) by

\[ \mathcal{I}_Y(U) = \{ \text{holomorphic functions in } U \text{ that are vanishing on } Y \cap U \}. \]

We have the exact sequence:

\[ 0 \longrightarrow \mathcal{I}_Y \xrightarrow{i} \mathcal{O}_X \xrightarrow{r} i_* \mathcal{O}_Y \longrightarrow 0, \]
where \( i \) is the inclusion and \( r \) is defined by restriction. Here we define

\[ i_* \mathcal{O}_Y(U) = \{ \text{holomorphic functions on } Y \cap U \}, \]
which as a sheaf defined on \( X \) is the extension of \( \mathcal{O}_Y \) by zero outside \( Y \). Hence we may also abuse notation to write it as \( \mathcal{O}_Y \).

**Exercise 1.13.** Prove that the above sequence is exact.

1.3. **Stalks and germs.** Let \( \mathcal{F} \) be a sheaf on a topological space \( X \) and \( p \in X \) a point. Suppose \( U \) and \( V \) are two open subsets, both containing \( p \), with two sections \( \alpha \in \mathcal{F}(U) \) and \( \beta \in \mathcal{F}(V) \). Define an equivalence relation \( \alpha \sim \beta \), if there exists an open subset \( W \) satisfying \( p \in W \subset U \cap V \) such that \( \alpha|_W = \beta|_W \). Define the stalk \( \mathcal{F}_p \) as the union of all sections in open neighborhoods of \( p \) modulo this equivalence relation. Namely, \( \mathcal{F}_p := \lim_{\longrightarrow} \mathcal{F}(U) = \left( \bigsqcup_{U \ni p} \mathcal{F}(U) \right)/\sim \).

Note that \( \mathcal{F}_p \) is also a group, by adding representatives of two equivalence classes. There is a group homomorphism \( r_U : \mathcal{F}(U) \rightarrow \mathcal{F}_p \) mapping a section \( \alpha \in \mathcal{F}(U) \) to its equivalence class. The image is called the germ of \( \alpha \).
Show that the quotient sheaf \( O \) the exact sequence using stalks. First note that given a sheaf map \( \phi \), for \( p \in U \), where \( A \) is an abelian group. The restriction maps are either the identity map \( A \to A \) or the zero map. For \( q \neq p \), the stalk \( \mathcal{F}_q = \{0\} \). At \( p \), we have \( \mathcal{F}_p = A \). Note that \( \mathcal{F} \) can also be obtained by extending the constant sheaf \( A \) at \( p \) by zero to \( X \setminus \{p\} \).

**Exercise 1.15.** Let \( X \) be a Riemann surface and \( p \in X \) a point. Let \( \mathcal{A}_p \) be the ideal sheaf of \( p \) in \( X \) parameterizing holomorphic functions vanishing at \( p \). We have the exact sequence

\[
0 \longrightarrow \mathcal{A}_p \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0.
\]

Show that the quotient sheaf \( \mathcal{O}_p \) is isomorphic to the skyscraper sheaf with stalk \( C \) at \( p \).

It is more convenient to verify injectivity and surjectivity for maps of sheaves by using stalks. First note that given a sheaf map \( \phi: \mathcal{E} \to \mathcal{F} \) and a point \( p \in X \), one can define the induced map \( \phi_p: \mathcal{E}_p \to \mathcal{F}_p \) on the stalks of \( p \) by \( \phi_p([\sigma]) = [\phi_U(\sigma)] \) for a neighborhood \( U \) of \( p \) and a section \( \sigma \in \mathcal{E}(U) \). One can check that the definition does not depend on the choice of the representative \( \sigma \) and the neighborhood \( U \).

**Proposition 1.16.** Let \( \phi: \mathcal{E} \to \mathcal{F} \) be a map for sheaves \( \mathcal{E} \) and \( \mathcal{F} \) on a topological space \( X \).

1. \( \phi \) is injective if and only if the induced map \( \phi_p: \mathcal{E}_p \to \mathcal{F}_p \) is injective for the stalks at every point \( p \).
2. \( \phi \) is surjective if and only if the induced map \( \phi_p: \mathcal{E}_p \to \mathcal{F}_p \) is surjective for the stalks at every point \( p \).
3. \( \phi \) is an isomorphism if and only if the induced map \( \phi_p: \mathcal{E}_p \to \mathcal{F}_p \) is an isomorphism for the stalks at every point \( p \).

**Proof.** The claim (3) follows from (1) and (2). Let us prove (1) only, and one can find the proof of (2) in standard textbooks.

Suppose \( \phi \) is injective. Take a section \( \sigma \in \mathcal{E}(U) \) in a neighborhood \( U \) of \( p \). If \( \phi_p([\sigma]) = 0 \in \mathcal{F}_p \), i.e. \( [\phi_U(\sigma)] = 0 \), then there exists a smaller open subset \( V \subset U \) such that \( 0 = \phi_U(\sigma)|_V = \phi_V(\sigma)|_V \in \mathcal{F}(V) \), hence \( \sigma|_V = 0 \in \mathcal{E}(V) \). Consequently the equivalence class \( [\sigma] = 0 \in \mathcal{E}_p \) and we conclude that \( \phi_p \) is injective.

Conversely, suppose \( \phi_p \) is injective for every point \( p \). Take a section \( \sigma \in \mathcal{E}(U) \) for an open subset \( U \). If \( \phi_U(\sigma) = 0 \in \mathcal{F}(U) \), then for every point \( p \in U \), \( [\phi_U(\sigma)] = 0 \in \mathcal{F}_p \). Since \( \phi_p \) is injective, it implies that \( [\sigma] = 0 \in \mathcal{E}_p \), i.e. there exists an open subset \( U_p \ni p \) such that \( \sigma|_{U_p} = 0 \in \mathcal{E}(U_p) \). Applying the gluing property to the open covering \( \{U_p\} \) of \( U \), we thus conclude that \( \sigma = 0 \in \mathcal{E}(U) \).

**Remark 1.17.** The image of \( \phi \) does not automatically form a sheaf. In general, it is only a presheaf. If the sheafification of \( \operatorname{Im}(\phi) \) equals \( \mathcal{F} \), we say that \( \phi \) is surjective. In particular, it does not mean \( \mathcal{E}(U) \to \mathcal{F}(U) \) is surjective for every open set \( U \). Sometimes one has to pass to a refined open covering in order to obtain a surjection between sections.

**Example 1.18.** Consider the exponential map \( \exp: \mathcal{O} \to \mathcal{O}^* \) on the punctured plane \( \mathbb{C} \setminus \{0\} \). As a map of sheaves it is surjective, but the section \( z \) over \( \mathbb{C} \setminus \{0\} \) does not have an inverse. Nevertheless, it does have an inverse over any contractible open subset.
1.4. Sheaf cohomology. Let $\mathcal{F}$ be a sheaf on a topological space $X$. Take an open covering $\mathcal{U} = \{ U_\alpha \}$ of $X$. Define the $k$-th cochain group

$$C^k(\mathcal{U}, \mathcal{F}) := \prod_{\alpha_0, \ldots, \alpha_k} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}).$$

An element $\sigma$ of $C^k(\mathcal{U}, \mathcal{F})$ consists of a section $\sigma_{\alpha_0, \ldots, \alpha_k} \in \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$ for every $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$, where $\alpha_0, \ldots, \alpha_k$ are ordered and distinct indices.

Define a coboundary map $\delta : C^k(\mathcal{U}, \mathcal{F}) \to C^{k+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta \sigma)_{\alpha_0, \ldots, \alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \sigma_{\alpha_0, \ldots, \alpha_j, \alpha_{j+1}}|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_k+1}}.$$ 

Example 1.19. Consider $\mathcal{U} = \{ U_1, U_2, U_3 \}$ as an open covering of $X$. Take a cochain element $\sigma \in C^0(\mathcal{U}, \mathcal{F})$, i.e. $\sigma$ is a collection of a section $\sigma_i \in \mathcal{F}(U_i)$ for every $i$. Then we have

$$(\delta \sigma)_{ij} = (\sigma_j - \sigma_i)|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_j).$$

Now take $\tau \in C^1(\mathcal{U}, \mathcal{F})$, i.e. $\tau$ is a collection of a section $\tau_{ij} \in \mathcal{F}(U_i \cap U_j)$ for every pair $i, j$. Then we have

$$(\delta \tau)_{123} = (\tau_{23} - \tau_{13} + \tau_{12})|_{U_1 \cap U_2 \cap U_3} \in \mathcal{F}(U_1 \cap U_2 \cap U_3).$$

A cochain $\sigma \in C^k(\mathcal{U}, \mathcal{F})$ is called a cocycle if $\delta \sigma = 0$. We say that $\sigma$ is a coboundary if there exists $\tau \in C^{k-1}(\mathcal{U}, \mathcal{F})$ such that $\delta \tau = \sigma$.

Lemma 1.20. A coboundary is a cocycle, i.e. $\delta \circ \delta = 0$.

Proof. Let us prove it for the above example. The same idea applies in general with heavier notation. Under the above setting, we have

$$((\delta \circ \delta) \sigma)_{123} = (\delta \sigma)_{23} - (\delta \sigma)_{13} + (\delta \sigma)_{12}$$

$$= (\sigma_3 - \sigma_2) - (\sigma_3 - \sigma_1) + (\sigma_2 - \sigma_1)$$

$$= 0 \in \mathcal{F}(U_1 \cap U_2 \cap U_3).$$

Here we omit the restriction notation, since it is obvious. \hfill \Box

Exercise 1.21. Prove in full generality that $\delta \circ \delta = 0$.

For the coboundary map $\delta_k : C^k(\mathcal{U}, \mathcal{F}) \to C^{k+1}(\mathcal{U}, \mathcal{F})$, define the $k$-th cohomology group (respect to $\mathcal{U}$) by

$$H^k(\mathcal{U}, \mathcal{F}) := \frac{\ker(\delta_k)}{\text{Im}(\delta_{k-1})}.$$ 

This is well-defined due to the above lemma.

Example 1.22. For $k = 0$, we have $H^0(\mathcal{U}, \mathcal{F}) = \ker(\delta_0)$. Take an element $\{ \sigma_i \in \mathcal{F}(U_i) \}$ in this group. Because it is a cocycle, it satisfies

$$\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_j).$$

By the gluing property of sheaves, there exists a global section $\sigma \in \mathcal{F}(X)$ such that $\sigma|_{U_i} = \sigma_i$. Conversely, if $\sigma$ is a global section, then define $\sigma_i = \sigma|_{U_i} \in \mathcal{F}(U_i)$. In this way we obtain a cocycle in $C^1(\mathcal{U}, \mathcal{F})$. From the discussion we see that $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$, which is independent of the choice of an open covering. Hence $H^0(\mathcal{U}, \mathcal{F})$ is called the group of global sections of $\mathcal{F}$, and we denote it by $H^0(X, \mathcal{F})$ or simply by $H^0(\mathcal{F})$. 
In general, we would like to define cohomology independent of open coverings. Take two open coverings \( U = \{ U_\alpha \}_{\alpha \in I} \) and \( V = \{ V_\beta \}_{\beta \in J} \). We say that \( U \) is a refinement of \( V \) if for every \( U_\alpha \), there exists a \( V_\beta \) such that \( U_\alpha \subset V_\beta \) and we write it as \( \overline{U} \subset V \). Then we have an index map \( \phi : I \rightarrow J \) sending \( \alpha \) to \( \beta \). It induces a map 
\[ \rho_\phi : C^k(\overline{U}, \mathcal{F}) \rightarrow C^k(U, \mathcal{F}) \]
given by
\[ \rho_\phi(\sigma)_{a_0, \ldots, a_k} = \sigma_{\phi(a_0), \ldots, \phi(a_k)}|_{U_{a_0} \cap \cdots \cap U_{a_k}}. \]
One checks that it commutes with the coboundary map \( \delta \), i.e. \( \delta \circ \rho_\phi = \rho_\phi \circ \delta \).
Therefore, it induces a map
\[ \rho : H^k(\overline{U}, \mathcal{F}) \rightarrow H^k(U, \mathcal{F}) \]
(which is independent of the choice of \( \phi \)). Finally, we define the \( k \)-th (Čech) cohomology group by passing to the direct limit:
\[ H^k(X, \mathcal{F}) := \lim_{\rightarrow} H^k(U, \mathcal{F}). \]
The definition involves direct limit, which is inconvenient to use in practice. Nevertheless, we can simplify the situation if the open covering \( \overline{U} \) is fine enough.
We say that \( \overline{U} = \{ U_i \}_{i \in I} \) is acyclic respect to \( \mathcal{F} \), if for any \( k > 0 \) and \( i_1, \ldots, i_k \in I \) we have
\[ H^k(U_{i_1} \cap \cdots \cap U_{i_k}, \mathcal{F}) = 0. \]

**Theorem 1.23** (Leray’s Theorem). If the open covering \( \overline{U} \) is acyclic respect to \( \mathcal{F} \), then \( H^*(\overline{U}, \mathcal{F}) \cong H^*(X, \mathcal{F}) \).

**Remark 1.24.** In the context of complex manifolds, if \( U_i \)'s are contractible, then \( \overline{U} \) is acyclic respect to the sheaves we will consider. While for algebraic varieties, if \( U_i \)'s are affine, then \( \overline{U} \) is acyclic.

**Example 1.25.** Let us compute the cohomology of the structure sheaf \( \mathcal{O} \) on the Riemann sphere \( \mathbb{P}^1 \). It is clear that \( H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C} \), since any holomorphic function on \( \mathbb{P}^1 \) is constant. For higher cohomology, use \([X, Y]\) to denote the homogeneous coordinates of \( \mathbb{P}^1 \). Take the standard open covering \( U = \{ [X, Y] : X \neq 0 \} \) and \( V = \{ [X, Y] : Y \neq 0 \} \). It is acyclic respect to the structure sheaf \( \mathcal{O} \) (morally because \( U, V \cong \mathbb{C} \) is contractible). Let \( s = Y/X \) and \( t = X/Y \) be affine coordinates of \( U \) and \( V \), respectively. Suppose \( h \) is an element in \( C^1([U, V], \mathcal{O}) \), i.e. \( h \in \mathcal{O}(U \cap V) \).
We can write
\[ h = \sum_{i = -\infty}^{\infty} a_is^i. \]
Now take
\[ f = -\sum_{i = 0}^{\infty} a_is^i \in \mathcal{O}(U), \]
\[ g = \sum_{i = -\infty}^{-1} a_is^i = \sum_{i = -\infty}^{-1} a_it^{-i} \in \mathcal{O}(V). \]
Then we have \((f, g) \in C^0([U, V], \mathcal{O})\) and \( \delta((f, g)) = g - f = h \). It implies that \( H^1(\mathbb{P}^1, \mathcal{O}) = 0 \). All other \( H^k(\mathbb{P}^1, \mathcal{O}) = 0 \) for \( k > 1 \), since there are only two open subsets in the covering.
Example 1.26. Let $\Omega$ denote the sheaf of holomorphic one-forms on a Riemann surface, i.e. locally a section of $\Omega$ can be expressed as $f(z)dz$, where $z$ is local coordinate and $f(z)$ a holomorphic function, consistent with change of coordinates.

Let us compute the cohomology of $\Omega$ on $\mathbb{P}^1$. Take the above open covering. Suppose $\omega$ is a global holomorphic one-form. Then in the open subset $U$, it can be written as

$$\left(\sum_{i=0}^{\infty} a_i s^i\right) ds.$$

Using the relation $s = 1/t$ and $ds = -dt/t^2$, in $V$ it can be expressed as

$$-\left(\sum_{i=0}^{\infty} a_i t^{-i-2}\right) dt,$$

which is holomorphic if and only if $a_i = 0$ for all $i$. Hence $\omega$ is the zero one-form and $H^0(\mathbb{P}^1, \Omega) = 0$. Now take $\omega \in C^1(\{U, V\}, \Omega)$, i.e. $\omega \in \Omega(U \cap V) = \Omega(\mathbb{C}^*)$, we express it as

$$\omega = \left(\sum_{i=-\infty}^{\infty} a_i t^i\right) dt.$$

Note that any $\alpha \in \Omega(U)$ and $\beta \in \Omega(V)$ can be written as

$$\alpha = \left(\sum_{i=0}^{\infty} b_i s^i\right) ds,$$

$$\beta = \left(\sum_{i=0}^{\infty} c_i t^i\right) dt.$$

Hence on $U \cap V$ we have

$$\delta((\alpha, \beta)) = \beta - \alpha = -\left(\sum_{i=0}^{\infty} b_i t^{-i-2}\right) dt + \left(\sum_{i=0}^{\infty} c_i t^i\right) dt.$$

Note that only the term $t^{-1}$ is missing from the expression. We conclude that $H^1(\mathbb{P}^1, \Omega) = \{a_{-1}t^{-1} dt\} \cong \mathbb{C}$.

Remark 1.27. In general, the rank of $H^1(X, \mathcal{O}) \cong H^0(X, \Omega)$ (by Serre Duality) is called the genus of a Riemann surface (or a complex algebraic curve) $X$.

Exercise 1.28. Let $D = p_1 + \cdots + p_n$ be a collection of $n$ points in $\mathbb{P}^1$. We say that $D$ is an effective divisor of degree $n$. Define the sheaf $\mathcal{O}(D)$ on $\mathbb{P}^1$ by

$$\mathcal{O}(D)(U) = \{f \in \mathcal{H}(U) : f \in \mathcal{O}(U \setminus \{p_1, \ldots, p_n\}) with at worst a simple pole at each p_i\}.$$ 

Assume that the standard covering of $\mathbb{P}^1$ is acyclic respect to $\mathcal{O}(D)$. Use it to calculate the cohomology groups $H^*(\mathbb{P}^1, \mathcal{O}(D))$.

As many other homology/cohomology theories, one can associate a long exact sequence of cohomology to a short exact sequence. Suppose we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Then $\alpha$ and $\beta$ induce maps

$$\alpha : C^k(U, \mathcal{E}) \rightarrow C^k(U, \mathcal{F}), \quad \beta : C^k(U, \mathcal{F}) \rightarrow C^k(U, \mathcal{G}).$$
Since the coboundary map $\delta$ is given by alternating sums of restrictions, $\alpha$ and $\beta$ commute with $\delta$, and hence they send a cocycle to cocycle and a coboundary to coboundary. Consequently they induce maps for cohomology

$$
\alpha_* : H^k(X, \mathcal{E}) \to H^k(X, \mathcal{F}), \quad \beta_* : H^k(X, \mathcal{F}) \to H^k(X, \mathcal{G}).
$$

Next we define the coboundary map

$$
\delta_* : H^k(X, \mathcal{G}) \to H^{k+1}(X, \mathcal{E}).
$$

For $\sigma \in C^k(U, \mathcal{G})$ satisfying $\delta \sigma = 0$, because $\beta$ is surjective, after refining $U$ (still denoted by $U$) there exists $\tau \in C^k(U, \mathcal{F})$ satisfying $\beta(\tau) = \sigma$. Then $\beta(\delta \tau) = \delta(\beta(\tau)) = \delta \sigma = 0$, hence after refining $U$ further there exists $\mu \in C^{k+1}(U, \mathcal{E})$ satisfying $\alpha(\mu) = \delta \tau$. Note that $\mu$ is a cocycle. This is because $\alpha(\delta \mu) = \alpha(\alpha(\mu)) = \delta \delta(\tau) = 0$ and $\alpha$ is injective, hence $\delta \mu = 0$ and $\mu \in \ker(\delta)$. We thus take $\delta_* \sigma := [\mu] \in H^{k+1}(X, \mathcal{E})$. One checks that this is independent of the choice of $\tau$ and $\mu$.

We say that a sequence of maps

$$
\cdots \longrightarrow A_{n-1} \overset{\alpha_{n-1}}{\longrightarrow} A_n \overset{\alpha_n}{\longrightarrow} A_{n+1} \longrightarrow \cdots
$$

is exact if $\text{Im}(\alpha_{n-1}) = \ker(\alpha_n)$.

**Proposition 1.29.** The long sequence of cohomology associated to a short exact sequence of sheaves is exact.

**Proof.** We prove it under an extra assumption that there exists an acyclic open covering $U$ such that for any $U = U_1 \cap \cdots \cap U_k$ we have the short exact sequence:

$$
0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \to 0.
$$

At least for sheaves considered in this course, this assumption always holds. It further implies the following sequence is exact:

$$
0 \to C^k(U, \mathcal{E}) \to C^k(U, \mathcal{F}) \to C^k(U, \mathcal{G}) \to 0.
$$

Let us prove that

$$
H^k(U, \mathcal{F}) \overset{\beta_*}{\longrightarrow} H^k(U, \mathcal{G}) \overset{\delta_*}{\longrightarrow} H^{k+1}(U, \mathcal{E})
$$

is exact. The other cases are easier.

Consider $\tau \in Z^k(U, \mathcal{F})$. In the definition of $\delta_*$, take $\sigma = \beta(\tau)$. Then there exists $\mu \in C^k(U, \mathcal{E})$ such that $\alpha(\mu) = \delta \tau = 0$. Then we have $\mu = 0$ since $\alpha$ is injective. Consequently $\delta_* \beta_* (\tau) = \delta_* (\sigma) = \mu = 0$, hence $\delta_* \beta_* = 0$ and $\text{Im}(\beta_*) \subset \ker(\delta_*)$.

Conversely, suppose $\delta_* \sigma = 0$ for $\sigma \in Z^k(U, \mathcal{G})$. In the definition of $\delta_*$, it implies that $\mu = 0 \in H^{k+1}(U, \mathcal{E})$, hence there exists $\gamma \in C^k(U, \mathcal{E})$ such that $\delta \gamma = \mu$. Since $\alpha(\mu) = \delta \tau$, we have $\delta \gamma = \delta \alpha(\gamma)$ and $\tau - \alpha(\gamma) \in Z^k(U, \mathcal{F})$ is a cocycle. Moreover, $\delta(\tau - \alpha(\gamma)) = \delta(\tau) = \sigma$, hence $\beta_*(\tau - \alpha(\gamma)) = \sigma$. We conclude that $\ker(\delta_*) \subset \text{Im}(\beta_*)$. \(\square\)

**Exercise 1.30.** Prove in general the cohomology sequence is exact.

**Example 1.31.** Consider the short exact sequence

$$
0 \longrightarrow \mathcal{I}_p \overset{i}{\longrightarrow} \mathcal{O}_{\mathbb{P}^1} \overset{r}{\longrightarrow} \mathcal{O}_p \longrightarrow 0.
$$

Its long exact sequence of cohomology is as follows:

$$
0 \to H^0(\mathcal{I}_p) \to H^0(\mathcal{O}_{\mathbb{P}^1}) \to H^0(\mathcal{O}_p) \to H^1(\mathcal{I}_p) \to H^1(\mathcal{O}_{\mathbb{P}^1}) \to 0.
$$

The last term is zero because $p$ is a point so it does not have higher cohomology. We have $H^0(\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$ because any holomorphic function on $\mathbb{P}^1$ is constant. Note
that $H^0(\mathcal{F}_p) = 0$, because vanishing at $p$ forces such a constant function to be zero. Moreover we have seen that $H^1(\mathcal{O}_{\mathbb{P}^1}) = 0$. Altogether it implies that $H^1(\mathcal{F}_p) = 0$, because $H^0(\mathcal{O}_{\mathbb{P}^1}) \to H^0(\mathcal{F}_p)$ is an isomorphism by evaluating at $p$.

**Exercise 1.32.** Let $D$ be an effective divisor of degree $n$ on $\mathbb{P}^1$. We have the short exact sequence

$$0 \longrightarrow \mathcal{F}_D \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_D \longrightarrow 0.$$  

Use the associated long exact sequence to calculate the cohomology $H^*(\mathbb{P}^1, \mathcal{F}(D))$.

2. **Vector bundles, line bundles and divisors**

2.1. **Holomorphic vector bundles.** Let $k$ be a positive integer. Consider a holomorphic map $\pi: E \to X$ of complex manifolds such that for every $x \in X$ the fiber $E_x = \pi^{-1}(x)$ is isomorphic to $\mathbb{C}^k$ and such that there exists a neighborhood $U$ of $x$ with a biholomorphism

$$\phi_U: E_U = \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^k$$

mapping $E_x$ to $\{x\} \times \mathbb{C}^k$ as an isomorphism of vector spaces. Then $E$ is called a (holomorphic) vector bundle of rank $k$ on $X$ and $\{(U, \phi_U)\}$ is called a trivialization of $E$. If $E$ is of rank one, we say that $E$ is a line bundle.

There is another characterization of vector bundles by transition functions. Suppose $\mathcal{U} = \{U_\alpha\}$ is an open covering of $X$. Given holomorphic functions

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \to \text{GL}(\mathbb{C}^k),$$

we can construct a vector bundle $E$ by gluing $U_\alpha \times \mathbb{C}^k$ together. More precisely, define

$$E = \sqcup(U_\alpha \times \mathbb{C}^k)/\sim$$

as a complex manifold by identifying $(x, v)$ with $(x, g_{\alpha\beta}(v))$ for $x \in U_\alpha \cap U_\beta$ and $v \in \mathbb{C}^k$ and define the map $E \to X$ by projection to the bases $U_\alpha$. We call $\{g_{\alpha\beta}\}$ transition functions of $E$. They have to satisfy the following compatibility conditions:

$$g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = I, \quad \text{for all } x \in U_\alpha \cap U_\beta,$$

$$g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = I, \quad \text{for all } x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

**Exercise 2.1.** Let $E$ and $F$ be two vector bundles on $X$ of rank $k$ and $l$, respectively. Define the direct sum $E \oplus F$, the tensor product $E \otimes F$, the dual $E^*$, and the wedge product $\wedge^r E$ for $r \leq k$. Calculate the ranks of these bundles and represent their transition functions in terms of the transition functions of $E$ and $F$.

A map between two vector bundles $E$ and $F$ on $X$ is a holomorphic map $f: E \to F$ such that $f(E_x) \subset F_x$ and $f_x = f|_{E_x}: E_x \to F_x$ is linear for all $x \in X$. Note that if $f(E_x)$ has the same rank for every $x$, then $\ker(f)$ and $\text{Im}(f)$ are naturally subbundles of $E$ and $F$, respectively. We say that $E$ and $F$ are isomorphic if $f_x$ is a linear isomorphism for every $x$. A vector bundle is called trivial if it is isomorphic to $X \times \mathbb{C}^k$.

**Exercise 2.2.** Give an example of a vector bundle map $f: E \to F$ on $X$ such that the image of $f$ is not a vector bundle.

**Exercise 2.3.** Let $L$ be a line bundle. Prove that $L \otimes L^*$ is trivial.
Define a section \( \sigma \) of a vector bundle \( E \) as a holomorphic map \( \sigma : X \to E \) such that \( \sigma(x) \in E_x \) for every \( x \in X \), i.e. \( \pi \circ \sigma \) is identity. If \( \sigma(x) = 0 \in E_x \), we say that \( \sigma \) is vanishing at \( x \).

**Exercise 2.4.** Let \( L \) be a line bundle on \( X \). Prove that \( L \) is trivial if and only if it possesses a nowhere vanishing section.

**Example 2.5** (Tangent bundles). Let \( X \) be an \( n \)-dimensional complex manifold. Suppose \( \{ \phi_U : U \to \mathbb{C}^n \} \) are coordinate charts of \( X \). Define the (holomorphic) tangent bundle \( T_X \) by setting \( T_X = \sqcup T_x \) with

\[
T_x = \mathbb{C}\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\} \cong \mathbb{C}^n
\]

as well as transition functions \( g_{UV} = J(\phi_V \phi_U^{-1}) \), where \( J \) denotes the Jacobian matrix \( (\frac{\partial \phi_U}{\partial \phi_V}) \) for \( 1 \leq i, j \leq n \). The dual bundle \( T_X^* \) is called the cotangent bundle of \( X \). The determinant \( \det(T_X^*) \) is called the canonical (line) bundle of \( X \).

**Exercise 2.6.** Prove that the canonical bundle of \( \mathbb{P}^1 \) has no nontrivial sections.

**Remark 2.7.** Alternatively, one can define vector bundles on a topological space, a differential manifold and an algebraic variety. The above definitions and properties go through word by word once replacing “holomorphic map” by “homomorphism”, “smooth map” or “regular map”.

### 2.2. Vector bundles and locally free sheaves

There is a one-to-one correspondence between isomorphism classes of vector bundles of rank \( n \) and isomorphism classes of locally free sheaves of rank \( n \). Here we briefly explain the idea.

Let \( \mathcal{O}_X \) be the structure sheaf of a complex manifold \( X \). Then \( \mathcal{O}_X(U) \) has a ring structure (not only a group) for any open subset \( U \). A sheaf of \( \mathcal{O}_X \)-modules is a sheaf \( \mathcal{F} \) on \( X \) such that for each open subset \( U \), the group \( \mathcal{F}(U) \) is an \( \mathcal{O}_X(U) \)-module. An \( \mathcal{O}_X \)-module \( \mathcal{F} \) is called free if it is isomorphic to a direct sum of \( \mathcal{O}_X \). It is called locally free if there is an open covering \( U = \{U_\alpha\} \) such that for each open subset \( U_\alpha \), \( \mathcal{F}|_{U_\alpha} \) is a free \( \mathcal{O}_X|_{U_\alpha} \)-module. The rank of \( \mathcal{F} \) on \( U \) is the number of copies of \( \mathcal{O} \) in the summand. If \( X \) is connected, the rank of \( \mathcal{F} \) is constant. In particular, a locally free sheaf of rank one is called an invertible sheaf.

Roughly speaking, if \( \mathcal{F} \) is locally free of rank \( n \), we can choose a set of \( n \) generators \( x_1, \ldots, x_n \) for the \( \mathcal{O}_X(U) \)-module \( \mathcal{F}(U) \). They span an \( n \)-dimensional vector space \( \mathbb{C}[x_1, \ldots, x_n] \) over \( U \). By change of generators we can write down the corresponding transition functions, which give \( \mathcal{F} \) a vector bundle structure. Conversely if \( F \) is a vector bundle on \( X \), locally we have \( F|_{U} \cong U \times \mathbb{C}^n \) with \( x_1, \ldots, x_n \) a basis (i.e. \( n \) linearly independent sections) of \( \mathbb{C}^n \) over \( U \). Then we can identify \( F|_{U} \) as an \( \mathcal{O}_X(U) \)-module of rank \( n \) using \( x_1, \ldots, x_n \) as generators.

**Example 2.8.** A hypersurface in \( \mathbb{P}^n \) is the zero locus of a homogeneous polynomial. Let \( X \subset \mathbb{P}^n \) be a submanifold and \( Y \subset X \) a hypersurface section, i.e. \( Y \) is cut out (transversely) by a hypersurface \( F \) in \( \mathbb{P}^n \) with \( X \). We have the short exact sequence

\[
0 \to \mathcal{I}_{Y/X} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.
\]

The ideal sheaf \( \mathcal{I}_{Y/X} \) is an invertible sheaf. Indeed, for an open subset \( U \subset X \), \( \mathcal{I}_{Y/X}(U) \) can be expressed as \( (F|_{U}) \cdot \mathcal{O}_X(U) \), hence is locally free of rank one. The sheaf \( \mathcal{O}_Y \) (extended to \( X \) by zero outside of \( Y \)) is not locally free. For \( U \cap Y = \emptyset \), \( \mathcal{O}_Y(U) = 0 \) and for \( U \cap Y \neq \emptyset \), \( \mathcal{O}_Y(U) \) is non-trivial.
2.3. **Divisors.** Let $X$ be a complex manifold. Suppose $Y \subset X$ is an irreducible subspace of codimension one satisfying that for every $p \in Y$ there exists an open neighborhood $U \subset X$ of $p$ such that $U \cap Y$ is cut out by a holomorphic function $f$ (and any two such functions differ by a unit). Then we say that $Y$ is an *irreducible divisor* of $X$. We call $f$ a local defining equation for $Y$ near $p$. A *divisor* $D$ on $X$ is a formal linear combination of irreducible divisors:

$$D = \sum_{i=1}^{n} a_i Y_i,$$

where $a_i \in \mathbb{Z}$ (or $\mathbb{Q}, \mathbb{R}$ depending on the context). If $a_i \geq 0$ for all $i$, then we say that $D$ is an *effective divisor* and denote it by $D \geq 0$. Divisors on $X$ form an additive group $\text{Div}(X)$.

Suppose $f$ is a local defining equation of an irreducible divisor $Y \subset X$ on an open subset $U \subset X$. For another local function $g$ on $U$, we can write

$$g = f^a \cdot h$$

such that $h \in \mathcal{O}_X(U)$ not divisible by $f$. We say that $a$ is the *vanishing order* of $g$ on $Y \cap U$. The vanishing order is a local constant, hence is independent of $U$. We denote by

$$\text{ord}_Y(g) = a$$

the vanishing order of $g$ on $Y$.

For two functions $g, h$ on $X$, we have

$$\text{ord}_Y(gh) = \text{ord}_Y(g) + \text{ord}_Y(h).$$

For a function $f = g/h$, we define

$$\text{ord}_Y(f) = \text{ord}_Y(g) - \text{ord}_Y(h).$$

If $\text{ord}_Y(f) > 0$, we say that $f$ has a zero on $Y$. If $\text{ord}_Y(f) < 0$, we say that $f$ has a pole on $Y$. We also define the *divisor associated to* $f$ by

$$(f) = \sum_Y \text{ord}_Y(f),$$

as well as the *divisor of zeros*

$$(f)_0 = \sum_Y \text{ord}_Y(g)$$

and the *divisor of poles*

$$(f)_\infty = \sum_Y \text{ord}_Y(h).$$

They satisfy that

$$(f) = (f)_0 - (f)_\infty.$$

If a divisor $D$ is the associated divisor of a meromorphic function $f$, i.e. if $D = (f)$, then $D$ is called a principal divisor.

Recall that $\mathcal{M}^*$ is the multiplicative sheaf of (not identically zero) meromorphic functions and $\mathcal{O}^*$ the multiplicative sheaf of nowhere vanishing regular functions, which is a subsheaf of $\mathcal{M}^*$.

**Proposition 2.9.** $\text{Div}(X) \cong H^0(X, \mathcal{M}^*/\mathcal{O}^*)$. 
Proof. Suppose that \( \{f_\alpha\} \) represents a global section of \( \mathcal{M}^*/\mathcal{O}^* \) with respect to an open covering \( U = \{U_\alpha\} \). Associate to it a divisor \( D_\alpha = \langle f_\alpha \rangle \) in \( U_\alpha \). We claim that \( D_\alpha = D_\beta \) in \( U_\alpha \cap U_\beta \). This is because

\[
\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta),
\]

hence \( f_\alpha \) and \( f_\beta \) define the same divisor. Consequently \( \{D_\alpha\} \) defines a global divisor. Moreover, if \( \{f_\alpha\} \) and \( \{g_\alpha\} \) define the same divisor, then \( f_\alpha/g_\alpha \in \mathcal{O}^*(U_\alpha) \), hence \( \{f_\alpha\} \) and \( \{g_\alpha\} \) represent the same section of \( \mathcal{M}^*/\mathcal{O}^* \). This shows an injection

\[
H^0(X, \mathcal{M}^*/\mathcal{O}^*) \hookrightarrow \text{Div}(X).
\]

Conversely, suppose \( D \) is a divisor on \( X \). We can choose an open covering \( U = \{U_\alpha\} \) such that \( D \) is locally defined by \( f_\alpha \in \mathcal{M}^*(U_\alpha) \). Since \( f_\alpha \) and \( f_\beta \) define the same divisor \( D|_{U_\alpha \cap U_\beta} \), we conclude that

\[
\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).
\]

Then \( \{f_\alpha\} \) defines a global section of \( \mathcal{M}^*/\mathcal{O}^* \). Moreover if \( D \) corresponds to the identity of \( \mathcal{M}^*/\mathcal{O}^* \) (which is 1 since the group structure is multiplicative), it implies that locally \( f_\alpha \in \mathcal{O}^*(U_\alpha) \) (after refining the open covering), hence \( f_\alpha \) does not have any zeros or poles. Consequently \( D|_{U_\alpha} = 0 \) for all \( U_\alpha \) and \( D \) is globally zero. This shows the other injection

\[
\text{Div}(X) \hookrightarrow H^0(X, \mathcal{M}^*/\mathcal{O}^*).
\]

\[\square\]

2.4. Line bundles. Recall that a line bundle \( L \) on \( X \) is a vector bundle of rank 1. Equivalently, it is a locally free sheaf of rank 1. Define the Picard group \( \text{Pic}(X) \) parameterizing isomorphism classes of line bundles on \( X \). The group law is given by tensor product. We can interpret \( \text{Pic}(X) \) as a cohomology group.

**Proposition 2.10.** \( \text{Pic}(X) \cong H^1(X, \mathcal{O}^*) \).

**Proof.** Take an open covering \( U = \{U_\alpha\} \) of \( X \) with respect to the trivialization of a line bundle \( L \). The transition function

\[
g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}
\]

can be regarded as a section of \( \mathcal{O}^*(U_\alpha \cap U_\beta) \), satisfying

\[
g_{\alpha\beta} \cdot g_{\beta\gamma} = 1,
\]

\[
g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1.
\]

Therefore, \( \{g_{\alpha\beta}\} \) is a cocycle in \( C^1(U, \mathcal{O}^*) \), hence it represents a cohomology class in \( H^1(X, \mathcal{O}^*) \).

Suppose \( M \) is another line bundle with transition functions \( \{h_{\alpha\beta}\} \). If \( M \) and \( L \) are isomorphic, then \( L \otimes M^* \) is trivial, i.e. \( \{g_{\alpha\beta}/h_{\alpha\beta}\} \) are transition functions of \( L \otimes M^* \), which has a nowhere vanishing section \( \sigma \). Suppose on \( U_\alpha \) we have \( \sigma_\alpha : U_\alpha \to \mathbb{C}^* \) as the restriction of \( \sigma \). Then on \( U_\alpha \cap U_\beta \) we have

\[
\frac{g_{\alpha\beta}}{h_{\alpha\beta}} \cdot \sigma_\alpha = \sigma_\beta.
\]
Therefore we conclude that
\[
\frac{g_{\alpha \beta}}{h_{\alpha \beta}} = \frac{\sigma_\beta}{\sigma_\alpha} \in \delta C^0(U, \mathcal{O}^*).
\]
\[
\square
\]

Now we describe another important correspondence between line bundles and divisors. Suppose \(D\) is a divisor on \(X\) with local defining equations \(\{f_\alpha \in \mathcal{M}^*(U_\alpha)\}\). Define
\[
g_{\alpha \beta} = \frac{f_\beta}{f_\alpha}.
\]
Then we have \(g_{\alpha \beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\). Moreover, \(\{g_{\alpha \beta}\}\) satisfy the compatibility conditions of transition functions, hence they define a line bundle, denoted by \(L = [D]\) or \(L = \mathcal{O}_X(D)\). We have a group homomorphism
\[
\text{Div}(X) \to \text{Pic}(X)
\]
induced by
\[
D + D' \mapsto [D] \otimes [D'].
\]
We say that \(D\) and \(D'\) are \textit{linearly equivalent}, if \([D]\) and \([D']\) are isomorphic line bundles. We denote linear equivalence by
\[
D \sim D'.
\]
The following result says that the kernel of the above map consists of principal divisors. In other words, two divisors \(D \sim D'\) if and only if \(D - D'\) is a principal divisor.

**Proposition 2.11.** The associated line bundle \([D]\) is trivial if and only if \(D\) is a principal divisor, i.e. \(D = (f)\) for some \(f \in \mathcal{M}^*(X)\).

**Proof.** Suppose \(D = (f)\) is the associated divisor of a meromorphic function \(f\) on \(X\). Then \(D\) has local defining equations \(\{f_\alpha = f|_{U_\alpha}\}\). The transition functions associated to \([D]\) are all equal to 1, hence \([D]\) is a trivial line bundle. Conversely, suppose \([D]\) is trivial. Then it has a nowhere vanishing section \(\sigma\) whose restriction to \(U_\alpha\) is denoted by \(\sigma_\alpha\). The transition functions \(g_{\alpha \beta} = f_\beta/f_\alpha\) defined above satisfy
\[
g_{\alpha \beta} \cdot \sigma_\alpha = \sigma_\beta,
\]
hence we have
\[
\frac{f_\alpha}{\sigma_\alpha} = \frac{f_\beta}{\sigma_\beta} \in \mathcal{M}^*(U_\alpha \cap U_\beta).
\]
We can glue \(\{f_\alpha/\sigma_\alpha\}\) to form a global function \(f \in \mathcal{M}^*(X)\). Since \(\sigma\) is nowhere vanishing, we obtain that \((f) = D\). \qed

Let us summarize the above discussion using the short exact sequence
\[
0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0.
\]
Recall that \(\text{Div}(X) \cong H^0(X, \mathcal{M}^*/\mathcal{O}^*)\) and \(\text{Pic}(X) \cong H^1(X, \mathcal{O}^*)\). Then we have the long exact sequence
\[
0 \to H^0(X, \mathcal{O}^*) \to H^0(X, \mathcal{M}^*) \overset{(1)}{\to} \text{Div}(X) \overset{[\cdot]}{\to} \text{Pic}(X) \to \cdots
\]
which encodes the information in the above discussion.
2.5. **Sections of a line bundle.** Let \( L \) be a line bundle on \( X \) with transition functions \( \{g_{\alpha\beta}\} \). A **holomorphic section** \( s \) of \( L \) is a collection \( \{s_\alpha \in \mathcal{O}(U_\alpha)\} \), satisfying that

\[
g_{\alpha\beta}s_\alpha = s_\beta.
\]

Conversely, a collection \( \{s_\alpha \in \mathcal{O}(U_\alpha)\} \) such that \( s_\beta/s_\alpha = g_{\alpha\beta} \) determines a section of \( L \).

Similarly, we define a **meromorphic section** \( s \) to be a collection

\[
\{s_\alpha \in \mathcal{M}(U_\alpha)\}
\]

such that \( g_{\alpha\beta}s_\alpha = s_\beta \). Suppose \( t \neq 0 \) is another meromorphic section with collection \( \{t_\alpha\} \). We have

\[
\frac{s_\beta}{t_\beta} = \frac{s_\alpha}{t_\alpha},
\]

hence the quotient \( s/t \) is a global meromorphic function. Conversely, if \( f \) is a global meromorphic function, then \( \{f \cdot s_\alpha\} \) defines another meromorphic section of \( L \).

**Proposition 2.12.** For any nontrivial section \( s \) of \( L \), we have \( L \cong [\langle s \rangle] \). A line bundle \( L \) is associated to a divisor \( D \) if and only if it has a meromorphic section \( s \) such that \( \langle s \rangle = D \). In particular, \( L \) has a holomorphic section if and only if it is associated to an effective divisor.

**Proof.** Suppose \( L \) has transition functions \( \{g_{\alpha\beta}\} \). For a meromorphic section \( s \neq 0 \), consider the divisor \( \langle s \rangle \) associated to the local sections \( s_\alpha \) in \( U_\alpha \). Since

\[
g_{\alpha\beta}s_\alpha = s_\beta,
\]

\( \{s_\alpha\} \) form a global divisor \( D = \langle s \rangle \) on \( X \).

Since the local defining equations of \( D \) are given by \( \{s_\alpha\} \), the transition functions of the associated line bundle \( [D] \) are \( \{g_{\alpha\beta} = s_\beta/s_\alpha\} \), which equal to those of \( L \), and consequently the collection \( \{s_\alpha\} \) gives rise to a meromorphic section of \( L \equiv [D] \).

Note that for a section \( s \) of \( L \), the divisor \( \langle s \rangle \) is effective if and only if \( s \) is a holomorphic section. \( \square \)

Now we consider a line bundle as a locally free sheaf of rank 1 and reinterpret the above correspondence. Let \( D \) be a divisor on \( X \). Define a sheaf \( \mathcal{O}_X(D) \) or simply \( \mathcal{O}(D) \) by

\[
\mathcal{O}(D)(U) = \{f \in \mathcal{M}(U) : (f) + D|_U \geq 0\}.
\]

It has a vector space structure, since \((f) + D|_U \geq 0\) and \((g) + D|_U \geq 0\) implies that \((af + bg) + D|_U \geq 0\) for \( a, b \in \mathbb{C} \).

**Proposition 2.13.** The space of holomorphic sections of \([D]\) can be identified with \( H^0(X, \mathcal{O}(D)) \).

**Proof.** A global section \( s \in H^0(X, \mathcal{O}(D)) \) is a meromorphic function satisfying that

\[
\langle s \rangle + D \geq 0.
\]

Suppose \( D \) is locally defined by \( \{f_\alpha\} \). The associated line bundle \([D]\) has transition functions

\[
\{g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}\}.
\]
Then the collection \( \{ s \cdot f_\alpha \} \) defines a section \( \sigma \) of \([D]\). Since
\[
(s) + (f_\alpha) \geq 0
\]
in every \( U_\alpha \), \( \sigma \) is a holomorphic section of \([D]\). Moreover, the associated divisor \( D' = (s) + D \) of the section is linearly equivalent to \( D \), since \( D' - D = (s) \) is a principal divisor.

Conversely, given a holomorphic section \( \sigma \) of \([D]\), i.e. a collection \( \{ h_\alpha \in \mathcal{O}(U_\alpha) \} \) such that
\[
\frac{h_\beta}{h_\alpha} = g_{\alpha \beta} = \frac{f_\beta}{f_\alpha}.
\]
Then \( \{ h_\alpha/f_\alpha \} \) defines a global meromorphic function \( g \). Since \( (h_\alpha) \geq 0 \) in every \( U_\alpha \), we have
\[
(g|_{U_\alpha}) + (f_\alpha) = (h_\alpha) \geq 0,
\]
hence \( (g) + D \geq 0 \) globally on \( X \) and \( g \in H^0(X, \mathcal{O}(D)) \).

\textbf{Remark 2.14.} Replacing \( X \) by any open subset \( U \), the proposition implies that the sheaf \( \mathcal{O}(D) \) can be regarded as gathering local holomorphic sections of the line bundle \([D]\). If \( D \sim D' \), i.e. \( D' - D = (f) \) for a global meromorphic function \( f \), then for any \( g \in \mathcal{O}(D')(U) \), we have
\[
0 \leq (g) + D' = (g) + (f) + (D) = (fg) + (D)
\]
restricted to \( U \). So we obtain an isomorphism
\[
\mathcal{O}(D')(U) \cong \mathcal{O}(D)(U)
\]
for any open subset \( U \), compatible with the sheaf restriction maps. In this sense, the sheaf \( \mathcal{O}(D) \) and the line bundle \([D]\) have a one-to-one correspondence up to isomorphism, assuming that every line bundle can be associated to a divisor.

Let \( |D| \) be the set of effective divisors that are linearly equivalent to \( D \). We call \( |D| \) the \textit{linear system} associated to \( D \).

\textbf{Proposition 2.15.} Let \( X \) be compact and \( D \) a divisor on \( X \). Then we have
\[
\mathbb{P}H^0(X, \mathcal{O}(D)) = |D|,
\]
i.e. an effective divisor in \([D]\) and a holomorphic section of \([D]\) (up to scale) determine each other.

\textit{Proof.} For any \( D' \in |D| \), by definition \( D' - D = (f) \) is principal for some \( f \in \mathcal{M}(X) \), hence \( (f) + D = D' \geq 0 \) and \( f \in H^0(X, \mathcal{O}(D)) \). Since \( X \) is compact, if \( g \) is another function such that \( D' - D = (g) \), then \( (f/g) = 0 \), i.e. \( f/g \) is holomorphic and nowhere vanishing, hence it is a non-zero constant since \( X \) is compact.

Conversely, any \( f \in H^0(X, \mathcal{O}(D)) \) defines an effective divisor \( D' = (f) + D \). If \( (f) + D = (g) + D \), then \( (f/g) = 0 \) and \( f/g \) is a non-zero constant. \( \square \)

\textbf{Exercise 2.16.} Let \( D = \sum a_i p_i \) be a divisor on \( \mathbb{P}^1 \) with \( a_i \in \mathbb{Z} \) and \( p_i \in \mathbb{P}^1 \). Define the \textit{degree} of \( D \) by \( \deg(D) = \sum a_i \).

(1) Prove that \( D \sim D' \) if and only if \( \deg(D) = \deg(D') \).

(2) Calculate the cohomology \( H^r(\mathbb{P}^1, \mathcal{O}(D)) \) in terms of \( \deg(D) \).

3. \textbf{Algebraic curves}

We identify (compact) Riemann surfaces with smooth complex algebraic curves.
3.1. The Riemann-Roch formula. Let $X$ be a compact connected Riemann
surface. Define its arithmetic genus by
$$g := h^1(\mathcal{O}_X) = \dim_{\mathbb{C}} H^1(\mathcal{O}_X).$$

**Theorem 3.1** (Riemann-Roch Formula). Let $D$ be a divisor on $X$ and $\mathcal{O}(D)$ the
associated line bundle (or invertible sheaf). Then we have
$$h^0(\mathcal{O}(D)) - h^1(\mathcal{O}(D)) = 1 - g + \deg(D).$$

**Remark 3.2.** Define the (holomorphic) Euler characteristic of a sheaf $\mathcal{F}$ by
$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i h^i(\mathcal{F}).$$
Then the Riemann-Roch formula can be written as
$$\chi(\mathcal{O}(D)) - \chi(\mathcal{O}_X) = \deg(D).$$

**Proof.** Let us first prove it for effective divisors of degree $\geq 0$. Do induction on $n$. The
formula obviously holds for $\mathcal{O}_X$. Suppose it is true for $\deg(D) < n$. Consider
$D = p + D'$ with $D'$ an effective divisor of degree $n - 1$. We have the short exact
sequence
$$0 \to \mathcal{O}(D') \to \mathcal{O}(D) \to \mathbb{C}_p \to 0,$$
where $\mathbb{C}_p$ is the skyscraper sheaf with one-dimensional stalk supported at $p$.

The exactness can be easily checked. The map $\mathcal{O}(D') \to \mathcal{O}(D)$ is an inclusion,
since $(f) + D' \geq 0$ implies that $(f) + D = (f) + D' + p \geq 0$. For the quotient sheaf,
if $\text{ord}_p(D') = m$, we can write $f = z^{-m}h(z)$ for $f \in \mathcal{O}(D')(U)$ and $f = z^{-m-1}h(z)$
for $f \in \mathcal{O}(D)(U)$ where $U$ is a small neighborhood of $\{p; z = 0\}$ and $h \in \mathcal{O}(U)$. So
the quotient sheaf is given by $\mathbb{C} \cdot \{z^{-m-1}\} \cong \mathbb{C}$ supported at $p$. Since the associated
cohomology sequence is long exact, we have
$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D')) + 1 = 1 - g + (n - 1) + 1 = 1 - g + n.$$

In general, write a divisor $D = D_1 - D_2$, where $D_1$ and $D_2$ are effective divisors
of degree $d_1$ and $d_2$, respectively, and $d_1 - d_2 = \deg(D)$. By the same idea, we have
the short exact sequence
$$0 \to \mathcal{O}(D) \to \mathcal{O}(D_1) \to \mathbb{C}^{d_2} \to 0.$$ Then we obtain that
$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D_1)) - d_2 = 1 - g + d_1 - d_2 = 1 - g + \deg(D).$$

**Remark 3.3.** Assuming the Serre duality
$$H^1(\mathcal{O}(D)) \cong H^0(K \otimes \mathcal{O}(-D)),$$
where $K$ is the canonical line bundle of $X$, then we can rewrite the Riemann-Roch
formula as
$$h^0(L) - h^0(K \otimes L^*) = 1 - g + \deg(L),$$
where $L$ is a line bundle on $X$. Note that $K$ has degree $2g - 2$ (to be discussed later). We thus conclude that
$$h^0(K) = g, \quad h^1(K) = h^0(\mathcal{O}) = 1.$$ It implies that the space of holomorphic one-forms on a genus $g$ Riemann surface
is $g$-dimensional.
3.2. The Riemann-Hurwitz formula. A branched cover \( \pi: X \to Y \) between two (compact and connected) Riemann surfaces is a (surjective) holomorphic map. For a general point \( q \in Y \), \( \pi^{-1}(q) \) consists of \( d \) distinct points, and we call \( d \) the degree of \( \pi \). Locally at \( p \mapsto q \), if the map is given by \( x \mapsto y = x^m \), where \( x, y \) are local coordinates of \( p, q \), respectively, then \( m \) is called the vanishing order of \( \pi \) at \( p \), and denote it by \( \text{ord}_p(\pi) = m \).

If \( \text{ord}_p(\pi) > 1 \), we say that \( p \) is a ramification point. If \( \pi^{-1}(q) \) contains a ramification point, then \( q \) is called a branch point. Define the pullback divisor
\[
\pi^*(q) = \sum_{p \in \pi^{-1}(q)} \text{ord}_p(\pi) \cdot p \in \text{Div}(X).
\]

Note that \( \pi^*(q) \) is an effective divisor of degree \( d \) in \( X \).

**Theorem 3.4 (Riemann-Hurwitz Formula).** Let \( \pi: X \to Y \) be a branched cover of Riemann surfaces. Then we have
\[
K_X \sim \pi^*K_Y + \sum_{p \in X} (\text{ord}_p(\pi) - 1) \cdot p,
\]
where \( K_X \) and \( K_Y \) are canonical divisor classes of \( X \) and \( Y \), respectively.

**Proof.** Recall that the canonical bundle as a sheaf is the sheaf of holomorphic one-forms. Take a one-form \( \omega \) on \( Y \) locally given by \( f(w)dw \) at a point \( q = \pi(p) \). Suppose the map \( \pi \) at \( p \) is given by \( z \mapsto w = z^m \),
then we have
\[
\pi^*(f(w)dw) = mf(z^m)z^{m-1}dz.
\]
Namely, the associated divisors satisfy the relation
\[
(\pi^*\omega)|_U = (\pi^*(\omega))|_U + (\text{ord}_p(\pi) - 1) \cdot p
\]
in a neighborhood \( U \) of \( p \). So globally it implies that
\[
(\pi^*\omega) = \pi^*\omega + \sum_{p \in X} (\text{ord}_p(\pi) - 1) \cdot p.
\]

Since \( \pi^*\omega \) is a one-form on \( X \), \( \pi^*\omega \) is a canonical divisor of \( X \) and the claimed formula follows. \( \Box \)

We can interpret the (numerical) Riemann-Hurwitz formula from a topological viewpoint. Let \( \chi(X) \) denote the topological Euler characteristic of \( X \). If \( X \) is a Riemann surface of genus \( g \), take a triangulation of \( X \) and suppose the number of \( k \)-dimensional edges is \( c_k \) for \( k = 0, 1, 2 \). Then we have
\[
\chi(X) = c_0 - c_1 + c_2 = 2 - 2g.
\]

**Proposition 3.5.** Let \( \pi: X \to Y \) be a degree \( d \) branched cover of Riemann surfaces. Then we have
\[
\chi(X) = d \cdot \chi(Y) - \sum_{p \in X} (\text{ord}_p(\pi) - 1).
\]
Proof. Take a cell decomposition of $Y$ such that every branch point is a vertex. Pull it back as a cell decomposition of $X$. Note that it pulls back a face to $d$ faces, an edge to $d$ edges and a vertex $v$ to $|\pi^{-1}(v)|$ vertices. Moreover if

$$\pi^{-1}(v) = \sum_{i=1}^{k} m_i p_i$$

for distinct points $p_i$, then $|\pi^{-1}(v)| = k$. In other words, we have

$$|\pi^{-1}(v)| = d - \sum_{p \in \pi^{-1}(v)} (\operatorname{ord}_p(\pi) - 1).$$

Then the claimed formula follows right away. □

**Corollary 3.6** (Numerical Riemann-Hurwitz Formula). Let $\pi: X \to Y$ be a degree $d$ branched cover of Riemann surfaces of genus $g$ and $h$, respectively. Then we have

$$2g - 2 = d(2h - 2) + \sum_{p \in X} (\operatorname{ord}_p(\pi) - 1).$$

In particular, if $g < h$, such a branched cover does not exist.

**Corollary 3.7.** The canonical bundle of a genus $g$ Riemann surface $X$ has degree equal to $2g - 2$.

Proof. Every Riemann surface $X$ possesses a non-trivial meromorphic function, say by the Riemann-Roch formula. It induces a branched cover $\pi: X \to \mathbb{P}^1$ of some degree $d$. By the Riemann-Hurwitz Formula, we know that

$$\deg(K_X) = d(-2) + \sum_{p \in X} (\operatorname{ord}_p(\pi) - 1),$$

since we have seen that $\deg(K_{\mathbb{P}^1}) = -2$. By the Numerical Riemann-Hurwitz Formula, we have

$$2 - 2g = 2d - \sum_{p \in X} (\operatorname{ord}_p(\pi) - 1).$$

Then the claim follows immediately. □

**Exercise 3.8.** Write down an explicit one-form on $\mathbb{P}^1$, analyze its zeros and poles, and use the information to determine the degree of $K_{\mathbb{P}^1}$.

**Exercise 3.9.** Let $X$ be a Riemann surface of genus $g$. If $X$ admits a branched cover of degree two to $\mathbb{P}^1$, we say that $X$ is hyperelliptic. Prove that every Riemann surface of $g \leq 2$ is hyperelliptic.

### 3.3. Genus formula of plane curves.

Suppose $F(Z_0, Z_1, Z_2)$ is a general degree $d$ homogeneous polynomial whose vanishing locus is a complex curve $C \subset \mathbb{P}^2$. Since $F$ is general, $C$ is a (smooth) Riemann surface. More precisely, the singularities of $C$ locate at the common zeros of $F = 0$ and $\partial F / \partial Z_i = 0$ for all $i$, which are empty for a general $F$. On the other hand, if $F$ is special, then $C$ can be singular. For example, let $F = Z_0^2 - Z_1^2$. Then $C$ is a union of two lines, hence has a (nodal) singularity at the intersection of the two lines. If $F = Z_0 Z_1^2 - Z_2^3$. Then $C$ has a (cuspidal) singularity at $[1, 0, 0]$.

**Exercise 3.10.** A plane cubic is a plane curve of degree three. Find a plane cubic such that it has a unique singular point, and moreover, such that the singularity has a unique branch (i.e. not several branches locally meeting at the point).
Theorem 3.11. In the above setting, the genus $g$ of $C$ is given by

$$g = \frac{(d - 1)(d - 2)}{2}.$$ 

Proof. We give two proofs. The first one is more algebraic. Suppose $C_1$ and $C_2$ are two plane curves of degree $d$, defined by $F_1$ and $F_2$. Then $F_1/F_2$ is a meromorphic function on $\mathbb{P}^2$, hence $C_1$ and $C_2$ are linearly equivalent. It follows that all degree $d$ plane curves are linearly equivalent. Hence it makes sense to use $\mathcal{O}_{\mathbb{P}^2}(d)$ to denote the line bundle associated to a plane curve of degree $d$. In particular, $\mathcal{O}(1)$ is the line bundle associated to a line $L$ in $\mathbb{P}^2$. The ideal sheaf of $L$ has sections given by holomorphic functions vanishing along $L$, hence it can be identified with $\mathcal{O}(−L) = \mathcal{O}(−1)$, the dual of $\mathcal{O}(1)$. Then we have the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(−1) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_L \to 0.$$ 

Tensor it with $\mathcal{O}_{\mathbb{P}^2}(1−m)$. We obtain that

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1−m) \to \mathcal{O}_{\mathbb{P}^2}(−(m − 1)) \to \mathcal{O}_{\mathbb{P}^2}(1−m)|_L \to 0.$$ 

Since $\mathcal{O}_{\mathbb{P}^2}(1−m)|_L$ is the line bundle associated to a degree $1 − m$ divisor on $L \cong \mathbb{P}^1$, we conclude that

$$\chi(\mathcal{O}_{\mathbb{P}^2}(1−m)) − \chi(\mathcal{O}_{\mathbb{P}^2}(−m)) = \chi(\mathcal{O}_{\mathbb{P}^2}(1−m)) = 2 − m,$$

where we apply the Riemann-Roch formula to $\mathbb{P}^1$ in the last equality. Then we obtain that

$$\chi(\mathcal{O}_{\mathbb{P}^2}) − \chi(\mathcal{O}_{\mathbb{P}^2}(−d)) = \sum_{m=1}^{d} (2 − m) = -\frac{d(d−3)}{2}.$$ 

Now by the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(−d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0,$$

we have

$$1 − g = \chi(\mathcal{O}_C) = -\frac{d(d−3)}{2},$$

hence the genus formula follows. Here we assumed that for any line bundle $L$ on a Riemann surface $X$, we have $h^i(X, L) = 0$ for $i \geq 2$.

The other proof is an application of the Riemann-Hurwitz formula. Without loss of generality, suppose $o = [0, 0, 1] \notin C$. Let $L$ be the line cut out by $Z_2 = 0$. Project $C$ to $L$ from $o$, i.e.,

$$[Z_0, Z_1, Z_2] \mapsto [Z_0, Z_1].$$

In affine coordinates $x = Z_1/Z_0$ and $y = Z_2/Z_0$, this map is given by vertical projection

$$(x, y) \mapsto x,$$

i.e., we project $C$ vertically to the $x$-axis. This yields a degree $d$ branched cover

$$\pi: C \to L \cong \mathbb{P}^1.$$ 

A point $p$ is a ramification point of $\pi$ if and only if there exists a vertical line tangent to $C$ at $p$, i.e., $p$ is a common zero of $F$ and $\partial F/\partial Z_2$. Since $F$ and $\partial F/\partial Z_2$ have degree $d$ and $d − 1$, respectively, they intersect at $d(d−1)$ points. By Riemann-Hurwitz, we have

$$2g − 2 = d(−2) + d(d−1),$$
hence the genus formula follows. In order to ensure that all of the ramification points are simple, we can choose a general projection direction such that it is different from those of the (finitely many) lines with higher tangency order to $C$. □

**Remark 3.12.** In the first proof, indeed we did not use the smoothness assumption of $C$. So the (arithmetic) genus formula holds for an arbitrary plane curve, even if it is singular. Similarly in the second proof, even if the projection has higher ramification points, a detailed local study plus Riemann-Hurwitz can provide the same formula.

**Exercise 3.13.** A conic is a smooth plane curve of degree two. Let $C$ be the conic defined by $Z_1Z_2 - Z_0^2 = 0$. Prove directly that $C$ is isomorphic to $\mathbb{P}^1$.

3.4. **Base point free and very ample line bundles.** Let $L$ be a line bundle on a complex manifold $X$. We say that $L$ has a base point at $p \in X$ if $p$ belongs to the vanishing locus of every holomorphic section of $L$. If the base locus of $L$ is empty, then $L$ is called base point free.

For a base point free line bundle $L$, let $\sigma_0, \ldots, \sigma_n$ be a basis of the space $H^0(X, L)$ of holomorphic sections. Locally at a point $p \in X$, consider $\sigma_i$ as a holomorphic function and associate to $p$ the point

$$[\sigma_0(p), \ldots, \sigma_n(p)] \in \mathbb{P}^n.$$ 

This is well-defined, since if we take a different chart for the trivialization of $L$, we get the same point

$$[g_{\alpha\beta}\sigma_0(p), \ldots, g_{\alpha\beta}\sigma_n(p)] \in \mathbb{P}^n$$

where $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ are transition functions of $L$. Therefore, we obtain a holomorphic map

$$\phi_L : X \to \mathbb{P}^n.$$ 

Note that different choices of basis of $H^0(X, L)$ give maps that differ by the action of $\text{PGL}(n+1)$, hence this map is intrinsically defined up to automorphisms of $\mathbb{P}^n$.

**Remark 3.14.** We can give a more conceptual and coordinate free description of $\phi_L$. Since $L$ is base point free, the space of holomorphic sections $\sigma$ vanishing at $p$ forms a hyperplane $H_p \subset H^0(X, L) \cong \mathbb{C}^{n+1}$. Then one can define $\phi_L(p) = [H_p] \in (\mathbb{P}^n)^*$ in the dual projective space that parameterizes hyperplanes.

**Proposition 3.15.** Let $X$ be a compact complex manifold. Then there is a one-to-one correspondence between (the pullback of) hyperplane sections of $X$ and effective divisors in the linear system $|L|$.

**Proof.** This is just a reformulation of the one-to-one correspondence

$$|L| = \mathbb{P}H^0(X, L),$$

which we proved before. In other words, an effective divisor in $|L|$ uniquely determines a holomorphic section $\sigma = \sum_{i=0}^n a_i\sigma_i$ up to scaling, which defines a hyperplane in $\mathbb{P}H^0(X, L)$. □

**Example 3.16.** If $L = \mathcal{O}$, then $H^0(X, \mathcal{O}) = \mathbb{C}$, hence $\phi_{\mathcal{O}}$ maps $X$ to a point.
Example 3.17. Let $X = \mathbb{P}^1$ and $L = \mathcal{O}(2p)$ where $p = [0, 1]$. Then $H^0(\mathbb{P}^1, L)$ is 3-dimensional and we can choose a basis as

$$1, \quad \frac{Y}{X}, \quad \frac{Y^2}{X^2}.$$ 

Hence we obtain that

$$\phi_L([X,Y]) = [X^2, XY, Y^2],$$

which is a smooth conic in $\mathbb{P}^2$, i.e. the image is cut out by $Z_0Z_2 - Z_1^2 = 0$. The genus formula for plane curves also implies that the image has genus zero.

Exercise 3.18. A submanifold $X \subset \mathbb{P}^n$ is called non-degenerate if it is not contained in any hyperplane. If $X$ is isomorphic to $\mathbb{P}^1$, we call it a smooth rational curve. For any complex curve $X$ in $\mathbb{P}^n$, the intersection number of $X$ with a general hyperplane is called the degree of $X$.

1) Show that any non-degenerate smooth rational curve in $\mathbb{P}^n$ has degree $\geq n$.

2) For $d \geq n \geq 3$, show that there exist non-degenerate smooth degree $d$ rational curves in $\mathbb{P}^n$.

Example 3.19. Let $E$ be a complex curve of genus one, i.e. an elliptic curve, and $L = \mathcal{O}(2p)$ for a point $p \in E$. By Riemann-Roch, $h^0(E, L) = 2$. We claim that $L$ is base point free. Otherwise if $q$ is a base point, then $q$ has to be $p$ and there exists another effective divisor $p + r \in |2p|$ such that $p + r \sim 2p$. But this implies that $r - p$ is a principal divisor, whose associated meromorphic function induces a one-to-one isomorphism $E \cong \mathbb{P}^1$, leading to a contradiction. We thus conclude that $\phi_L: E \to \mathbb{P}^1$ is a branched cover of degree two. Two points $s$ and $t$ lie in the same fiber of $\phi_L$ if and only if $s + t \sim 2p$.

The above example indicates that $\phi_L$ is not always an embedding. We say that $L$ is very ample if $\phi_L$ is an embedding and that $L$ is ample if $L^\otimes m$ is very ample for some $m > 0$. We give a criterion for base point free and very ample line bundles.

Proposition 3.20. Let $L$ be a line bundle on a Riemann surface $X$.

1) $L$ is base point free if and only if

$$h^0(X, L \otimes \mathcal{O}(-p)) = h^0(X, L) - 1$$

for all $p \in X$.

2) $L$ is very ample if and only $L$ is base point free and moreover for all $p, q \in X$ (not necessarily distinct)

$$h^0(X, L \otimes \mathcal{O}(-p - q)) = h^0(X, L \otimes \mathcal{O}(-p)) - 1 = h^0(X, L \otimes \mathcal{O}(-q)) - 1.$$

Proof. Treat $L$ as a locally free sheaf of rank one. By the short exact sequence

$$0 \to L \otimes \mathcal{O}(-p) \to L \to \mathcal{O}_p \to 0,$$

we have

$$h^0(X, L) - 1 \leq h^0(X, L \otimes \mathcal{O}(-p)) \leq h^0(X, L).$$

Then $L$ has a base point at $p$ if and only if all holomorphic sections of $L$ vanish at $p$, i.e., if and only if $H^0(X, L \otimes \mathcal{O}(-p)) = H^0(X, L)$. This proves (1).

For (2), a very ample line bundle is necessarily base point free by definition. If $p \neq q \in X$ have the same image under $\phi_L$, it is equivalent to saying that the subspace of sections vanishing at $p$ is the same as the subspace of sections vanishing at $q$, which is further equivalent to

$$h^0(X, L \otimes \mathcal{O}(-p)) = h^0(X, L \otimes \mathcal{O}(-p - q)) = h^0(X, L \otimes \mathcal{O}(-q)).$$
Moreover, \( \phi_L \) induces an injection restricted to the tangent space \( T_p(X) \) if and only if there exists a hyperplane such that it cuts out \( X \) locally a simple point at \( p \), namely, if and only if there is a section vanishing at \( p \) with multiplicity one, i.e.,

\[
h^0(X, L \otimes \mathcal{O}(-2p)) < h^0(X, L \otimes \mathcal{O}(-p)).
\]

But we have seen that

\[
h^0(X, L \otimes \mathcal{O}(-2p)) \geq h^0(X, L \otimes \mathcal{O}(-p)) - 1
\]

(using \( L \otimes \mathcal{O}(-p) \) as \( L \) before). Hence (2) follows from combining the two cases. \( \square \)

**Remark 3.21.** In (2), for \( p \neq q \) the condition geometrically means that the sections of \( L \) separate any two distinct points. When \( p = q \), the condition means that the sections of \( L \) separate tangent vectors at \( p \).

**Exercise 3.22.** Prove that the line bundle \( \mathcal{O}(d) \) is very ample on \( \mathbb{P}^1 \) if and only if \( d > 0 \). (In this case the induced map \( \phi \) embeds \( \mathbb{P}^1 \) into \( \mathbb{P}^d \) as a degree \( d \) smooth rational curve, which is called a *rational normal curve*.)

**Example 3.23.** Let \( E \) be an elliptic curve, i.e., a Riemann surface of genus one. Fix a point \( p \in E \). Consider the morphism

\[
\tau: E \to \text{Pic}^0(E)
\]

by \( \tau(q) = [q - p] \), where \( \text{Pic}^0(E) \) is the Picard group of isomorphism classes of line bundles of degree zero. For \( q_1 \neq q_2 \), we have \( q_1 - p \neq q_2 - p \), hence \( \tau \) is injective. For any line bundle \( L \) of degree zero, by Riemann-Roch \( h^0(L \otimes \mathcal{O}(p)) \geq 1 \), hence \( L \otimes \mathcal{O}(p) \) has a section vanishing at a single point \( q \). It implies that \( L \cong [q - p] \), hence \( \tau \) is surjective. Therefore, we thus conclude that \( \tau \) is an isomorphism. This defines a group law on \( E \) with respect to \( p \), i.e., \( q + r = s \), where \( s \in E \) is the unique point satisfying that

\[
(q - p) + (r - p) \sim s - p.
\]

Now consider the linear system \( |3p| \) on \( E \). Since

\[
h^0(E, \mathcal{O}(3p)) = 3, \quad h^0(E, \mathcal{O}(2p)) = 2, \quad h^0(E, \mathcal{O}(p)) = 1,
\]

the above proposition implies that \( \mathcal{O}(3p) \) is very ample. It induces an embedding of \( E \) into \( \mathbb{P}^2 \) as a plane cubic curve. A line in \( \mathbb{P}^2 \) cuts out a divisor of degree three in \( E \), say, \( q + r + s \) (not necessarily distinct) if and only if

\[
q + r + s \sim 3p.
\]

Note that the tangent line \( L \) of \( E \) at \( p \) is a *flex line*, i.e., the contact order \( (L \cap E)_p = 3 \). Such \( p \) is called a *flex point*.

**Exercise 3.24.** Show that there are in total nine flex points in a smooth plane cubic curve.

Let \( V \subset |L| \) be a linear subspace. We say that \( V \) is a *linear series* of \( L \). The linear system \( |L| \) is also called a *complete linear series*. The above definitions and properties go through similarly for the induced map \( \phi_V \).

**Exercise 3.25.** Write down a linear series of \( |\mathcal{O}(3)| \) on \( \mathbb{P}^1 \) such that it maps \( \mathbb{P}^1 \) into \( \mathbb{P}^2 \) as a *singular* plane cubic curve. How many different types of such singular plane cubics can you describe? (up to automorphisms of \( \mathbb{P}^2 \))
3.5. **Canonical maps.** Let \( K \) be the canonical line bundle on a Riemann surface \( X \). If \( X \) is \( \mathbb{P}^1 \), \( \deg(K) = -2 \) and \( K \) is not effective. If \( X \) is an elliptic curve, then \( K \cong \mathcal{O} \) and the induced map \( \phi_K \) is onto a point. From now on, assume that the genus of \( X \) satisfies \( g \geq 2 \).

We say that \( X \) is **hyperelliptic** if it admits a branched cover of degree two to \( \mathbb{P}^1 \). Two points \( p, q \in X \) are called **conjugate** if they have the same image in \( \mathbb{P}^1 \). A ramification point of the double cover is called a **Weierstrass point**, i.e., it is self conjugate. By Riemann-Hurwitz, a hyperelliptic curve of genus \( g \geq 2 \) has \( 2g + 2 \) Weierstrass points.

**Lemma 3.26.** If \( X \) is a hyperelliptic curve of genus \( g \geq 2 \), then \( X \) admits a unique double cover of \( \mathbb{P}^1 \).

**Proof.** If the claim is not true, then there exist \( q \neq r \in X \) such that \( h^0(X, \mathcal{O}(p + q)) = 2 \) and \( h^0(X, \mathcal{O}(p + r)) = 2 \), which would induce two different double covers of \( \mathbb{P}^1 \). Let \( L = \mathcal{O}_X(p + q + r) \). If \( h^0(X, L) = 3 \), since \( h^0(X, L(-x - y)) \leq 1 \) and \( h^0(X, L(-x)) \leq 2 \) by degree reason, \( \phi_L \) would map \( X \) as a plane cubic of genus one, leading to a contradiction. Then we conclude that

\[
H^0(X, \mathcal{O}(p + q)) = H^0(X, \mathcal{O}(p + r)) = H^0(X, \mathcal{O}(p + q + r)),
\]

which implies that both \( q \) and \( r \) are base points of \( \mathcal{O}(p + q + r) \), hence \( h^0(X, \mathcal{O}(p)) = 2 \) and \( X \cong \mathbb{P}^1 \), leading to a contradiction. \( \square \)

**Proposition 3.27.** Let \( X \) be a Riemann surface of genus \( g \geq 2 \). Then the canonical line bundle \( K \) is base point free. Moreover, the induced map

\[
\phi_K : X \to \mathbb{P}^{g-1}
\]

is an embedding if and only if \( X \) is not hyperelliptic. If \( X \) is hyperelliptic, then \( \phi_K \) is a double cover of a rational normal curve in \( \mathbb{P}^{g-1} \) (i.e. \( \mathbb{P}^1 \) of degree \( g-1 \) embedded in \( \mathbb{P}^{g-1} \)).

**Proof.** First, we show that \( K \) is base point free. For any point \( p \in X \), by Riemann-Roch we have

\[
h^0(X, K \otimes \mathcal{O}(-p)) - h^0(X, \mathcal{O}(p)) = 1 - g + (2g - 3),
\]

\[
h^0(X, K \otimes \mathcal{O}(-p)) = g - 1 = h^0(X, K) - 1.
\]

Hence \( K \) satisfies the criterion of base point freeness.

Next, \( K \) fails to separate \( p \) and \( q \) (not necessarily distinct) if and only if

\[
h^0(X, K \otimes \mathcal{O}(-p - q)) = h^0(X, K \otimes \mathcal{O}(-p)) = g - 1,
\]

which is equivalent to, by Riemann-Roch again, that

\[
h^0(X, \mathcal{O}(p + q)) = 2.
\]

In other words, if and only if the linear system \( |p + q| \) induces a double cover \( X \to \mathbb{P}^1 \), i.e., \( X \) is hyperelliptic.

Finally, if \( X \) is hyperelliptic of genus \( g \geq 2 \), it admits a unique double cover of \( \mathbb{P}^1 \). By the above analysis, two points \( p \) and \( q \) have the same image under the canonical map \( \phi_K \) if and only if \( h^0(X, \mathcal{O}(p + q)) = 2 \), i.e., \( p \) and \( q \) are conjugate. Then \( \phi_K \) is a double cover of a rational curve of degree \( \deg(K)/2 = g - 1 \) in \( \mathbb{P}^{g-1} \), i.e., the
image of \( \phi_K \) a rational normal curve in \( \mathbb{P}^{g-1} \). A hyperplane section of \( \phi_K(X) \) pulls back to \( X \) a divisor of type 
\[
\sum_{i=1}^{g-1} (p_i + q_i),
\]
where \( p_i \neq q_i \) are either conjugate or \( p_i = q_i \) is a Weierstrass point. \( \square \)

**Remark 3.28.** If \( X \) is not hyperelliptic, \( \phi_K \) is called the canonical embedding of \( X \) and its image is called a canonical curve.

**Example 3.29.** Let \( X \) be a curve of genus two. Then \( h^0(X, K) = 2 \), hence \( X \) is hyperelliptic and the double cover of \( \mathbb{P}^1 \) in its definition is induced by the canonical line bundle, as we have seen.

**Example 3.30.** A non-hyperelliptic curve of genus three admits a canonical embedding to \( \mathbb{P}^2 \) as a plane quartic. An effective canonical divisor corresponds to a line section of the quartic. By the genus formula, any smooth plane quartic also has genus equal to three. Moreover, a smooth plane quartic \( X \) has a line bundle \( L \) of degree four by restricting \( \mathcal{O}_{\mathbb{P}^2}(1) \) to \( X \). By Riemann-Roch, \( h^0(X, K \otimes L^*) \geq 1 \), but \( \deg(K \otimes L^*) = 0 \), hence \( L \cong K \). We thus conclude that any smooth plane quartic is a canonical embedding of a non-hyperelliptic curve of genus three.

### 3.6. Dimension of linear systems

Let \( D = p_1 + \cdots + p_d \) be an effective divisor of degree \( d \) on a genus \( g \) smooth complex curve \( X \). Recall that the linear system \( |D| \) can be identified with \( \mathbb{P}H^0(X, \mathcal{O}(D)) \) parameterizing effective divisors linearly equivalent to \( D \). Suppose as a projective space the dimension is 
\[
r = \dim |D| = h^0(X, \mathcal{O}(D)) - 1.
\]
By Riemann-Roch and Serre Duality, we have 
\[
\dim |K \otimes \mathcal{O}(-D)| = r + g - d - 1.
\]
Note that \( |K \otimes \mathcal{O}(-D)| \) can be identified with the linear system of effective canonical divisors that contain \( D \). By the canonical map 
\[
\phi_K : X \to \mathbb{P}^{g-1},
\]
it says that the space of hyperplanes of \( \mathbb{P}^{g-1} \) that contain the points \( \phi_K(D) = \{ \phi_K(p_1), \ldots, \phi_K(p_d) \} \) is \( (r + g - d - 1) \)-dimensional. In other words, the linear subspace in \( \mathbb{P}^{g-1} \) spanned by \( \phi_K(p_1), \ldots, \phi_K(p_d) \) has dimension 
\[
(g - 2) - (r + g - d - 1) = (d - 1) - r.
\]
Since we expect in general \( d \) points to span a \( (d - 1) \)-dimensional linear subspace, geometrically it says that \( \phi_K(D) \) fails to impose 
\[
r = \dim |D|
\]
independent conditions. We summarize the discussion as the following geometric version of the Riemann-Roch formula.

**Theorem 3.31 (Geometric Riemann-Roch).** In the above setting, let \( \overline{\phi_K(D)} \) be the linear subspace in \( \mathbb{P}^{g-1} \) spanned by the image of \( D \) under the canonical map. Then we have 
\[
\dim |D| = \deg(D) - 1 - \dim \overline{\phi_K(D)}.
\]
Remark 3.32. Even if $D$ contains points with multiplicity, the above formulation still holds. Say, if $D$ contains $2p$, then $2p$ spans the tangent line at $p$. If $D$ contains $3p$, then $3p$ spans an osculating 2-plane at $p$, etc.

Example 3.33. Let us revisit the canonical embedding of a non-hyperelliptic curve $X$ of genus three in $\mathbb{P}^2$. Consider $D = p + q + r$. Then $h^0(\mathcal{O}(D)) \leq 2$, i.e. $\dim |D| = 1$ or 0. Note that $\dim |D| = 1$ if and only if
\[ \dim \phi_K(D) = 3 - 1 - 1 = 1 \]
by Geometric Riemann-Roch, i.e., if and only if $p$, $q$ and $r$ are collinear in $\mathbb{P}^2$.

Let us study in detail the dimension of a linear system.

Lemma 3.34. Let $D$ be a divisor on a smooth complex curve $X$. Then $\dim |D| \geq k$ if and only if for every $k$ points $p_1, \ldots, p_k \in X$ there exists an effective divisor in $|D|$ containing all of them.

Proof. First, suppose for every $k$ points $p_1, \ldots, p_k \in X$ there exists an effective divisor in $|D|$ containing all of them. Since $\sum_{i=1}^k p_i$ varies in a complex $k$-dimensional family, then $\dim |D| \geq k$ obviously holds. Alternatively, we can prove it by induction. Suppose it holds for $\leq k$. Assume for every $p_1, \ldots, p_{k+1}$, there exists $D' \in |D|$ containing all of them. Then we conclude that $\dim |D - p| \geq k$ for any $p \in X$. Choose a point $p$ not in the base locus of $|D|$. Consequently we have
\[ \dim |D| = \dim |D - p| + 1 \geq k + 1. \]

Conversely, suppose $\dim |D| \geq k$. Then we have
\[ h^0 \left( X, \mathcal{O} \left( D - \sum_{i=1}^k p_i \right) \right) \geq h^0(X, \mathcal{O}(D)) - k \geq 1. \]
It implies that there exists a meromorphic function $f$ such that
\[ (f) + D - \sum_{i=1}^k p_i \geq 0, \]
hence $(f) + D = D'$ is an effective divisor in $|D|$ containing $p_1, \ldots, p_k$. \hfill $\square$

Corollary 3.35. For any two effective divisors $D_1$ and $D_2$ in $X$, we have
\[ \dim |D_1| + \dim |D_2| \leq \dim |D_1 + D_2|. \]

Proof. Suppose $\dim |D_i| = k_i$ for $i = 1, 2$. Take any $k_1 + k_2$ points
\[ p_1, \ldots, p_{k_1}, q_1, \ldots, q_{k_2} \]
in $X$. By the above lemma, there exist $D'_1 \in |D_1|$ and $D'_2 \in |D_2|$ such that $D'_1$ contains all the $p_i$ and $D'_2$ contains all the $q_j$. Then $D'_1 + D'_2 \in |D_1 + D_2|$ contains all the $p_i, q_j$, hence we obtain that
\[ \dim |D_1 + D_2| \geq k_1 + k_2 \]
by using the lemma again. \hfill $\square$

Note that if $h^0(X, K \otimes (-D)) = 0$, then Riemann-Roch implies that
\[ h^0(X, \mathcal{O}(D)) = 1 - g + \deg(D). \]
Some subtlety occurs if
\[ h^0(X, K \otimes (-D)) > 0 \]
and we call such a divisor $D$ a \textit{special divisor} and the associated linear system $|D|$ a \textit{special linear system}. By Riemann-Roch, any divisor $D$ with $\deg(D) > 2g - 2$ is non-special. By Geometric Riemann-Roch, $D$ is non-special if and only if the linear span of $\phi_K(D)$ is the entire ambient space $\mathbb{P}^{g-1}$.

\textbf{Theorem 3.36} (Clifford’s Theorem). \textit{Let $D$ be an effective divisor such that $\deg(D) \leq 2g - 2$ on a genus $g$ smooth complex curve $X$. Then we have}

$$\dim |D| \leq \frac{1}{2} \cdot \deg(D).$$

\textit{Proof.} If $D$ is non-special, we have

$$\dim |D| = \deg(D) - g < \frac{1}{2} \deg(D).$$

If $D$ is special, then there exists an effective divisor $D'$ such that $D + D' \sim K$. By the above lemma we have

$$\dim |D| + \dim |D'| \leq \dim |K| = g - 1.$$  

By Riemann-Roch and Serre Duality, we have

$$\dim |D| - \dim |D'| = 1 - g + \deg(D).$$

The desired inequality follows by combining the two relations. \hfill \square

\textbf{Remark 3.37.} The above equality holds if $D = 0$, $D = K$ or $X$ is hyperelliptic. If $D = 0$ or $D = K$, one easily checks that the equality holds. If $X$ is hyperelliptic, we can take, say $D = p + q$, where $p$ and $q$ are conjugate and $\dim |p+q| = 1$. Indeed these are the only possibilities.

\textbf{Exercise 3.38.} Let $X$ be a hyperelliptic curve of genus $\geq 2$. For $0 < 2k \leq g$, find an effective divisor $D$ of degree $2k$ on $X$ such that $\dim |D| = k$. Classify all such divisors up to linear equivalence.

4. \textbf{Chern classes}

Chern classes are certain characteristic classes associated to vector bundles. They can be defined in various ways. Here our approach is to treat them as elements in the Chow ring of the base manifold.

4.1. \textbf{Chow ring and rational equivalence.} Let $X$ be an $n$-dimensional (projective) manifold (or with mild singularities). Let $Z^d(X)$ be the free abelian group generated by irreducible cycles (i.e. cut out by polynomial equations) of codimension $d$. Elements of $Z^d(X)$ are called $d$-cocycles or $(n-d)$-cycles.

Let $Y \subset X$ be a $(k+1)$-dimensional cycle. For a $k$-cycle $Z$, if there exists $f \in \mathcal{M}(Y)$ such that $(f) = Z$, we say that $Z$ is a principal $k$-cycle. Define rational equivalence $\sim$ between two $k$-cycles if their difference is a principal $k$-cycle. Define the \textit{Chow group} of codimension-$d$ cycle classes of $X$ as

$$A^d(X) = Z^d(X)/\sim$$

and use $[Z]$ to denote the equivalence class of $Z$ (or simply by $Z$). Sometimes we also write $A_{n-d}(X) = A^d(X)$ for the Chow group of $(n-d)$-dimensional cycle classes.
Remark 4.1. Geometrically, if two $k$-cycles $Z$ and $W$ are rationally equivalent, then $f$ induces a map $Y \to \mathbb{P}^1$ such that the fibers over 0 and $\infty$ are $Z$ and $W$, respectively.

**Proposition 4.2.** Let $X$ be an $n$-dimensional irreducible manifold. Then we have

1. $A^0(X) \cong \mathbb{Z}$ is generated by $[X]$.
2. $A^d(X) = 0$ for $d > n$.
3. $A^1(X) \cong \text{Pic}(X)$.
4. There exists a degree map $A^n(X) \to \mathbb{Z}$.

**Proof.** The first three claims hold by definition. For (4), note that $A^n(X)$ is generated by rational equivalence classes of points. Since the degree of a principal divisor on an algebraic curve is zero, the degree map $Z^n(X) \to \mathbb{Z}$ factors through $A^n(X)$.

Remark 4.3. Rational equivalence can be subtle even for 0-dimensional cycles. For instance, any two points in $\mathbb{P}^1$ are rationally equivalent. But two distinct points in a Riemann surface of positive genus are not rationally equivalent.

Example 4.4. One can show that any two $k$-dimensional linear subspaces of $\mathbb{P}^n$ are rationally equivalent. Moreover, any two hypersurfaces of the same degree are rationally equivalent in $\mathbb{P}^n$. In general, let $Y$ be a cycle in $\mathbb{P}^n$ of degree $k$, i.e. it intersects a general linear subspace of complementary dimension at $k$ points. Take a general point $p$ and project $Y$ from $p$ to a hyperplane $\mathbb{P}^{n-1}$. We can choose $p$ such that the projection $\pi: Y \to \mathbb{P}^{n-1}$ is generically finite, i.e. for a general point $q$ in the image of $\pi$, the inverse of $q$ consists of finitely many points. Suppose $p$ has coordinate $[1,0,\ldots,0]$ and the hyperplane is given by $X_0 = 0$. Then $\pi$ is given by $[X_0,\ldots,X_n] \mapsto [0,X_1,\ldots,X_n]$.

It implies that $[X_0,\ldots,X_n] \in Y$ varies in a $\mathbb{P}^1$-family $[X_0,X_1,\ldots,X_n] \in Y_t$ to its projection image as $t$ varies from 1 to 0. Then we conclude that $Y$ is rationally equivalent to $\text{deg}(\pi) \cdot \pi(Y)$.

Let $[H]$ be the hyperplane class of $\mathbb{P}^n$. By induction on $n$ we thus know that $A^d(\mathbb{P}^n) = \mathbb{Z}$ generated by $[H^d]$, where $[H^d]$ is the class of a linear subspace of codimension $d$. This notation makes sense because we can take $d$ (general) hyperplanes $H_1,\ldots,H_d$ such that their intersection is a linear subspace of codimension $d$. Moreover, $[H_i] = [H]$ for all $i$.

This example suggests that the union of $A^d(X)$ for all $d$ possesses a graded ring structure induced by intersection product between cycles.

**Exercise 4.5.** Prove directly that two linear subspaces of $\mathbb{P}^n$ of the same dimension are rationally equivalent.

Let $V$ and $W$ be two irreducible subvarieties. If $\text{codim}(V \cap W) = \text{codim}(V) + \text{codim}(W)$, we say that $V$ and $W$ intersect properly. This induces an intersection product $A^d(X) \times A^e(X) \to A^{d+e}(X)$ by $[V] \cdot [W] = [V \cap W]$ with appropriate multiplicities on each component of the intersection. Define the *Chow ring* of $X$ as

$$A^*(X) = \bigoplus_{d=0}^n A^d(X).$$
This is a graded ring with unit 1 = [X].

**Remark 4.6.** The fact that the intersection product is well-defined is based on Chow’s moving lemma. It says that if Y and Z are two cycles, then there exists Y’ rationally equivalent to Y such that Y’ and Z intersect properly. Moreover, if Y” is another such cycle, then Y’ ∩ Z and Y” ∩ Z are rationally equivalent.

**Example 4.7.** Let H be the hyperplane class of \( \mathbb{P}^n \). Then we have
\[
A^*(\mathbb{P}^n) \cong \mathbb{Z}[H]/H^{n+1}
\]
as as truncated polynomial ring. Note that the Chow ring and the cohomology ring of \( \mathbb{P}^n \) coincide in this case. But in general they can be different, e.g. for a Riemann surface of positive genus.

**4.2. Formal calculations of Chern classes.** Let E be a vector bundle of rank m on X. Assign to E an element \( c_k(E) \in A^k(X) \) for each 1 ≤ k ≤ m. Call \( c_k(E) \) the k-th Chern class of E and define the total Chern class as
\[
c(E) = 1 + c_1(E) + \cdots + c_m(E).
\]
We want these assignments to satisfy the following properties:

1. If L is a line bundle associated to a divisor D, then \( c_1(L) = [D] \).
2. If there exists a short exact sequence 0 → E → F → G → 0, then \( c(F) = c(E)c(G) \).
3. If \( f: Y \to X \) is a map and E a vector bundle on X, then \( c(f^*E) = f^*c(E) \).

The actual definition of Chern classes will be introduced later by using degeneracy loci. Here we carry out some calculations first.

**Example 4.8.** Let \( E_i = \mathcal{O}(d_i) \) be a line bundle of degree \( d_i \) on \( \mathbb{P}^n \) for \( i = 1, 2 \). Then we have
\[
c(E_1 \oplus E_2) = c(E_1)c(E_2) = (1 + d_1H)(1 + d_2H) = 1 + (d_1 + d_2)H + d_1d_2H^2.
\]
Hence we conclude that
\[
c_1(E_1 \oplus E_2) = (d_1 + d_2)H,
\]
\[
c_2(E_1 \oplus E_2) = d_1d_2H^2.
\]

**Example 4.9.** We compute the Chern class of the tangent bundle of \( \mathbb{P}^n \). Let S be the tautological line bundle such that the fiber \( S|_{[L]} \) represents the 1-dimensional subspace \( L \subset \mathbb{C}^{n+1} \). Then \( S^* \) is the universal line bundle such that the fiber \( S^*|_{[L]} \) is the space of linear functionals \( \text{Hom}(L, \mathbb{C}) \). Take a non-zero linear map \( \phi: \mathbb{C}^{n+1} \to \mathbb{C} \) and define \( \Phi([L], v) = \phi(v) \), where v is a vector in L. Then \( \Phi \) can be regarded as a section of \( S^* \). Moreover, \( \Phi \) is the zero map at \([L]\) if and only if \( L \subset \ker(\phi) \), i.e. the zero locus of \( \Phi \) in \( \mathbb{P}^n \) is a hyperplane given by (the projectivization of) \( \ker(\phi) \). It implies that
\[
S^* = \mathcal{O}_{\mathbb{P}^n}(1), \quad S = \mathcal{O}_{\mathbb{P}^n}(-1).
\]
We have an exact sequence
\[
0 \to S \to V \to Q \to 0,
\]
where $V = \mathbb{C}^{n+1} \times \mathbb{P}^n$ is the trivial bundle of rank $n+1$, $S \to V$ is induced by the inclusion $L \subset \mathbb{C}^{n+1}$ and $Q$ is the quotient bundle with fiber $Q_{|L} = \mathbb{C}^{n+1}/L$. Tensor with $S^*$ and apply the identification $S = \mathcal{O}_{\mathbb{P}^n}(-1)$. We thus obtain that

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^n}(1) \to S^* \otimes Q \to 0.$$ 

Note that the fiber of the tangent bundle $T_{\mathbb{P}^n} |_{[L]}$ is canonically isomorphic to $\text{Hom}(L, \mathbb{C}^{n+1}/L)$. More precisely, given $\phi \in \text{Hom}(L, \mathbb{C}^{n+1})$ and $v \in L$, one can define an arc $v(t) = v + t\phi(v) \in \mathbb{C}^{n+1}$. The part $\phi(v)$ modulo $L$ is a tangent vector at $[L]$. It thus implies that

$$T_{\mathbb{P}^n} \cong S^* \otimes Q.$$ 

Hence we obtain the following Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^n}(1) \to T_{\mathbb{P}^n} \to 0.$$ 

The total Chern class of $T_{\mathbb{P}^n}$ is

$$c(T_{\mathbb{P}^n}) = c(\mathcal{O}_{\mathbb{P}^n}(1))^{n+1} = (1 + H)^{n+1} = 1 + (n+1)H + \binom{n+1}{2}H^2 + \cdots + \binom{n+1}{n}H^n.$$ 

4.3. **The splitting principle.** In an ideal world, a vector bundle would split as a direct sum of line bundles. Then we could calculate its Chern class using the formal properties. In reality there exist many non-splitting vector bundles. However for the purpose of computing Chern classes, one may still pretend that a vector bundle splits and carry out the calculation formally.

To illustrate what this means, consider a vector bundle $E$ of rank $n$. Suppose $E$ splits as a direct sum of line bundles

$$E = L_1 \oplus \cdots \oplus L_n$$ 

and let $a_i = c_1(L_i)$. Then we have

$$c(E) = \prod_{i=1}^{n} (1 + a_i) = 1 + \left( \sum_{i=1}^{n} a_i \right) + \left( \sum_{i \neq j} a_ia_j \right) + \cdots$$

hence we conclude that

$$c_1(E) = \sum_{i=1}^{n} a_i, \quad c_2(E) = \sum_{i \neq j} a_ia_j, \quad \ldots$$
Suppose $L$ is a line bundle with first Chern class $c_1(L) = b$. Then
\[
c(E \otimes L) &= c((L_1 \otimes L) \oplus \cdots \oplus (L_n \otimes L)) \\
&= (1 + a_1 + b) \cdots (1 + a_n + b) \\
&= 1 + \left( \sum_{i=1}^{n} a_i + nb \right) + \left( \sum_{i \neq j} a_i a_j + \frac{n}{2} b^2 + (n - 1) \left( \sum_{i=1}^{n} a_i \right) b \right) + \cdots \\
&= 1 + (c_1(E) + nc_1(L)) + \left( c_2(E) + \frac{n}{2} c_1^2(L) + (n - 1)c_1(E)c_1(L) \right) + \cdots
\]
Hence we conclude that
\[
c_1(E \otimes L) = c_1(E) + nc_1(L), \\
c_2(E \otimes L) = c_2(E) + \frac{n}{2} c_1^2(L) + (n - 1)c_1(E)c_1(L),
\]
etc.

The splitting principle says that even if $E$ is not decomposable as a direct sum of line bundles, the above calculation still holds. In other words, one can write
\[
c(E) = \prod_{i=1}^{n} (1 + a_i),
\]
where $a_i$’s are formal roots of the Chern polynomial
\[
c_t(E) = 1 + c_1(E)t + \cdots + c_n(E)t^n
\]
with $t$ as a formal parameter. We call $a_1, \ldots, a_n$ the Chern roots of $E$. If $E$ splits as a direct sum of line bundles, then $a_i$’s are just the first Chern classes of the summands, hence they belong to $A^1(X)$. In general, $a_i$ may not correspond to a divisor class. But the Chern classes of $E$ are represented by symmetric functions of $a_i$’s, hence in the end one can get rid of $a_i$’s and express the final result of such calculations by the Chern classes of the original vector bundles.

Let us use the splitting principle to study the Chern class of the canonical line bundle.

**Proposition 4.10.** Let $K$ be the canonical line bundle and $T$ the tangent bundle on a complex manifold $X$. Then we have
\[
c_1(K) = -c_1(T).
\]

**Proof.** If $T$ splits as $\bigoplus_{i=1}^{n} L_i$, then
\[
\bigwedge^n T = \bigotimes_{i=1}^{n} L_i.
\]
Hence we conclude that
\[
c_1\left( \bigwedge^n T \right) = c_1\left( \bigwedge^n T \right) = \sum_{i=1}^{n} c_1(L_i) = c_1(T).
\]
Therefore, the associated divisor of $K$ is the dual, i.e. $-c_1(T)$. The splitting principle ensures that the result holds regardless of whether $T$ is decomposable. □

**Corollary 4.11.** On $\mathbb{P}^n$ we have $K = \mathcal{O}(-n - 1)$. 


Proof. We have seen that
\[ c(T_{\mathbb{P}^n}) = (1 + H)^{n+1}. \]
In particular, we have \[ c_1(T_{\mathbb{P}^n}) = (n+1)H. \] Then the claim follows from the previous proposition. \qed

Finally let us say a word about why the splitting principle works. Roughly speaking, for a vector bundle \( E \) on \( X \), there exists a map \( f: \tilde{X} \to X \) such that \( f^*E \) splits and \( f^*: A^*(X) \to A^*(\tilde{X}) \) is injective. Then one can carry out the calculation in \( A^*(X) \) regarded as a subring of \( A^*(\tilde{X}) \).

Exercise 4.12. Let \( E \) and \( F \) be vector bundles of rank two on \( X \). Use their Chern classes to calculate the Chern classes of \( E^*, \text{Sym}^2(E), \bigwedge^2 E, E \oplus F \) and \( E \otimes F \).

4.4. Determinantal varieties. Let \( M = M(m,n) \) be the space of \( m \times n \) matrices. Note that \( M \) is isomorphic to \( \mathbb{C}^{mn} \). For \( 0 \leq k \leq \min(m,n) \), denote by \( M_k = M_k(m,n) \) the locus of matrices of rank at most \( k \), that is, cut out by all \( (k+1) \times (k+1) \) minors. We say that \( M_k \) is the \( k \)-th generic determinantal variety.

Proposition 4.13. \( M_k \) is an irreducible subvariety of \( M \) of codimension \( (m-k)(n-k) \).

Proof. Define the incidence correspondence \( \tilde{M}_k = \{(A,W) \mid A \cdot W = 0\} \subset M \times G(n-k,n) \).

Then \( \tilde{M}_k \) admits two projections \( p_1 \) and \( p_2 \) to \( M \) and \( G(n-k,n) \), respectively. The map \( p_1 \) is onto \( M_k \) and generically one to one. In other words, \( p_1 \) has positive dimensional fiber over \( A \) if and only if the rank of \( A \) is at most \( k - 1 \). On the other hand, for a fixed \( W \in G(n-k,n) \), the fiber of \( p_2 \) over \( W \) is isomorphic to \( \mathbb{A}^{mk} \).

Hence we conclude that
\[
\dim M_k = \dim \tilde{M}_k = \dim G(n-k,n) + mk = (m+n)k - k^2.
\]

Therefore, the codimension of \( M_k \) in \( M \) is equal to
\[
mn - (m+n)k + k^2 = (m-k)(n-k).
\]

Moreover, \( \tilde{M}_k \) is a vector bundle on \( G(n-k,n) \), hence is irreducible. So is \( M_k \). \qed

Remark 4.14. As a vector bundle, \( \tilde{M}_k \) is smooth, but the contraction \( M_k \) can be singular. With a more detailed study of the tangent space, one can show that the singular locus of \( M_k \) is exactly \( M_{k-1} \).

Determinantal varieties are useful for the study of maps between vector bundles. Let \( E \) and \( F \) be two vector bundles on \( X \) of rank \( n \) and \( m \), respectively. Suppose \( \phi: E \to F \) is a bundle map. Choose a local trivialization of \( E \) and \( F \). Then \( \phi \) locally corresponds to an \( m \times n \) matrix, whose entries are holomorphic functions. Denote by \( \Phi_k \) the locus in \( X \) where the rank of \( \phi \) is at most \( k \). Then \( \Phi_k \) is the inverse image of the generic determinantal variety \( M_k \), hence it has expected codimension \( (m-k)(n-k) \) in \( X \).
4.5. **Degeneracy loci.** Let $E$ be a vector bundle of rank $n$ on a projective variety $X$. Moreover, suppose $E$ is *globally generated*, i.e. for any $p \in X$, the fiber $E|_p$ is spanned by the global sections of $E$. We will first define Chern classes for such $E$. In general, one can twist a vector bundle by a very ample line bundle to obtain a globally generated vector bundle, define its Chern classes and then twist back to the original bundle.

For $1 \leq i \leq \min\{n, \dim X\}$, let $s_1, \ldots, s_i$ be general global sections of $E$. At a point $p \in X$, we expect these sections to be linearly independent. Hence we denote by $D_i$ the *degeneracy locus* in $X$ where $s_1(p), \ldots, s_i(p)$ are linearly dependent for $p \in D_i$. Equivalently, $D_i$ is the locus of $p$ in $X$ such that

$$s_1(p) \wedge \cdots \wedge s_i(p) = 0.$$  

Note that $D_i$ has a determinantal structure. Take a trivialization of $E$ over $U$ and let $e_1, \ldots, e_n$ be sections that generate $E|_U$. Then one can write

$$s_j = \sum_{l=1}^n a_{jl} e_l$$

for $j = 1, \ldots, i$, where $a_{jl} \in \mathcal{O}(U)$. Then $D_i|_U$ is defined by all $i \times i$ minors of the $i \times n$ matrix $(a_{jl})$. Adapting the dimension of the corresponding determinantal variety, we conclude that the (expected) codimension of $D_i$ in $X$ is $n - i + 1$. We thus define the Chern class

$$c_{n-i+1}(E) := [D_i] \in A^{n-i+1}(X).$$

**Remark 4.15.** If the sections $s_1, \ldots, s_i$ are general, then $D_i$ has the right codimension as expected. Moreover, different choices of general sections give rise to rationally equivalent degeneracy loci.

The definition by degeneracy loci satisfies the formal properties of Chern classes. For instance, suppose $E = L_1 \oplus L_2$ such that $L_i$ has a section $s_i$ with $(s_i) = S_i$ as an effective divisor. Then $D_1$ is the locus where the section $(s_1, s_2)$ of $E$ is zero, i.e.

$$c_2(E) = [D_1] = [S_1] \cdot [S_2] = c_1(L_1)c_1(L_2).$$

Moreover, take $(s_1, 0)$ and $(0, s_2)$ as two sections of $E$. Then $D_2$ is the locus where either $s_1$ or $s_2$ is zero, i.e.

$$c_1(E) = [D_2] = [S_1] + [S_2] = c_1(L_1) + c_1(L_2).$$

Hence we recover that $c(E) = c(L_1)c(L_2)$ as required by the formal properties of Chern classes.

Let us consider some applications along this circle of ideas.

**Example 4.16** (Bézout’s Theorem). Consider the vector bundle $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ on $\mathbb{P}^2$ with $a, b > 0$. We know that

$$c(E) = 1 + (a+b)L + abL^2,$$

where $L$ is the line class of $\mathbb{P}^2$. Let $F$ and $G$ be two general homogenous polynomials in three variables of degree $a$ and $b$, respectively. Then $(F, G)$ is a general section
of $E$. The degeneracy locus $D_1$ associated to $(F,G)$ consists of points that are common zeros of $F$ and $G$. In other words, let $C$ and $D$ be the corresponding plane curves defined by $F$ and $G$, respectively. Then we conclude that

$$\#(C \cap D) = \deg(c_2(E)) = ab.$$ 

4.6. **Grassmannians and Schubert calculus.** As the dual projective space parameterizes hyperplanes, one can consider in general the Grassmannian $G(k,n)$ parameterizing $k$-dimensional linear subspaces of $\mathbb{P}^n$, or equivalently $(k+1)$-dimensional linear subspaces of $\mathbb{C}^{n+1}$.

Recall that $\dim G(k,n) = (k+1)(n-k)$. As we showed before for $\mathbb{P}^n$, there is a similar exact sequence on $G(k,n)$:

$$0 \to S \to V \to Q \to 0,$$

where $S$ is the tautological bundle of rank $k+1$ whose fiber over $W \in G(k,n)$ represents the subspace $W$, $V$ is the trivial bundle of rank $n+1$ and $Q$ is the quotient bundle of rank $n-k$. The tangent bundle $T_G$ is isomorphic to $S^* \otimes Q$, hence it fits in the Euler exact sequence

$$0 \to S \otimes S^* \to \bigoplus_{i=0}^{n+1} S^* \to T_G \to 0.$$ 

In order to compute the Chern classes of $T_G$, we need to know the Chow ring of $G(k,n)$. In general, the Chow ring of a Grassmannian is generated by its subvarieties parameterizing $k$-dimensional linear subspaces satisfying certain interpolation conditions. Below we will use an example to illustrate the idea.

Consider $G(1,3)$ parameterizing lines in $\mathbb{P}^3$. This is a 4-dimensional variety. Fix a flag $(p \in L \subset \Lambda)$, where $p$ is a point, $L$ is a line and $\Lambda$ is a plane in $\mathbb{P}^3$. Define the *Schubert cycles* of $G(1,3)$ as follows:

- $\sigma_L$: the locus of lines that intersect $L$;
- $\sigma_p$: the locus of lines that passes through $p$;
- $\sigma_\Lambda$: the locus of lines that are contained in $\Lambda$.

One can choose another flag and the resulting cycles are rationally equivalent to the original ones. So those $\sigma_\bullet$ give elements in $A^*(G(1,3))$.

**Exercise 4.17.** Show that the codimensions of $\sigma_L$, $\sigma_p$ and $\sigma_\Lambda$ in $G(1,3)$ are 1, 2 and 2, respectively.

Since $\sigma_L \in A^1(G(1,3))$, let us compute $\sigma_L^2$ in $A^2(G(1,3))$. Modulo rational equivalence, we can take two different lines $L_1, L_2$ and compute $\sigma_{L_1} \cap \sigma_{L_2}$ instead. Indeed, we specialize $L_1, L_2$ such that they intersect at one point $p$. Then they also span a plane $\Lambda$. In this special configuration, $[L_0] \in \sigma_{L_1} \cap \sigma_{L_2}$ means either $L_0$ is contained in $\Lambda$ or $L_0$ passes through $p$. It implies that

$$\sigma_L^2 = \sigma_p + \sigma_\Lambda$$

in $A^*(G(1,3))$. Of course one has to check that it does not cause higher multiplicities during the specialization. But it can be verified by a further study of the tangent space of Schubert cycles. In general, the study of intersection products between Schubert cycles is called *Schubert calculus*.

**Exercise 4.18.** Carry out the following calculation in $A^*(G(1,3))$.

(1) Show that $\sigma_p \sigma_\Lambda = 0$, $\sigma_p^2 = 1$ and $\sigma_\Lambda^2 = 1.$
(2) Calculate the degree of $\sigma_L$ in $A^4(\mathbb{G}(1, 3)) \cong \mathbb{Z}$. In other words, calculate the number of lines that intersect four general lines in $\mathbb{P}^3$.

Back to the exact sequence $0 \to S \to V \to Q \to 0$ applied to $\mathbb{G}(1, 3)$. We know that $S^*|_{W} = \text{Hom}(W, \mathbb{C})$, where $W \subset \mathbb{C}^4$ is a 2-dimensional linear subspace. Let us use the degeneracy loci to calculate the Chern classes of $S^*$.

Take two general linear maps $\phi_i: \mathbb{C}^4 \to \mathbb{C}$ for $i = 1, 2$. They induce two general sections $\Phi_i$ of $S^*$ by $\Phi_i(W)(w) = \phi_i(w)$ for $w \in W$, $i = 1, 2$. By definition, the vanishing locus of $\Phi_1$ (or $\Phi_2$) is $c_2(S^*)$. Note that $\Phi_1$ is the zero map on $W$ if and only if $W \subset \ker(\phi_1)$, that is, the line $W$ is contained in the plane $\Lambda$ obtained by projectivizing $\ker(\phi_1)$. We thus obtain that

$$c_2(S^*) = \sigma_\Lambda.$$

Similarly the locus where $\Phi_1$ and $\Phi_2$ fail to be linearly independent is $c_1(S^*)$. This happens if and only if $W \cap (\ker(\phi_1) \cap \ker(\phi_2)) \neq 0$, i.e. $[W]$ intersects the line $L$ given by the projectivization of $\ker(\phi_1) \cap \ker(\phi_2)$. We thus obtain that

$$c_1(S^*) = \sigma_L.$$

In summary, we conclude that

$$c(S^*) = 1 + \sigma_L + \sigma_\Lambda.$$

Then we have

$$c(S) = 1 - \sigma_L + \sigma_\Lambda,$$

$$c(Q) = \frac{1}{c(S)} = 1 + \sigma_L + \sigma_p,$$

hence one can readily calculate $c(T_G) = c(S^* \otimes Q)$.

5. **Algebraic surfaces**

Throughout this section $X$ will be a smooth complex projective surface. Recall that $\text{Div}(X)$ is the group of all divisors in $X$ whose quotient modulo linear equivalence is the Picard group $\text{Pic}(X)$ of isomorphism classes of line bundles. A complex algebraic curve in $X$ is also a divisor.

5.1. **Intersection theory on surfaces.** This part can be viewed as a special case of the intersection product in the Chow ring we studied previously, but we go over it again with more details proofs. Let $C, D$ be two curves on $X$ and $p$ a point in their intersection. We use the notation $\mathcal{O}_C(D)$ to denote the line bundle $\mathcal{O}_X(D)$ restricted to $C$. If $C$ and $D$ meet transversally at $r$ distinct points, it is natural to define the (geometric) intersection number $C \cdot D = r$. We want to generalize this intersection product to any two divisors in $\text{Div}(X)$.

**Theorem 5.1.** There is a unique intersection pairing $\text{Div}(X) \times \text{Div}(X) \to \mathbb{Z}$, denoted by $\cdot$, for two divisors $C, D$ such that

1. If $C$ and $D$ meet transversally everywhere, then $C \cdot D = \#(C \cap D)$,
2. It is symmetric: $C \cdot D = D \cdot C$,
3. It is additive: $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$, and
4. If $C_1 \sim C_2$, then $C_1 \cdot D = C_2 \cdot D$.

The proof of the theorem is decomposed into several steps.
Lemma 5.2. Suppose two curves $C$ and $D$ meet transversally. Then
\[ \#(C \cap D) = \deg(O_C(D)). \]

Proof. Note that $O_C(D)$ is a line bundle on $C$ whose associated divisor is $C \cap D$. Since the intersection is transverse everywhere, this divisor thus consists of $\#(C \cap D)$ points, hence its degree equals the degree of the line bundle. □

Proof of Theorem [5.1]. First let us show the uniqueness. Since $X$ is projective, let $H$ be a fixed very ample divisor (i.e. $H$ is a hyperplane section of $X$ embedded in a projective space). For any two divisors $C, D$ in $X$, we can find $n \gg 0$ such that $C + nH, D + nH$ and $nH$ are all very ample. Then one can choose general $C' \in |C + nH|, D' \in |D + nH|, E' \in |nH|$ and $F' \in |nH|$ such that they are all smooth, $D'$ transversal to $C'$, $E'$ transversal to $D'$, and $F'$ transversal to $C', E'$. Since $C \sim C' - E'$ and $D \sim D' - F'$, by the properties we have
\[ C \cdot D = \#(C' \cap D') - \#(C' \cap E') - \#(E' \cap D') + \#(E' \cap F'). \]

It implies that $C \cdot D$ is determined by the geometric intersection numbers.

For the existence, first we define for two very ample divisors $C, D$ by setting $C \cdot D = \#(C' \cap D')$ where $C' \in |C|$ and $D' \in |D|$ such that $C', D'$ meet transversally. Since $D$ and $D'$ (resp. $C$ and $C'$) represent the same line bundle, by Lemma [5.2] the above setting does not depend on the choices of $C'$ and $D'$. Now we have a well-defined pairing between very ample divisor classes. To define it for two arbitrary divisors $C, D$, we can write $C \sim C' - E'$ and $D \sim D' - F'$ such that $C', D', E', F'$ are all very ample. Then $C \cdot D$ follows from the intersection pairing between these very ample divisors. One can check that if we choose different expressions for $C$ and $D$, then the resulting pairings are the same. Moreover, it is easy to check that this definition satisfies all the desired properties. □

It would be useful to define the intersection number locally at a point. Suppose $C$ and $D$ are two curves that meet at finitely many points, i.e. they do not share any positive dimensional components. At an intersection point $p$, suppose they are defined by $f(x, y)$ and $g(x, y)$, respectively, where $x$ and $y$ are local coordinates at $p = (0, 0)$. Let $O_{p,x} = \mathbb{C}[[x, y]]$ be the (analytic) local ring of formal power series. Define the intersection multiplicity $(C \cdot D)_p$ to be the length of $O_{p,x}/(f, g)$, i.e. the dimension of the $\mathbb{C}$-vector space $\mathbb{C}[[x, y]]/(f, g)$.

Example 5.3. Let $f = y$ and $g = y - x^2$ over $\mathbb{C}$. Then $\mathbb{C}[[x, y]]/(f, g) \cong \mathbb{C}[[x]]/x^2$ is generated by $1$ and $x$ as a $\mathbb{C}$-vector space, hence the length is two. Geometrically we see that the two curves defined by $f$ and $g$ have contact order two at the origin.

Proposition 5.4. If $C$ and $D$ are two curves in $X$ that meet at finitely many points, then
\[ C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p. \]

Proof. Recall the exact sequence
\[ 0 \to O_C(-D) \to O_C \to O_{C \cap D} \to 0, \]
where the geometric structure (not just as a set) of $C \cap D$ is defined by the ideal $(f, g)$ locally. In other words, sections of $O_{C \cap D}$ at $p$ corresponds to elements in $O_{p,x}/(f, g)$. Then we have
\[ h^0(O_{C \cap D}) = \chi(O_C) - \chi(O_C(-D)). \]
The left-hand side equals \( \sum_{p \in C \cap D} (C \cdot D)_p \) by definition. The right-hand side only depends on the classes of \( C \) and \( D \), hence we can replace them by differences of very ample curves that are transversal to each other. Therefore, the claim reduces to the identity \( \deg(\mathcal{O}_C(D)) = C \cdot D \) as in the definition of the intersection pairing. \( \square \)

**Exercise 5.5.** Consider two plane curves \( Q : YZ - X^2 = 0 \) and \( C : Y^2Z - X^3 = 0 \). Calculate the intersection multiplicity of \( Q \) and \( C \) at every intersection point and confirm that they satisfy Proposition 5.4.

**Example 5.6** (Self-intersection). Even if \( C \) is nonsingular, we cannot directly see how to define \( C^2 = C \cdot C \). Instead, we use \( C^2 = \deg(\mathcal{O}_C(C)) \), where \( \mathcal{O}_C(C) \) is called the normal bundle of \( C \) in \( X \), also denoted by \( N_{C/X} \) or \( N_C \).

Let us study normal bundles in a more general setting. Suppose \( X \) is a smooth complex manifold and \( D \subset X \) a smooth submanifold of codimension \( k \). Denote by \( N_{D/X} \) or simply \( N_D \) the normal bundle of \( D \) in \( X \) as

\[
N_D := (T_X|_D)/T_D.
\]

Similarly, let \( N_D^* \) be the conormal bundle that fits in the exact sequence

\[
0 \rightarrow N_D^* \rightarrow \Omega_X|_D \rightarrow \Omega_D \rightarrow 0,
\]

where \( \Omega = T^* \) is the cotangent bundle, i.e. the sheaf of differentials. Both \( N_D \) and \( N_D^* \) are of rank \( k \) equal to the codimension of \( D \) in \( X \).

**Proposition 5.7** (Adjunction formula I). Suppose \( D \) is a smooth effective divisor in \( X \). Then we have

\[
N_D \cong \mathcal{O}_X(D)|D, \quad N_D^* \cong \mathcal{O}_X(-D)|D.
\]

**Proof.** Suppose \( D \) is defined locally by \((U_{\alpha}, f_\alpha)\). Then the line bundle \( \mathcal{O}_X(D) \) has transition functions \( g_{\alpha\beta} = f_\alpha/f_\beta \). Since \( f_\alpha \equiv 0 \) on \( D \cap U_{\alpha}, \) the differential \( df_\alpha \) defines a local section of \( N_D^* \) on \( U_{\alpha} \) by the defining exact sequence of \( N_D^* \). Moreover,

\[
df_\alpha = g_{\alpha\beta} \cdot df_\beta
\]
on \( D \cap U_{\alpha\beta} \), hence the union of \((U_{\alpha}, df_\alpha)\) forms a global section of \( N_D^* \otimes \mathcal{O}_X(D) \), which is nowhere zero, for otherwise \( D \) would be singular at the vanishing locus \( df = 0 \). (For example if \( f = y^2 - x^3 \), then \( df = 0 \) at the origin, and geometrically one sees a cuspidal singularity.) It implies that \( N_D^* \otimes \mathcal{O}_X(D) \) is a trivial line bundle, because it has a nowhere vanishing section. \( \square \)

**Proposition 5.8** (Adjunction formula II). In the above setting, suppose \( D \) is a divisor and let \( K \) be the canonical line bundle of \( X \). Then

\[
K_D \cong (K_X \otimes \mathcal{O}_X(D))|_D.
\]

**Proof.** By the defining sequence of \( N_D^* \) and Proposition 5.7, we have

\[
c_1(\mathcal{O}_D(-D)) = c_1(N_D^*) = c_1(K_X|_D) - c_1(K_D),
\]
hence \( K_D \cong (K_X \otimes \mathcal{O}_X(D))|_D \). \( \square \)

**Example 5.9** (Canonical line bundle of a hypersurface). Let \( X \) be a degree \( k \) smooth hypersurface in \( \mathbb{P}^n \). Then we have

\[
K_X \cong (K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(k))|_X = \mathcal{O}_X(k - n - 1),
\]
as we saw previously that \( K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n - 1) \).
Now we specialize to the case of surfaces.

**Proposition 5.10** (Adjunction formula for surfaces). If $C$ is a smooth curve of genus $g$ on $X$ and if $K_X$ is the canonical divisor class of $X$, then

$$2g - 2 = C.(C + K_X).$$

*Proof.* We know that

$$K_C \cong (K_X \otimes \mathcal{O}(C))|_C.$$

Take the degree of both sides. The left-hand side is $2g - 2$ and the right-hand side is $C.(C + K_X)$. Hence the desired formula follows. $\square$

**Example 5.11** (Plane curves). Suppose $D$ and $E$ are plane curves of degree $d$ and $e$, respectively. Then $D \cdot E = de$ as we saw in Bézout’s theorem. Moreover, if $C$ is a smooth plane curve of degree $d$, then by Proposition 5.10 we can recover that

$$g = (d - 1)(d - 2).$$

**Example 5.12** (Quadric surfaces). Let $Q$ be a smooth quadric surface in $\mathbb{P}^3$. In terms of coordinates, $Q$ is the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ by

$$[X_1,Y_1] \times [X_2,Y_2] \mapsto [X_1X_2, X_1Y_2, Y_1X_2, Y_1Y_2]$$

so that the image is cut out by the quadratic equation $Z_0Z_3 - Z_1Z_2 = 0$.

One can show that $\text{Pic}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by lines in the two families of rulings (draw a picture), where each summand $\mathbb{Z}$ is from the corresponding $\mathbb{P}^1$ in $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Denote the generators by $f_1 = (1,0)$ and $f_2 = (0,1)$. We have

$$f_1^2 = f_2^2 = 0, \quad f_1 \cdot f_2 = 1,$$

because two lines in the same family are disjoint and two lines in different families meet transversally at a point (again using the picture). A hyperplane section of $Q$ has class $(1,1)$ and the canonical line bundle $K$ of $Q$ has class $(-2,-2)$ by the adjunction formula.

Let $C$ be a smooth curve of class $(a,b)$ in $Q$. Then $C + K$ has class $(a - 2,b - 2)$. By Proposition 5.10, we have

$$2g - 2 = (a,b).((a - 2,b - 2) = 2ab - 2a - 2b,$$

$$g = (a - 1)(b - 1).$$

In particular, if $g = 0$ and if $C$ has degree $d = a + b$ in $\mathbb{P}^3$, we have $(a,b) = (1,d - 1)$ or $(d - 1,1)$.

Recall the Riemann-Roch formula on curves. It says that $\chi(L) - \chi(C) = \text{deg}(L)$ for a line bundle $L$ on a curve $C$. In other words, for fixed degree the holomorphic Euler characteristic of $L$ only depends on the topology of $C$. An analogous result holds for surfaces (as well as for higher dimensional varieties).

**Theorem 5.13** (Riemann-Roch for surfaces). Let $D$ be a divisor on a surface $X$. Then we have

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = \frac{1}{2}D \cdot (D - K_X).$$
Proof. Since the result only depends on the class of $D$, we can write $D \sim C - E$ for two nonsingular curves $C$ and $E$. By the exact sequences
\[ 0 \to \mathcal{O}_X(C - E) \to \mathcal{O}_X(C) \to \mathcal{O}_E(C) \to 0, \]
\[ 0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0, \]
we obtain that
\[ \chi(\mathcal{O}_X(C - E)) - \chi(\mathcal{O}_X) = \chi(\mathcal{O}_C(C)) - \chi(\mathcal{O}_E(C)). \]
Moreover, by Riemann-Roch on curves we have
\[ \chi(\mathcal{O}_C(C)) = 1 - g_C + C^2, \]
\[ \chi(\mathcal{O}_E(C)) = 1 - g_E + C.E. \]
Finally, by Proposition 5.10 we know
\[ g_C = \frac{1}{2} C \cdot (C + K_X) + 1, \]
\[ g_E = \frac{1}{2} E \cdot (E + K_X) + 1. \]
Combining the above, we thus obtain the desired formula. \qed

Exercise 5.14. Suppose that $X$ is a smooth surface of degree $d$ in $\mathbb{P}^3$ and suppose it contains a line $L$. Prove that $L^2 = 2 - d$ on $X$.

Exercise 5.15. Let $X$ be a smooth surface of degree $d$ in $\mathbb{P}^3$. Calculate the self-intersection number $K_X^2$. 
