MATH8808: ALGEBRAIC TOPOLOGY

DAWEI CHEN

Contents

Date: Fall 2018.
Most of the material is from [H].

1. Underlying Geometric Notions

1.1. **Homotopy.**

Definition 1.1. Let X be a (topological) space and A a subspace of X. A *deformation retraction* of X onto A is a (continuous) family of maps $f_t : X \to X$ for $t \in I = [0,1]$, such that $f_0 = \mathbb{1}$, $f_1(X) = A$, and $f_t|_A = \mathbb{1}$ for all t. The family f_t is continues in the sense that the map $X \times I \to X$ by $(x,t) \to f_t(x)$ is continuous.

Example 1.2. Thick letters, Möbius strap, punctured disks.

A deformation retraction is a special case of the notion of a homotopy.

Definition 1.3. A *homotopy* is a family of maps $f_t : X \to Y$ for $t \in I$ such that $F : X \times I \to Y$ by $F(x,t) = f_t(x)$ is continuous.

Two maps $f_0, f_1 : X \to Y$ are *homotopic* if there exists a homotopy f_t connecting them, and in this case we write $f_0 \simeq f_1$.

Suppose X deformation retracts onto A. Denote by $r = f_1 : X \to A$ the retraction and $i : A \to X$ the inclusion. Then $r \circ i = \mathbb{1}$ and $i \circ r \simeq \mathbb{1}$, where the latter is given by f_t. Hence one can generalize the situation in a more symmetric way as follows.

Definition 1.4. Two spaces X and Y are *homotopy equivalent* (or simply *homotopic*) if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathbb{1}$ and $g \circ f \simeq \mathbb{1}$. In this case we write $X \simeq Y$.

It is easy to check that homotopy equivalence is an equivalence relation. A space A homotopic to a point a is called *contractible*. Namely, the map $p : A \to a \in A$ is homotopic to the identity map $\mathbb{1}$ on A. In other words, there exists a map $f : A \times I \to A$ such that $f_0 = \mathbb{1}$ and $f_1 = p$. In particular, one can visualize the contraction from A to $a \in A$ by changing A to $f_t(A) \subset A$ at time t.

Example 1.5. The graphs given by a pair of glasses, figure 8, and a capsule are all homotopic, since they are deformation retracts of a 2-punctured disk.

Example 1.6. An n-disk is contractible, since it deformation retracts to its center.

1.2. **Cell Complexes.** Denote an n-disk by D^n. The boundary of D^n is an $(n-1)$-dimensional sphere S^{n-1}.

Example 1.7. Construct a torus by gluing one 0-cell, two 1-cells, and one 2-cell.

In general, one can construct a space by the following inductive procedure:

(1) Start with a discrete set X^0, whose points are 0-cells.
(2) Suppose we have constructed $X^0, \dots, X^{n-1}$. Let $\{D^n_\alpha\}$ be a collection of n-disks and $\{\varphi_\alpha\}$ a collection of maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. Define X^n to be the quotient space of $X^{n-1} \sqcup_\alpha D^n_\alpha$ under the identifications $x \simeq \varphi_\alpha(x)$ for all $x \in \partial D^n_\alpha$. Hence as a set, $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$ where e^n_α is an open n-disk.

A space X constructed in the above way is called a *cell complex* or *CW complex*.[1] Each X^n is called the *n-skeleton* of X and each e^n_α is called an n-cell of X. If $X = X^n$ for some $n \in \mathbb{Z}^{\geq 0}$, X is called *finite-dimensional* and $\dim X = n$.

Example 1.8. A 1-dimensional cell complex corresponds to a *graph*, where vertices are 0-cells and edges are 1-cells.

Example 1.9. The sphere S^n is a cell complex. One can glue two cells e^0 and e^n by identifying the boundary of e^n as a single point.

Example 1.10. We describe a cell complex structure for the real projective space $\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\}/\sim$, where $v \sim \lambda v$ for scalars $\lambda \neq 0$. Note that $\mathbb{RP}^n = S^n/v \sim (-v)$, which is equivalent to saying that $\mathbb{RP}^n$ is the quotient space of a hemisphere D^n with opposite points of ∂D^n identified. Since $\partial D^n = S^{n-1}$, we see that $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching an n-cell, where the attaching map is given by $S^{n-1} \to \mathbb{RP}^{n-1}$. By induction, we thus conclude that $\mathbb{RP}^n$ has a cell complex structure with one call e^i in each dimension $0 \leq i \leq n$.

Definition 1.11. If a closed space $A \subset X$ is a union of cells of X, we say that A is a *subcomplex* of X and the pair (X, A) is called a *CW pair*.

Example 1.12. Each skeleton X^n is a subcomplex of X.

Example 1.13. In the above construction of $\mathbb{RP}^n$, each $\mathbb{RP}^i$ is a subcomplex of $\mathbb{RP}^n$ for $i \leq n$.

Remark 1.14. In general, the closure of a cell may not be a subcomplex. For example, start with S^1 with its minimal cell structure. Attach to it a 2-cell by a map $S^1 \to S^1$ whose image is a nontrivial arc of S^1 (say, a semicircle by folding). Then the closure of the 2-cell is not a subcomplex since it contains only part of the 1-cell.

1.3. **Operations on Cell Complexes.** We describe several standard operations on cell complexes.

Products. Suppose X and Y are cell complexes with cells e^m_α and e^n_β respectively. Then the products $e^m_\alpha \times e^n_\beta$ give a cell complex structure for $X \times Y$. For example, the cell complex structure of a torus described before is the product $S^1 \times S^1$ where S^1 is with its standard cell structure.

Quotients. Suppose (X, A) is a CW pair. Then the quotient space X/A has a cell structure whose cells are those of $X - A$ plus one new 0-cell as the image of A in

[1] The C stands for "closure-finite", and the W for "weak" topology.

X/A. For a cell e^n_α of $X - A$ with attaching map $S^{n-1} \to X^{n-1}$, the corresponding cell in X/A is attached by the composition $S^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}$.

For example, given S^{n-1} with any cell structure, the quotient D^n/S^{n-1} is S^n with its standard cell structure. For another example, let $X = S^1 \times S^1$ be the torus with the product structure, and $A \subset X$ be the complement of the unique 2-cell. Then X/A consists of a 0-cell and a 2-cell, and hence X/A is S^2.

Cone and Suspension. For a cell complex X, define the *cone* $CX = (X \times I)/(X \times \{0\})$ with the standard product and quotient structure. One can visualize CX as a cone, say, if $X = S^1$.

Define the *suspension* SX to be the quotient of $X \times I$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. In other words, SX is the union of two copies of the cone CX. One can visualize SX as a suspension, say, if $X = S^1$.

Join. Note that a cone CX is the union of all line segments joining points of X to a vertex, and a suspension SX is the union of all line segments joining points of X to two vertices. In general, given two spaces X and Y, define the *join* $X * Y$ to be the space of line segments joining points in X to points in Y. More precisely, $X * Y$ is defined as the quotient space $X \times Y \times I$ under the identifications

$$(x, y_1, 0) \sim (x, y_2, 0), \quad (x_1, y, 1) \times (x_2, y, 1).$$

In other words, we collapse $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y. For example, if X and Y are both closed intervals, then $X * Y$ becomes a tetrahedron.

A nice way to write $X * Y$ is as formal linear combinations $t_1 x + t_2 y$ with $0 \leq t_i \leq 1$, $t_1 + t_2 = 1$, and the identifications $0x + 1y = y$ and $1x + 0y = x$. Similarly, an iterated join $X_1 * \cdots * X_n$ can be viewed as the space of formal linear combinations $t_1 x_1 + \cdots + t_n x_n$ with $0 \leq t_i \leq 1$, $t_1 + \cdots + t_n = 1$, and $0x_i$ being omitted. An important case is when each X_i is a point. The join of n points can be viewed as a convex polyhedron of dimension $n - 1$, called a *simplex*:

$$\Delta^{n-1} = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_1 + \cdots + t_n = 1 \text{ and } t_i \geq 0\}.$$

For example, the join of two points is a line segment, the join of three points is a triangle, and the join of four points is a tetrahedron.

Wedge Sum. Given $x \in X$ and $y \in Y$, define the *wedge sum* $X \vee Y$ to be the quotient $X \sqcup Y/x \sim y$. For example, $S^1 \vee S^1$ is the figure 8.

Smash Product. Given $x \in X$ and $y \in Y$, consider the subspaces $X \times \{y\}$ and $\{x\} \times Y$ in the product $X \times Y$. They only intersect at one point $\{x\} \times \{y\}$, hence their union can be identified with the wedge sum $X \vee Y$. Then we define *Smash product* $X \wedge Y$ to be the quotient $X \times Y/X \vee Y$.

If x and y are 0-cells in X and Y respectively, then $X \wedge Y$ is also a cell complex. For example, $S^m \wedge S^n$ has a standard cell structure with only two cells, of dimension 0 and $m+n$, hence $S^m \wedge S^n = S^{m+n}$. One can clearly visualize this when $m = n = 1$ by collapsing the boundary of a square to a point to produce a 2-sphere.

1.4. **Criteria for Homotopy Equivalence.** One way to produce homotopy equivalent spaces is via collapsing subspaces.

Proposition 1.15. *If (X, A) is a CW pair with A contractible, then $X \simeq X/A$.*

Before we prove it, let us look at some examples first.

Example 1.16. A pair of glasses and a capsule are homotopic to the figure 8, since their middle bars are contractible subcomplexes.

More generally, suppose X is a graph. If the two endpoints of an edge of X are distinct, then it is contractible, hence we can collapse it to a point and produce a homotopy equivalent graph with fewer edges. Consequently any graph is homotopy equivalent to a graph whose edges are all loops. In particular, any connected component of X is either an isolated vertex or a wedge sum of circles.

Example 1.17. Let X be S^2 attached with an external arc A connecting its north and south poles. Let $B \subset S^2$ be an internal arc connecting the two poles. Since A and B are both contractible, we conclude that $X/A \simeq X \simeq X/B$. Note that X/A is a nodal sphere and $X/B \simeq S^2 \vee S^1$, and it may not be directly obvious that $X/A \simeq X/B$.

Another way to produce homotopy equivalent spaces is via attaching one space to another and continuously varying how the parts are attached together. Let $A \subset X_1$ be a subspace, and $f : A \to X_0$ a map to another space. Define $X_0 \sqcup_f X_1$ as the quotient space $X_0 \sqcup X_1$ by identifying each point $a \in A$ with $f(a) \in X_0$. We say that $X_0 \sqcup_f X_1$ is the space X_0 with X_1 *attached along A via f*.

Example 1.18. If $(X_1, A) = (D^n, S^{n-1})$, this is the case of attaching an n-cell to X_0 via $f : S^{n-1} \to X_0$.

Example 1.19. Let $f : X \to Y$ be a map, and $A = X \times \{1\} \subset X \times I$ is a copy of X on the bottom of the cylinder $X \times I$. Still denote by f the map $A \to Y$ by $(x, 1) \mapsto f(x)$. Then $M_f = (X \times I) \sqcup_f Y$ is called the *mapping cylinder* of f. In particular, M_f deformation retracts to Y by sliding each point (x, t) along the segment $\{x\} \times I$ to the endpoint $f(x) \in Y$ (draw a picture).

The quotient space $C_f = M_f/X \times \{0\}$ is called the *mapping cone* of f. Equivalently, $C_f = CX \sqcup_f Y$ where CX is the cone $(X \times I)/(X \times \{0\})$ (draw a picture).

Proposition 1.20. *If (X_1, A) is a CW pair and the two attaching maps $f, g : A \to X_0$ are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$.*

Before we prove it, let us again look at some examples first.

Example 1.21. Recall that a nodal sphere is homopotic to $S^1 \vee S^2$. Let $f : A \to S^1$ be the map that wraps an arc $A \subset S^2$ around S^1. Since A is contractible, f is homotopic to a constant map $g : A \to \{*\}$. Hence $S^1 \sqcup_f S^2 \simeq S^1 \sqcup_g S^2$, where the former is a nodal sphere and the latter is $S^1 \vee S^2$.

Example 1.22. Let (X, A) be a CW pair. Denote by $X \cup CA$ the mapping cone of the inclusion $A \hookrightarrow X$. Then we have $X/A = (X \cup CA)/CA \simeq X \cup CA$, where the latter homotopy equivalence is by Proposition 1.15 since CA is contractible.

Suppose A is contractible in X, i.e., $A \hookrightarrow X$ is homotopic to a constant map $A \mapsto x \in X$. Then by Proposition 1.20, the mapping cone $X \cup CA$ of the inclusion $A \hookrightarrow X$ is homotopic to the mapping cone of the constant map, which is $X \vee SA$ where SA is the suspension of A (draw a picture). Hence by the previous paragraph we have $X/A \simeq X \cup CA \simeq X \vee SA$ in this case. For example, S^0 (as a discrete set of two points) is contractible in S^2, hence we see again that $S^2/S^0 \simeq S^2 \vee S^1$, where S^2/S^0 is the nodal sphere and S^1 is the suspension of S^0.

1.5. **The Homotopy Extension Property.** Suppose $f_0 : X \to Y$ is a map, and $A \subset X$ is a (closed) subspace. Given a homotopy $f_t : A \to Y$ of $f_0|_A$, we would like to extend it to a homotopy $f_t : X \to Y$ of the given f_0. If this extension problem can always be solved, we say that (X, A) has the *homotopy extension property.* In other words, (X, A) has the homotopy extension property if every pair of maps $X \times \{0\} \to Y$ and $A \times I \to Y$ that agree on $A \times \{0\}$ can be extended to a map $X \times I \to Y$, i.e., if any map $X \times \{0\} \cup A \times I \to Y$ can be extended to a map $X \times I \to Y$.

A map $r : X \to X$ is called a *retraction* of X onto A if $r(X) = A$ and $r|_A = \mathbb{1}$. Then a deformation retraction of X onto A is thus a homotopy from the identity map of X to a retraction of X onto A.

Proposition 1.23. *A pair (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times I$ is a retraction of $X \times I$.*

Proof. If (X, A) has the homotopy extension property, then by definition the identity map $X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$ can be extended to a map $X \times I \to X \times \{0\} \cup A \times I$, hence $X \times I$ retracts onto $X \times \{0\} \cup A \times I$.

Conversely, suppose $X \times \{0\} \cup A \times I$ is a retraction of $X \times I$. Given a map $X \times \{0\} \cup A \times I \to Y$, composing it with the retraction map $X \times I \to X \times \{0\} \cup A \times I$, we thus obtain the desired extension $X \times I \to Y$. □

Remark 1.24. Even if A is not closed in X, Proposition 1.23 still holds.

Proposition 1.25. *If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a (deformation) retraction of $X \times I$, hence (X, A) has the homotopy extension property.*

Proof. See the proof of [H, Proposition 0.16]. □

Recall that Proposition 1.15 says that collapsing a contractible subcomplex is a homotopy equivalence. Now we prove a generalization of this assertion.

Proposition 1.26. *If the pair (X, A) satisfies the homotopy extension property and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

Proof. We need to find an inverse homotopy equivalent map for q. For $t \in I$, suppose $f_t : X \to X$ is a homotopy extending a contraction of A, where $f_0 = \mathbb{1}$. Since $f_t(A) \subset A$ for all t, let $\bar{f}_t : X/A \to X/A$ be the map induced by f_t. Then we have $qf_t = \bar{f}_t q$ (draw a commutative diagram). Note that $f_1(A)$ is a point, so f_1 induces a map $g : X/A \to X$ such that $f_1 = gq$ (draw a commutative diagram). It follows that $qg = \bar{f}_1$. Therefore, g and q are inverse homotopy equivalences, since $gq = f_1 \simeq f_0 = \mathbb{1}$ via f_t and $qg = \bar{f}_1 \simeq \bar{f}_0 = \mathbb{1}$ via $\bar{f}_t$. □

Next we generalize Proposition 1.20. First, for two pairs (W, Y) and (Z, Y), if there exist maps $\varphi : W \to Z$ and $\psi : Z \to W$, restricting to the identity on Y, such that $\varphi\psi \simeq \mathbb{1}$ and $\psi\varphi \simeq \mathbb{1}$ via homotopies that restrict to the identity on Y at all times, we say that W and Z are *homotopic relative to* Y and denote it by $W \simeq Z \ \text{rel}\ Y$.

Proposition 1.27. *If (X_1, A) is a CW pair and we have attaching maps $f, g : A \to X_0$ that are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \ \text{rel}\ X_0$.*

Proof. Let $F : A \times I \to X_0$ be a homotopy from f to g. Consider the space $X_0 \sqcup_F (X_1 \times I)$ that glues X_0 and $X_1 \times I$ with the attaching map F. It contains both $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ as subspaces. Applying Proposition 1.25 to the CW pair (X_1, A), there is a deformation retraction of $X_1 \times I$ onto $X_1 \times \{0\} \cup A \times I$, which induces a deformation retraction of $X_0 \sqcup_F (X_1 \times I)$ onto $X_0 \sqcup_{F|_0} X_1 = X_0 \sqcup_f X_1$ (why?). Similarly, $X_0 \sqcup_F (X_1 \times I)$ deformation retracts onto $X_0 \sqcup_g X_1$. Note that both deformation retractions restrict to the identity on X_0, hence together they give a homotopy equivalence $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0. □

2. The Fundamental Group

2.1. **Basic Constructions.** A *path* in a space X is a continuous map $f : I \to X$, where $I = [0, 1]$. A *homotopy* of paths is a family $f_t : I \to X$, $0 \le t \le 1$, such that

(1) The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.
(2) The map $F : I \times I \to X$ given by $F(s, t) = f_t(s)$ is continuous.

When two paths f_0 and f_1 are connected by such a homotopy f_t, we say that they are *homotopic* (rel the endpoints) and denote it by $f_0 \simeq f_1$. Geometrically speaking, it means that f_0 can continuously deform to f_1.

Example 2.1. Any two paths f_0 and f_1 in $\mathbb{R}^n$ (with the same endpoints) are homotopic via the *linear homotopy*

$$f_t(s) = (1 - t)f_0(s) + tf_1(s).$$

Proposition 2.2. *The relation of homotopy on paths with fixed endpoints is an equivalence relation.*

Consequently for a path f, we denote by $[f]$ the *homotopy class* of f.

Proof. A path $f \simeq f$ via the constant homotopy $f_t = f$ for all $0 \le t \le 1$, which verifies reflexivity.

If $f_0 \simeq f_1$ via the homotopy $f_t(s)$, then $f_1 \simeq f_0$ via $f_{1-t}(s)$, which verifies symmetry.

Finally suppose $f_0 \simeq f_1$ via f_t and $g_0 \simeq g_1$ via g_t, where $f_1 = g_0$. Define $h_t = f_{2t}$ for $0 \le t \le 1/2$ and $h_t = g_{2t-1}$ for $1/2 \le t \le 1$ (draw a picture). Then $f_0 \simeq g_1$ via h_t, which verifies transitivity. □

Given two paths $f, g : I \to X$ such that $f(1) = g(0)$, define the *composition* $f \cdot g$ (from left to right) by $f \cdot g(s) = f(2s)$ for $0 \le s \le 1/2$ and $f \cdot g(s) = g(2s - 1)$ for $1/2 \le s \le 1$. If $f_0 \simeq f_1$ via f_t and $g_0 \simeq g_1$ via g_t such that $f_0(1) = g_0(0)$ and $f_1(1) = g_1(0)$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ via $f_t \cdot g_t$ (draw a picture). Therefore, this operation preserves homotopy classes.

For a path $f : I \to X$ with $f(0) = f(1) = x_0 \in X$, we say that f is a *loop* (or a *closed path*) and call x_0 the *base point*. The set of all homotopy classes $[f]$ of loops at the base point x_0 is denoted by $\pi_1(X, x_0)$.

Proposition 2.3. $\pi_1(X, x_0)$ *is a group with respect to the product* $[f][g] = [f \cdot g]$.

This group $\pi_1(X, x_0)$ is called the *fundamental group* of X at the base point x_0, which is not necessarily commutative.

Proof. For two loops f, g with the same base point x_0, $f \cdot g$ is obviously a loop with base point x_0. We have also verified that the product $[f][g] = [f \cdot g]$ is well-defined. It remains to verify the group axioms for $\pi_1(X, x_0)$.

Some preparation first. Suppose $\varphi : I \to I$ is any continuous map such that $\varphi(0) = 0$ and $\varphi(1) = 1$. For a path f, we call the new path $f\varphi$ a *reparameterization* of f. Note that $f\varphi \simeq f$ via the homotopy $f\varphi_t$ where $\varphi_t : I \to I$ is defined by $\varphi_t(s) = (1-t)\varphi(s)+ts$, so that $\varphi_0 = \varphi$ and $\varphi_1 = \mathbb{1}$. In other words, we have $\varphi \simeq \mathbb{1}$ on I.

Given paths f, g, h with $f(1) = g(0)$ and $g(1) = h(0)$, we want to verify that $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$. Note that

$$(f \cdot g) \cdot h = \begin{cases} (f \cdot g)(2s), & 0 \le s \le 1/2 \\ h(2s-1), & 1/2 \le s \le 1 \end{cases} = \begin{cases} f(4s), & 0 \le s \le 1/4 \\ g(4s-1), & 1/4 \le s \le 1/2 \\ h(2s-1), & 1/2 \le s \le 1 \end{cases}$$

and

$$f \cdot (g \cdot h) = \begin{cases} f(2s), & 0 \le s \le 1/2 \\ (g \cdot h)(2s-1), & 1/2 \le s \le 1 \end{cases} = \begin{cases} f(2s), & 0 \le s \le 1/2 \\ g(4s-2), & 1/2 \le s \le 3/4 \\ h(4s-3), & 3/4 \le s \le 1. \end{cases}$$

Take a piecewise linear function φ that maps the three subintervals in the latter to the three subintervals in the former, respectively. Composing with φ reparameterizes $(f \cdot g) \cdot h$ as $f \cdot (g \cdot h)$. This verifies that the product operation in $\pi_1(X, x_0)$ is associative.

Given a path $f : I \to X$, define a constant path c at $f(1)$ by $c(s) = f(1)$ for all $s \in I$. Then one can similarly check as above that $f \cdot c \simeq f \simeq c \cdot f$. In particular, the homotopy class of the constant path at x_0 is a two-sided identity in $\pi_1(X, x_0)$.

For a path f starting at x_0, define the *inverse path* $\overline{f}$ by $\overline{f}(s) = f(1-s)$. Then one checks that $f \cdot \overline{f}$ is homotopic to the constant path at x_0. In particular if f is a loop with base point x_0, then $[\overline{f}]$ is a two-sided inverse for $[f]$ in $\pi_1(X, x_0)$. □

Example 2.4. Suppose X is a convex set in $\mathbb{R}^n$ with a base point $x_0 \in X$. Then any two loops f_0 and f_1 based at x_0 are homotopic via the linear homotopy

$$f_t(s) = (1-t)f_0(s) + tf_1(s).$$

We thus conclude that $\pi_1(X, x_0) = 0$.

For any two points $x_0, x_1 \in X$, if there exists a path h such that $h(0) = x_0$ and $h(1) = x_1$, we say that X is *path connected*. For any loop f based at x_1, we associate the loop $h \cdot f \cdot \overline{h}$ based at x_0 (draw a picture). It induces a change of base point map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$.

Proposition 2.5. *The map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is an isomorphism.*

Proof. Since $[h][\overline{h}]$ is the identity, we have

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h} \cdot h \cdot g \cdot \overline{h}] = \beta_h[f]\beta_h[g],$$

so β_h is a homomorphism. Moreover, $\beta_{\overline{h}}$ is the inverse of β_h, since

$$\beta_h\beta_{\overline{h}}[f] = \beta_h[\overline{h} \cdot f \cdot h] = [h \cdot \overline{h} \cdot f \cdot h \cdot \overline{h}] = [f]$$

and similarly $\beta_{\overline{h}}\beta_h[f] = [f]$. □

Therefore, if X is path connected, then the group $\pi_1(X, x_0)$ is independent of the base point, up to isomorphism. In this case we simply write $\pi_1(X)$ instead. If X is path connected and if $\pi_1(X) = 0$, we say that X is *simply connected*.

Proposition 2.6. *A space X is simply connected iff there is a unique homotopy class of paths connecting any two points in X.*

Proof. If X is simply connected, then any two points x_0 and x_1 in X can be connected by a path. If f and g are two paths joining x_0 and x_1, then $f \simeq f \cdot (\overline{g} \cdot g) \sim (f \cdot \overline{g}) \cdot g \simeq g$, since $f \cdot \overline{g}$ is homotopic to a constant loop by the assumption on $\pi_1(X)$.

Conversely, suppose for any two points in X there is a unique homotopy class of paths connecting them. Then X is path connected. Moreover, any two loops connecting x_0 to itself are homotopic, hence $\pi_1(X, x_0) = 0$. □

2.2. **The Fundamental Group of S^1.** In this section we will prove the following result.

Theorem 2.7. *$\pi_1(S) \cong \mathbb{Z}$ generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ based at $(1, 0)$.*

For $n \in \mathbb{Z}$, set $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$. Note that $[\omega]^n = [\omega_n]$. Then the theorem says that every loop in S^1 based at $(1, 0)$ is homotopic to ω_n for a unique n.

To prove the theorem, we will use the map $p : \mathbb{R} \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$ as a cover of S^1. So we first introduce the notion of covering space in more generality.

Give a space X, a *covering space* of X is a space $\widetilde{X}$ with a map $p : \widetilde{X} \to X$ satisfying the following condition:

- For each point $x \in X$, there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which maps homeomorphically onto U via p.

Such U in the above is called *evenly covered*. For example, any open arc in S^1 is evenly covered by p.

Given a covering $p : \widetilde{X} \to X$, we have the following facts:

(a) For each path $f : I \to X$ starting at $x_0 \in X$ and each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f} : I \to \widetilde{X}$ starting at $\widetilde{x}_0$.
(b) For each homotopy $f_t : I \to X$ of paths starting at $x_0 \in X$ and each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lifted homotopy $\widetilde{f}_t : I \to \widetilde{X}$ of paths starting at $\widetilde{x}_0$.

These facts can be deduced from a more general fact (c), see [H, p. 30].

Proof of Theorem 2.7. Suppose $f : I \to S^1$ is a loop at the base point $x_0 = (1, 0)$, representing an element of $\pi_1(S^1, x_0)$. By (a) there is a lift $\widetilde{f}$ to $\mathbb{R}$ starting at $0 \in p^{-1}(1, 0) = \mathbb{Z}$. Suppose $\widetilde{f}$ ends at some integer n. Note that $\widetilde{\omega}_n : I \to \mathbb{R}$ given by $\widetilde{\omega}_n(s) = ns$ is another path joining 0 to n, and $\widetilde{f} \simeq \widetilde{\omega}_n$ via the linear homotopy. Composing it with p gives a homotopy $f \simeq \omega_n$, so $[f] = [\omega_n]$.

Now suppose $\omega_n \simeq \omega_m$. Let f_t be a homotopy from $\omega_m = f_0$ to $\omega_n = f_1$. By (b) this homotopy lifts to a homotopy $\widetilde{f}_t$ of paths starting at 0. The uniqueness part of (a) implies that $\widetilde{\omega}_m = \widetilde{f}_0$ and $\widetilde{\omega}_n = \widetilde{f}_1$. Since $\widetilde{f}_t$ is a homotopy of paths, the endpoint of $\widetilde{f}_t(1)$ is independent of t. For $t = 0$ the endpoint is m and for $t = 1$ the endpoint is n, so $m = n$. □

We discuss an application.

Theorem 2.8 (Brouwer Fixed Point Theorem)**.** *Every continuous map $h : D^2 \to D^2$ has a fixed point, where D^2 is the closed unit disk.*

The upshot is that there is no retraction of D^2 onto S^1.

Proof. Suppose on the contrary that $h(x) \neq x$ for all $x \in D^2$. Define a map $r : D^2 \to S^1$ by sending x to the intersection of the ray $\overline{h(x)x}$ with $\partial D = S^1$ (draw a picture). Clearly r is continuous, and $r(x) = x$ for all $x \in S^1$, so r is a retraction of D^2 onto S^1.

Suppose f_0 is any loop in S^1 based at $x_0 \in S^1$. Since D^2 is convex, the linear homotopy $f_t(s) = (1-t)f_0(s)+tx_0$ gives a homotopy from f_0 to the constant loop at x_0. Since r is the identity on S^1, the composition rf_t is a homotopy from $rf_0 = f_0$ to rf_1 which is the constant loop at x_0. This contradicts that $\pi_1(S^1) \neq 0$. □

Now we study the fundamental group of a product space.

Proposition 2.9. *Suppose X and Y are path connected. Then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.*

Proof. A path $f : I \to X \times Y$ is of the form $f(s) = (g(s), h(s))$, hence it is equivalent to a pair of paths g in X and h in Y. Similarly a homotopy f_t in $X \times Y$ is equivalent to a pair of homotopies g_t in X and h_t in Y. Thus we obtain an isomorphism $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ given by $[f] \mapsto ([g], [h])$. □

Example 2.10. The torus has fundamental group $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

We mention a useful result for future use.

Theorem 2.11 (Borsuk-Ulam Theorem)**.** *For every continuous map $f : S^n \to \mathbb{R}^n$, there exists a pair of antipodal points x and $-x$ in S^n such that $f(x) = f(-x)$.*

Remark 2.12. For $n = 1$, the proof is simple since $f(x) - f(-x)$ changes sign as x goes half-way around the circle, hence the difference must be zero for some x. In general, the theorem implies that there is no continuous one-to-one map from S^n to $\mathbb{R}^n$, so S^n is not homeomorphic to any subspace of $\mathbb{R}^n$.

Proof. See the proof of [H, Theorem 1.10]. □

Corollary 2.13. *If $S^2 \subset \mathbb{R}^3$ is expressed as the union of three closed sets A_1, A_2, and A_3, then at least one of these sets contain a pair of antipodal points $\{x, -x\}$.*

Proof. Let $d_i : S^2 \to \mathbb{R}$ be the distance function to A_i, i.e.,

$$d_i(x) = \inf_{y \in A_i} |x - y|.$$

Clearly d_i is a continuous function, so the map $S^2 \to \mathbb{R}^2$ by $x \mapsto (d_1(x), d_2(x))$ is continous. By the Borsuk-Ulam theorem, there exist $x \in S^2$ such that $d_1(x) = d_1(-x)$ and $d_2(x) = d_2(-x)$.

If either of these two distances is zero, then x and $-x$ both lie in the same set A_1 or A_2 since they are closed sets. If both distances are nonzero, then x and $-x$ are not in $A_1 \cup A_2$, so they both lie in A_3. □

2.3. **Induced Homomorphisms.** Suppose $\varphi : (X, x_0) \to (Y, y_0)$ is a map sending x_0 to y_0. For any loop $f : I \to X$ with base point x_0, define $\varphi_*[f] = [\varphi f]$, which induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. One checks that φ_* is well-defined and satisfies the properties of being a homomorphism. Moreover for $\psi : (Y, y_0) \to (Z, z_0)$, we have $(\psi\varphi)_* = \psi_*\varphi_*$. Finally if $\mathbb{1} : X \to X$ is the identity map, then $\mathbb{1}_*$ is the identity on $\pi_1(X, x_0)$.

As an application, if φ is a homeomorphism with inverse ψ, then $\varphi_*\psi_* = (\varphi\psi)_* = \mathbb{1}_* = \mathbb{1}$ and similarly $\psi_*\varphi_* = \mathbb{1}$, so φ_* and ψ_* are isomorphisms. In other words, two (path connected) homeomorphic spaces have the same fundamental group.

We now study the fundamental groups of higher dimensional spheres.

Theorem 2.14. *For $n \geq 2$, $\pi_1(S^n) = 0$.*

We introduce the following lemma first, which will also be used later.

Lemma 2.15. *Suppose $\{A_\alpha\}$ is an open cover of a space X such that each A_α contains the base point x_0, each A_α is path connected, and each intersection $A_\alpha \cap A_\beta$ is path connected. Then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.*

Proof. Given a loop $f : I \to X$ with base point x_0, we want to find a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of I such that each subinterval $[s_{i-1}, s_i]$ is mapped by f to a single A_α. Since f is continuous, for each $s \in I$, there is a (closed) subinterval V_s containing s mapped by f to some A_α. Since I is compact (any open cover has a finite subcover), there are a finite number of these subintervals that cover I. Then the endpoints of these intervals define the desired partition of I.

Denote by A_i the subinterval containing $f([s_{i-1}, s_i])$ in the above, and let f_i be the restriction of f to $[s_{i-1}, s_i]$. Then $f = f_1 \cdots f_m$ with f_i a path in A_i. Since $A_i \cap A_{i+1}$ is path connected, we may choose a path g_i in $A_i \cap A_{i+1}$ from x_0 to $f(s_i) \in A_i \cap A_{i+1}$. Consider the loop

$$(f_1 \cdot \overline{g}_1) \cdot (g_1 \cdot f_2 \cdot \overline{g}_2) \cdot (g_2 \cdot f_3 \cdot \overline{g}_3) \cdots (g_{m-1} \cdot f_m),$$

which is homotopic to f (draw a picture). Each loop in this composition lies in a single A_i. □

Proof of Theorem 2.14. Take A_1 and A_2 to be the complements of two antipodal points in S^n, so S^n is a union of the two open sets A_1 and A_2. The intersection $A_1 \cap A_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, and choose a base point x_0 in $A_1 \cap A_2$. If $n \geq 2$, then $A_1 \cap A_2$ is path connected. By the above lemma, every loop in S^n based at x_0 is homotopic to a product of loops in A_1 or A_2. Since A_i is homeomorphic to $\mathbb{R}^n$ for $i = 1, 2$, we have $\pi_1(A_i) = 0$, and the claim thus follows. □

Corollary 2.16. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.*

Proof. Suppose on the contrary $f : \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism. If $n = 1$, since $\mathbb{R}^2 - \{0\}$ is path connected but $\mathbb{R}^1 - \{f(0)\}$ is not, the claim follows. When $n > 2$, for a point $x = f(0) \in \mathbb{R}^n$, the complement $\mathbb{R}^n - \{x\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, so Proposition 2.9 implies that

$$\pi_1(\mathbb{R}^n - \{x\}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}),$$

which is trivial by using Theorem 2.14. But $\pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$, leading to a contradiction. □

Remark 2.17. More generally for $m \neq n$, $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$. This can be proved similarly by using either higher homotopy groups or homology groups.

Now we study the behavior of fundamental groups under retractions.

Proposition 2.18. *If a space X retracts onto a subspace A, then $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion is injective. Moreover if A is a deformation retract of X, then i_* is an isomorphism.*

Proof. Let $r : X \to A$ be a retraction. Since $ri = \mathbb{1}$, we have $r_* i_* = \mathbb{1}$ on $\pi_1(A, x_0)$, hence i_* is injective. If $r_t : X \to X$ is a deformation retraction of X onto A, where $t \in I$, then $r_0 = \mathbb{1}$ on X, $r_t|_A = \mathbb{1}$, and $r_1(X) = A$. For any loop $f : I \to X$ based at $x_0 \in A$, the composition $r_t f$ gives a homotopy of f to a loop in A, so in this case i_* is also surjective. □

Next we consider more generally the fundamental groups of two homotopic spaces.

Proposition 2.19. *If $\varphi : X \to Y$ is a homotopy equivalence, then $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism for all $x_0 \in X$.*

Let us prove the following lemma first.

Lemma 2.20. *If $\varphi_t : X \to Y$ is a homotopy and h is the path $\varphi_t(x_0)$ formed by the images of a base point $x_0 \in X$, then the following diagram commutes:*

$$\varphi_{0*} = \beta_h \varphi_{1*}$$

where $\beta_h = [h \cdot () \cdot \overline{h}]$ is the change of base point map (draw a diagram).

Proof. Let h_t be the restriction of h to $[0, t]$, reparameterized such that the domain of h_t is still $[0, 1]$. More explicitly we can take $h_t(s) = h(ts)$. Note that $h(s) = \varphi_s(x_0)$, so $h_t(s) = h(ts) = \varphi_{ts}(x_0)$. If f is a loop in X based at x_0, then $h_t \cdot (\varphi_t f) \cdot \overline{h}_t$ gives a homotopy of loops based at $\varphi_0(x_0)$ (draw a picture). Its restrictions to $t = 0$ and $t = 1$ are $\varphi_0 f$ and $\beta_h(\varphi_1 f)$, so we conclude that $\varphi_{0*}([f]) = \beta_h(\varphi_{1*}([f]))$. □

Proof of Proposition 2.19. Let $\psi : Y \to X$ be a homotopy inverse of φ, so that $\varphi\psi \simeq \mathbb{1}$ and $\psi\varphi \simeq \mathbb{1}$. Consider the maps

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi\psi\varphi(x_0)).$$

Applying the above lemma to the homotopy $\psi\varphi \simeq \mathbb{1}$, we see that $\psi_*\varphi_* = \beta_h \mathbb{1}_* = \beta_h$ for some h, which is an isomorphism. It follows that φ_* is injective. Similarly ψ_* is injective. But their composition is an isomorphism, so φ_* (and ψ_*) must be surjective as well. □

3. Van Kampen Theorem

The Van Kampen Theorem gives a method to compute the fundamental group of a space that can be decomposed into simpler spaces with known fundamental groups.

3.1. **Free Products of Groups.**

Before we state the theorem, we first recall the definition of taking free products of groups. Given a collection of groups $\{G_\alpha\}$, we define the *free product group* $*_\alpha G_\alpha$ as follows:

- The elements are words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geq 0$ such that $g_i \in G_{\alpha_i}$ and is not the identity element and that any two adjacent $\alpha_i \neq \alpha_{i+1}$. Such words are called *reduced*.
- The group operation is $(g_1 \cdots g_m)(h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$ if g_m and h_1 are not in the same G_α. Otherwise we combine $(g_m h_1)$ as a single letter.
- $(g_1 \cdots g_m)^{-1} = g_m^{-1} \cdots g_1^{-1}$.
- The empty word is the identity element.

3.2. **The Van Kampen Theorem.** Suppose X is the union of path-connected open sets A_α, such that all A_α contain the base point $x_0 \in X$ (omitted in the notations below). The inclusion $A_\alpha \hookrightarrow X$ induces a homomorphism $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$. They extend to a homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$. Denote by $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$.

Note that $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, both induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$. It follows that $\ker \Phi$ contains all the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$, because

$$\Phi(i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}) = j_\alpha(i_{\alpha\beta}(\omega)) \cdot j_\beta(i_{\beta\alpha}(\omega)^{-1})) = (j_\alpha i_{\alpha\beta})(\omega) \cdot ((j_\beta i_{\beta\alpha})(\omega))^{-1}.$$

Theorem 3.1. *If X is the union of path-connected open sets A_α each containing the base point $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective.*

If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$.

Note that if $N = \ker \Phi$, then N is automatically a normal subgroup. The surjectivity part of the theorem has been proved already in Lemma 2.15. The other part is more technical, see [H] for a detailed proof.

Let us look at some examples.

Example 3.2. In the setting of the Van Kampen Theorem, if there are only two open sets A_α and A_β in the cover of X, then the condition on triple intersections is redundant, and one obtains an isomorphism $\pi_1(X) \cong (\pi_1(A_\alpha) * \pi_1(A_\beta))/N$, under the assumption that $A_\alpha \cap A_\beta$ is path connected.

Example 3.3. Consider the wedge sum $\vee_\alpha X_\alpha$ of a collection of path connected spaces X_α with base points x_α to be the quotient of the disjoint union $\sqcup_\alpha X_\alpha$ in which all the base points x_α are identified to a single point. If each x_α is a deformation retract of an open neighborhood U_α in X_α, then X_α is a deformation retract of its open neighborhood $A_\alpha = X_\alpha \vee_{\beta \neq \alpha} U_\beta$, which is path connected, since X_α is. The intersection of two or more distinct A_α's is $\vee_\alpha U_\alpha$, which deformation retracts to a point, hence it (is path connected and) has trivial fundamental group. The Van Kampen Theorem thus implies that $\Phi : *_\alpha \pi_1(X_\alpha) \to \pi_1(\vee_\alpha X_\alpha)$ is an isomorphism.

For instance, a wedge sum $\vee_\alpha S^1_\alpha$ of circles has fundamental group isomorphic to the free product of copies of $\mathbb{Z}$, one for each S^1_α.

Example 3.4. Let A be a circle in $\mathbb{R}^3$. Consider the complement $\mathbb{R}^3 - A$. It can deformation retract to S^2 union a diameter, by pushing points outside S^2 to S^2 and points inside S^2 to S^2 or the diameter away from A (draw a picture). Since S^2 union a diameter is homotopic to $S^2 \vee S^1$, by the Van Kampen Theorem, $\pi_1(\mathbb{R}^3 - A) \cong \pi_1(S^2) * \pi_1(S^1) \cong \mathbb{Z}$, since $\pi_1(S^2)$ is trivial. Indeed when one bends the diameter, one can see directly that $\mathbb{R}^3 - A$ deformation retracts to $S^2 \vee S^1$ (draw a picture).

Similarly if A and B are two unlinked circles in $\mathbb{R}^3$, then $\mathbb{R}^3 - A \cup B$ deformation retracts to $S^1 \vee S^1 \vee S^2 \vee S^2$, hence $\pi_1(\mathbb{R}^3 - A \cup B) \cong \mathbb{Z} * \mathbb{Z}$.

If A and B are two linked circles in $\mathbb{R}^3$, then $\mathbb{R}^3 - A \cup B$ deformation retracts to the wedge sum of S^2 and a torus $S^1 \times S^1$ separating A and B (draw a picture), hence $\pi_1(\mathbb{R}^3 - A \cup B) \cong \pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.

3.3. **Applications to cell complexes.** Let us study how the fundamental group is affected by attaching 2-cells. Suppose we attach a collection of 2-cells $\{e^2_\alpha\}$ to a path connected space X via maps $\varphi_\alpha : S^1 \to X$, producing a space Y. If s_0 is a base point of S^1, then φ_α determines a loop based at $\varphi_\alpha(s_0)$ in X, which we still denote by φ_α. Since $\varphi_\alpha(s_0)$ may not coincide for all α, choose a base point $x_0 \in X$ and a path γ_α from x_0 to $\varphi_\alpha(s_0)$. Then $\gamma_\alpha\varphi_\alpha\overline{\gamma}_\alpha$ is a loop at x_0. Let $N \subset \pi_1(X, x_0)$ be the subgroup generated by $\gamma_\alpha\varphi_\alpha\overline{\gamma}_\alpha$ for all α.

Proposition 3.5. *The inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is N, i.e. $\pi_1(Y) \cong \pi_1(X)/N$.*

It follows that N is independent of the choice of the paths γ_α. One can also see this directly. If we replace γ_α by another path η_α, then $\gamma_\alpha\varphi_\alpha\overline{\gamma}_\alpha$ changes to

$$\eta_\alpha\varphi_\alpha\overline{\eta}_\alpha = (\eta_\alpha\overline{\gamma}_\alpha)\gamma_\alpha\varphi_\alpha\overline{\gamma}_\alpha(\gamma_\alpha\overline{\eta}_\alpha),$$

so $\gamma_\alpha\varphi_\alpha\overline{\gamma}_\alpha$ and $\eta_\alpha\varphi_\alpha\overline{\eta}_\alpha$ define conjugate elements of $\pi_1(X, x_0)$.

Proof. We first expand Y to a larger space Z that deformation retracts to Y. For each α, take a rectangular strip $S_\alpha = I \times I$ and attach it to Y by gluing the lower edge $I \times \{0\}$ along γ_α, gluing the right edge $\{1\} \times I$ along an arc in e^2_α and identifying the left edge $\{0\} \times I$ to a common arc above x_0 for all α (draw a picture). We do not attach anything to the top edges. Clearly the resulting space Z deformation retracts to Y by shrinking the height of S_α.

In each e^2_α choose a point y_α not in the arc attached to S_α. Let $A = Z - \cup_\alpha\{y_\alpha\}$ and $B = Z - X$. Then A deformation retracts to X and B is contractible, hence $\pi_1(B) = 0$. Applying Van Kampen's Theorem to the cover $\{A, B\}$ of Z, we see that

$$\pi_1(Y) \cong \pi_1(Z) \cong \pi_1(X)/N',$$

where N' is the (normal) subgroup generated by the image of the map $\pi_1(A \cap B) \to \pi_1(A) \cong \pi_1(X)$.

It remains to show that $\pi_1(A \cap B)$ is generated by loops in $A \cap B$ that are homotopic to the loops $\gamma_\alpha\varphi_\alpha\overline{\gamma}_\alpha$. We apply Van Kampen's Theorem one more time to the cover of $A \cap B = Z - X - \cup_\alpha\{y_\alpha\}$ (the strips with bottom removed union the open punctured 2-cells) by the open sets $A_\alpha = A \cap B - \cup_{\beta \neq \alpha} e^2_\beta$. Since A_α deformation retracts to a circle in $e^2_\alpha - \{y_\alpha\}$ (a punctured open disk), we conclude that $\pi_1(A_\alpha) \cong \mathbb{Z}$ is generated by a loop homotopic to $\gamma_\alpha\varphi_\alpha\overline{\gamma}_\alpha$. $\square$

Example 3.6. Let M_g be the orientable surface of genus g, constructed by using one 0-cell, $2g$ 1-cells, and one 2-cell (draw a picture). The 1-skeleton is a wedge sum of $2g$ circles, hence its fundamental group is the free product $\mathbb{Z} * \cdots * \mathbb{Z}$ with $2g$ copies of $\mathbb{Z}$. The 2-cell is attached along the loop given by the product of the commutators $[a_1, b_1] \cdots [a_g, b_g]$. Hence by the preceding proposition, we have

$$\pi_1(M_g) \cong \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

Note that the abelianization of $\pi_1(M_g)$ is the direct sum of $2g$ copies of $\mathbb{Z}$. In particular for $g \neq h$, it implies that M_g is not homeomorphic, or even homotopic to M_h.

Corollary 3.7. *For every group G there is a 2-dimensional cell complex X_G with $\pi_1(X_G) \cong G$.*

Proof. Choose a presentation $G = \langle g_\alpha \mid r_\beta \rangle$, which means G is generated by g_α's with relation r_β's. It exists because every group is a quotient of a free group. Now construct X_G from $\vee_\alpha S^1_\alpha$ by attaching 2-cells e^2_β by the loops specified by the words r_β's. □

4. Covering Spaces

Recall that a *covering space* of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ such that

- For each point $x \in X$, there is an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by p.

Such a U is called *evenly covered* and the disjoint open sets in $\widetilde{X}$ that map homeomorphically onto U are called *sheets* of $\widetilde{X}$ over U.

4.1. **Lifting Properties.** A *lift* of a map $f : Y \to X$ is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$. Recall the following *homotopy lifting property* we discussed before.

Proposition 4.1. *Given a covering space $p : \widetilde{X} \to X$, a homotopy $f_t : Y \to X$ and a map $\widetilde{f}_0 : Y \to \widetilde{X}$ lifting f_0, then there exists a unique homotopy $\widetilde{f}_t : Y \to \widetilde{X}$ of $\widetilde{f}_0$ that lifts f_t.*

The special cases when Y is a point and $Y = I$ give the path lifting property and path homotopy lifting property.

Below is a simple application.

Proposition 4.2. *The map $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ induced by a covering space $p : (\widetilde{X}, x_0) \to (X, x_0)$ is injective.*

The image subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x}_0$ are loops.

Proof. Suppose $\widetilde{f}_0 : I \to \widetilde{X}$ is a loop at $\widetilde{x}_0$ in $\ker p_*$. Let $f_t : I \to X$ be a homotopy connecting $f_0 = p\widetilde{f}_0$ to the trivial loop f_1 at x_0. By the homotopy lifting property, there is a lifted homotopy $\widetilde{f}_t$ starting with $\widetilde{f}_0$ and ending with a constant loop. Hence $[\widetilde{f}_0] = 0$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$ and p_* is injective.

For the other statement, loops at x_0 lifting to loops at $\widetilde{x}_0$ certainly represent elements of the image of p_*. Conversely, if a loop represents an element of the image of p_*, then it is homotopic to a loop having such a lift, hence by the homotopy lifting property, the loop itself must have such a lift. □

Proposition 4.3. *Suppose X and $\widetilde{X}$ are path connected. Then the number of sheets of the covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ equals the index of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$.*

Proof. For a loop g in X based at x_0, denote by $\widetilde{g}$ its lift to $\widetilde{X}$ starting at $\widetilde{x}_0$. Let $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. A product $h \cdot g$ with $[h] \in H$ has the lift $\widetilde{h} \cdot \widetilde{g}$ ending at the same point as $\widetilde{g}$ since $\widetilde{h}$ is a loop by the preceding proposition. Thus we define a function Φ from all cosets $H[g]$ to $\pi^{-1}(x_0)$ by

$$H[g] \mapsto \widetilde{g}(1).$$

It suffices to show that Φ is an isomorphism.

Since $\widetilde{X}$ is path connected, $\widetilde{x}_0$ can be joined to any point in $p^{-1}(x_0)$ by a path $\widetilde{g}$ projecting to a loop g at x_0, hence Φ is surjective.

If $\Phi(H[g_1]) = \Phi(H[g_2])$, then $\widetilde{g}_1$ and $\widetilde{g}_2$ end at the same point, hence $g_1 \cdot \overline{g}_2$ lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$, so $[g_1][g_2]^{-1} \in H$ by the preceding proposition, and hence $H[g_1] = H[g_2]$, which implies that Φ is injective. □

If for each point $x \in X$ and each neighborhood U of x there exists an open neighborhood $V \subset U$ of x such that V is path connected, we say that X is *locally path connected.*

Example 4.4. Consider the graph of $y = \sin(\pi/x)$ for $0 < x < 1$, together with a closed arc joining $(1, 0)$ to $(0, 0)$ and joining $(0, 1)$ to $(0, 0)$. Then this space is path connected, but it is not locally path connected.

To conclude this section, we mention a more general lifting criterion.

Proposition 4.5. *Given a covering space* $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ *and a map* $f : (Y, y_0) \to (X, x_0)$ *with* Y *path connected and locally path connected, a lift* $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ *of* f *exists if and only if* $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

Proposition 4.6. *Given a covering space* $p : \widetilde{X} \to X$ *and a map* $f : Y \to X$*, if two lifts* $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$ *of* f *agree at one point of* Y *and if* Y *is connected, then* $\widetilde{f}_1$ *and* $\widetilde{f}_2$ *agree on all of* Y.

4.2. **Classification of Covering Spaces.** Since we talk about paths all the time, here we focus on path connected and locally path connected spaces X as well as path connected covering spaces. The motivating question is whether every subgroup of $\pi_1(X, x_0)$ can be realized as $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ for some covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. In particular to realize the trivial subgroup, since p_* is injective, it amounts to asking whether X has a simply connected covering space.

Let us first develop a necessary condition. Suppose $p : \widetilde{X} \to X$ is a covering space with $\widetilde{X}$ simply connected. Every point $x \in X$ has a neighborhood U with a lift $\widetilde{U} \subset \widetilde{X}$ projecting homeomorphically to U by p. Each loop in U lifts to a loop in $\widetilde{U}$, and the lifted loop has trivial homotopy class since $\pi_1(\widetilde{X}) = 0$. Therefore, composing it with p implies that the original loop in U is trivial in $\pi_1(X)$. To summarize, if each $x \in X$ has a neighborhood U such that the inclusion-induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial, we say that X is *semilocally simply connected.*

Remark 4.7. A locally simply connected space is certainly semilocally simply connected. Moreover, CW complexes satisfy the stronger property of being locally contractible.

Conversely, given a path connected, locally path connected and semilocally simply connected space X, we shall construct a simply connected covering space $\widetilde{X}$ of X. To motivate the construction, suppose $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a simply connected covering space. Then each $\widetilde{x} \in \widetilde{X}$ can be joined to $\widetilde{x}_0$ by a unique homotopy class of paths. Consequently by the homotopy lifting property, homotopy classes of paths in $\widetilde{X}$ starting at $\widetilde{x}_0$ are the same as homotopy classes of paths in X starting at x_0, since such a lift has unique homotopy class. This gives a way of describing $\widetilde{X}$ in terms of X.

More precisely, define

$$\widetilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}.$$

The function $p : \widetilde{X} \to X$ given by $[\gamma] \mapsto \gamma(1)$ is well-defined, since homotopic paths have the same endpoints. Since X is path connected, $\gamma(1)$ can be any point of X, so p is surjective.

Next we want to define a topology on $\widetilde{X}$. Some observations first. Let $\mathcal{U}$ be the collection of path connected open sets $U \subset X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. Since U is path connected, if the map $\pi_1(U) \to \pi_1(X)$ is trivial for one choice of base point in U, then it is trivial for all choices of base point. If $V \subset U$ is a path connected open subset, since the composition $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$ is trivial, it follows that V is also in $\mathcal{U}$. Therefore, $\mathcal{U}$ is a basis for the topology on X.

Given a set U in $\mathcal{U}$ and a path γ in X joining x_0 to a point in U, let

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}.$$

Clearly $U_{[\gamma]}$ depends only on the homotopy class $[\gamma]$. Since U is path connected, $p : U_{[\gamma]} \to U$ is surjective. Since $\pi_1(U) \to \pi_1(X)$ is trivial, different paths η in U joining $\gamma(1)$ to a fixed $x \in U$ are all homotopic in X, and hence $p : U_{[\gamma]} \to U$ is also injective. Moreover, we have

- $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. This is because if $[\gamma'] = [\gamma \cdot \eta]$, then elements of $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$ and hence lie in U_γ, while elements of $U_{[\gamma]}$ have the form $[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \overline{\eta} \cdot \mu] = [\gamma' \cdot \overline{\eta} \cdot \mu]$ and hence lie in $U_{[\gamma']}$.

The above property can be used to show that the sets $U_{[\gamma]}$ form a basis for a topology on $\widetilde{X}$. Given two such sets $U_{[\gamma]}$ and $V_{[\gamma']}$ and an element $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, by the above we have $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. So if $W \in \mathcal{U}$ is contained in $U \cap V$ and contains $\gamma''(1)$, then $W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']} = U_{[\gamma]} \cap V_{[\gamma']}$ and $[\gamma''] \in W_{[\gamma'']}$.

Now we show that the bijection $p : U_\gamma \to U$ is a homeomorphism, by showing that it gives a bijection between the subsets $V_{[\gamma']} \subset U_{[\gamma]}$ and the sets $V \in \mathcal{U}$ contained in U. In one direction we have seen that $p(V_{[\gamma']}) = V$. In the other direction, for any $[\gamma'] \in U_{[\gamma]}$ with endpoint in V, since $V_{[\gamma']} \subset U_{[\gamma']} = U_{[\gamma]}$ and $V_{[\gamma']}$ maps onto V by the bijection p, we have $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$.

Next we show that $p : \widetilde{X} \to X$ is a covering space. For fixed $U \in \mathcal{U}$, the sets $U_{[\gamma]}$ for varying $[\gamma]$ partition $p^{-1}(U)$, because if $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$, then $U_{[\gamma]} = U_{[\gamma'']} = U_{[\gamma']}$ by the above property.

We further show that $\widetilde{X}$ is path connected. For any point $[\gamma] \in \widetilde{X}$, let γ_t be the path in X such that it equals γ on $[0, t]$ and is stationary at $\gamma(t)$ on $[t, 1]$. Then the function $t \mapsto [\gamma_t]$ is a path in $\widetilde{X}$ lifting γ that starts at the constant path $[x_0]$ and ends at $[\gamma]$. This shows that $\widetilde{X}$ is path connected.

To show that $\widetilde{X}$ is simply connected, it suffices to show that $p_*(\pi_1(\widetilde{X}, [x_0])) = 0$ since p_* is injective. Recall that elements in the image of p_* are represented by loops γ at x_0 that lift to loops in $\widetilde{X}$ at $[x_0]$. In the preceding paragraph, we have seen that the path $t \mapsto [\gamma_t]$ lifts γ and starts at $[x_0]$. For this lifted path to be a loop, it means that $[\gamma_1] = [x_0]$. Since $\gamma_1 = \gamma$, it implies that $[\gamma] = [x_0]$ is trivial in $\pi_1(X, x_0)$.

The hypotheses for constructing a simply connected covering space of X in fact suffice for constructing covering spaces realizing arbitrary subgroups of $\pi_1(X)$.

Proposition 4.8. *Suppose X is path connected, locally path connected and semilocally simply connected. Then for every subgroup $H \subset \pi_1(X, x_0)$, there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$ for some base point $\widetilde{x}_0 \in X_H$.*

Proof. Recall the construction of a simply connected covering space $\widetilde{X}$ above. For points $[\gamma], [\gamma'] \in \widetilde{X}$, define $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot \overline{\gamma}'] \in H$. Since H is a subgroup, it is easy to see that this is an equivalence relation. Let X_H be the quotient space of $\widetilde{X}$ obtained by identifying $[\gamma]$ with $[\gamma']$ if $[\gamma] \sim [\gamma']$. Note that if $\gamma(1) = \gamma'(1)$, then $[\gamma] \sim [\gamma']$ if and only if $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$. It means that if any two points in basic neighborhoods $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified in X_H, then the whole neighborhoods are identified. Hence the projection $X_H \to X$ induced by $[\gamma] \to \gamma(1)$ is a covering space.

Choose $\widetilde{x}_0 \in X_H$ to be the equivalence class of the constant path c at x_0. For a loop γ in X based at x_0, its lift to $\widetilde{X}$ starting at $[c]$ ends at $[\gamma]$, so the image of this lifted path in X_H is a loop if and only if $[\gamma] \sim [c]$, i.e., $[\gamma] \in H$. It implies that the image of $p_* : \pi_1(X_H, \widetilde{x}_0) \to \pi_1(X, x_0)$ is exactly H. □

We are interested in classifying covering spaces up to isomorphism. For two covering spaces $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$, if $f : \widetilde{X}_1 \to \widetilde{X}_2$ is a homeomorphism such that $p_1 = p_2 f$, then we say that f is an *isomorphism* between the two covering spaces. In that case f^{-1} is also an isomorphism, and the composition of two isomorphisms is an isomorphism, so we have an equivalence relation.

Proposition 4.9. *Suppose X is path connected and locally path connected. Then two path connected covering spaces $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are isomorphic via an isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ taking a base point $\widetilde{x}_1 \in p_1^{-1}(x_0)$ to a base point $\widetilde{x}_2 \in p_2^{-1}(x_0)$ if and only if $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

Proof. If there is an isomorphism $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$, then from the relation $p_1 = p_2 f$ it follows that $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2*}(f_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1))) = p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.

Conversely, suppose that $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$. By the lifting criterion for maps, we may lift p_1 to a map $\widetilde{p}_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ with $p_2 \widetilde{p}_1 = p_1$. Similarly we obtain $\widetilde{p}_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ with $p_1 \widetilde{p}_2 = p_2$. Then $p_1(\widetilde{p}_2 \widetilde{p}_1) = (p_1 \widetilde{p}_2)\widetilde{p}_1 = p_2 \widetilde{p}_1 = p_1$, hence both $\widetilde{p}_2 \widetilde{p}_1$ and $\mathbb{1}$ are lifts of p_1 fixing the base point $\widetilde{x}_1$. By the unique lifting property, we have $\widetilde{p}_2 \widetilde{p}_1 = \mathbb{1}$, and similarly $\widetilde{p}_1 \widetilde{p}_2 = \mathbb{1}$, so $\widetilde{p}_1$ and $\widetilde{p}_2$ are inverse isomorphisms. □

We summarize the discussion by the following result.

Theorem 4.10. *Let X be path connected, locally path connected and semilocally simply connected. Then there is a bijection between the set of base point preserving isomorphism classes of path connected covering spaces $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ to the covering space $(\widetilde{X}, \widetilde{x}_0)$.*

If the base points are ignored, then this correspondence gives a bijection between isomorphism classes of path connected covering spaces $p : \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Proof. We have proved the first half of the result. For the other half, we will show that for a covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, changing the base point $\widetilde{x}_0$ within $p^{-1}(x_0)$ corresponds exactly to changing $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ to a conjugate subgroup of $\pi_1(X, x_0)$.

Suppose that $\widetilde{x}_1$ is another base point in $p^{-1}(x_0)$. Let $\widetilde{\gamma}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}_1$. Then $\widetilde{\gamma}$ projects to a loop γ in X representing an element $g \in \pi_1(X, x_0)$. Set

$H_i = p_*(\pi_1(\widetilde{X}, \widetilde{x}_i))$ for $i = 0, 1$. For a loop $\widetilde{f}$ at $\widetilde{x}_0$, $\overline{\widetilde{\gamma}} \cdot \widetilde{f} \cdot \widetilde{\gamma}$ is a loop at $\widetilde{x}_1$, which induces an inclusion $g^{-1}H_0g \subset H_1$. Similarly we have $gH_1g^{-1} \subset H_0$. Conjugating the latter by g^{-1} implies that $H_1 \subset g^{-1}H_0g$, hence $H_1 = g^{-1}H_0g$. Thus changing the base point from $\widetilde{x}_0$ to $\widetilde{x}_1$ changes H_0 to the conjugate subgroup $H_1 = g^{-1}H_0g$.

Conversely, to change H_0 to a conjugate subgroup $H_1 = g^{-1}H_0g$, choose a loop γ representing g, lift it to a path $\widetilde{\gamma}$ starting at $\widetilde{x}_0$, and let $\widetilde{x}_1 = \widetilde{\gamma}(1)$. Then the preceding argument shows that $H_1 = g^{-1}H_0g$. □

As a consequence of the lifting criterion, a simply connected covering space of a path connected and locally path connected space X is a covering space of every other path connected covering space of X. Therefore, a simply connected covering space of X is called the *universal cover*, since it is also unique up to isomorphism.

More generally, there is a *partial ordering* on the various path connected covering spaces of X, according to which ones cover which others. This corresponds to the partial ordering by inclusion of the corresponding subgroups of $\pi_1(X)$, or conjugacy classes of subgroups if base points are ignored.

Example 4.11. Consider $X = S^1 \vee S^1$ as an oriented loop $a \cdot b$. Now let $\widetilde{X}$ be any graph with four edges meeting at each vertex, and suppose the edges of $\widetilde{X}$ are assigned labels a and b with orientations such that locally near each vertex is the same as X, i.e., an a-edge oriented toward the vertex, an a-edge oriented away from the vertex, a b-edge oriented toward the vertex, and a b-edge oriented away from the vertex. We call such $\widetilde{X}$ a 2-oriented graph (draw a picture).

Given a 2-oriented graph $\widetilde{X}$, one can construct a map $p : \widetilde{X} \to X$ sending all vertices of $\widetilde{X}$ to the vertex of X and each edge of $\widetilde{X}$ to the edge of X with the same label by a map that is homeomorphism on the interior of the edge and preserves orientation. Clearly p is a covering. Conversely, it is easy to see that every covering space of X inherits such a 2-oriented graph structure (draw more graphs). Since $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective, to describe $\pi_1(\widetilde{X}, \widetilde{x}_0)$, one can instead describe its image subgroup in $\pi_1(X, x_0) \cong \mathbb{Z} * \mathbb{Z}$ generated by a and b (do for the graphs).

The universal cover of X can be constructed as follows (draw a picture). It is clearly simply connected and covers X, since it is an infinite 2-oriented graph without loops.

4.3. **Deck Transformations and Group Actions.** An isomorphism $\widetilde{X} \to \widetilde{X}$ of a covering space $p : \widetilde{X} \to X$ is called a *deck transformation*. Deck transformations of $\widetilde{X}$ form a group $G(\widetilde{X})$ under composition.

Example 4.12. For the universal cover $p : \mathbb{R} \to S^1$, the deck transformations are given by integral translations on $\mathbb{R}$, hence $G \cong \mathbb{Z}$ in this case.

Remark 4.13. Assuming $\widetilde{X}$ is path connected, by the unique lifting property, a deck transformation is completely determined by where it sends a single point. In particular, only the identity deck transformation can fix a point of $\widetilde{X}$.

If for each $x \in X$ and each pair of lifts $\widetilde{x}, \widetilde{x}'$ of x there is a deck transformation taking $\widetilde{x}$ to $\widetilde{x}'$, then $p : \widetilde{X} \to X$ is called *normal*. For example, the universal cover $p : \mathbb{R} \to S^1$ is normal. The term "normal" is inspired by the following result.

Proposition 4.14. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a path connected covering space of the path connected and locally path connected space X, and let H be the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subset \pi_1(X, x_0)$. Then*

- *$\widetilde{X}$ is normal covering if and only if H is a normal subgroup of $\pi_1(X, x_0)$.*
- *$G(\widetilde{X}) \cong N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.*

In particular, $G(\widetilde{X}) \cong \pi_1(X, x_0)/H$ if $\widetilde{X}$ is a normal covering. Hence for the universal cover $\widetilde{X} \to X$ we have $G(\widetilde{X}) \cong \pi_1(X)$.

Proof. We saw earlier that changing the base point $\widetilde{x}_0 \in p^{-1}(x_0)$ to $\widetilde{x}_1 \in p^{-1}(x_0)$ corresponds precisely to conjugating H by an element $[\gamma] \in \pi_1(X, x_0)$, where γ lifts to a path $\widetilde{\gamma}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. Hence $[\gamma] \in N(H)$ if and only if $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$, which by the lifting property is equivalent to the existence of a deck transformation τ taking $\widetilde{x}_0$ to $\widetilde{x}_1$. It follows that $\widetilde{X}$ is a normal covering space if and only if $N(H) = \pi_1(X, x_0)$, i.e., if and only if H is a normal subgroup of $\pi_1(X, x_0)$.

Define $\varphi : N(H) \to G(\widetilde{X})$ by sending $[\gamma]$ to the deck transformation τ taking $\widetilde{x}_0$ to $\widetilde{x}_1$ in the notation above. If γ' is another loop corresponding to a deck transformation τ' taking $\widetilde{x}_0$ to $\widetilde{x}_1'$, then $\gamma \cdot \gamma'$ lifts to $\widetilde{\gamma} \cdot (\tau(\widetilde{\gamma}'))$, which is a path from $\widetilde{x}_0$ to $\tau(\widetilde{x}_1') = \tau\tau'(\widetilde{x}_0)$, so $\tau\tau'$ is the deck transformation corresponding to $[\gamma][\gamma']$. Hence φ is a homomorphism. By the preceding paragraph, φ is surjective. Moreover, its kernel consists of classes $[\gamma]$ lifting to loops in $\widetilde{X}$, which are precisely the elements of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = H$. □

In general, given a group G and a space Y, an *action* of G on Y is a homomorphism from G to the group Homeo(Y) of all homeomorphisms from Y to itself. For $g \in G$, we will simply denote by g the induced homeomorphism.

We will be interested in actions satisfying the following condition:

($*$) Each $y \in Y$ has a neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint. In other words, $g_1(U) \cap g_2(U) \neq \emptyset$ implies $g_1 = g_2$.

Note that to satisfy ($*$), it suffices to take g_1 to be the identify, since $g_1(U) \cap g_2(U) \neq \emptyset$ is equivalent to $U \cap g_1^{-1} g_2(U) \neq \emptyset$.

The action of the deck transformation group $G(\widetilde{X})$ on $\widetilde{X}$ satisfies ($*$). To see this, take $\widetilde{U} \subset \widetilde{X}$ that projects homeomorphically to $U \subset X$. If $\widetilde{U} \cap g(\widetilde{U}) \neq \emptyset$ for some $g \in G(\widetilde{x})$, then $\widetilde{x}_1 = g(\widetilde{x}_2)$ for some $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{U}$. Since $\widetilde{x}_1$ and $\widetilde{x}_2$ lie in the same fiber $p^{-1}(x)$, which intersects $\widetilde{U}$ only at one point by assumption, we have $\widetilde{x}_1 = \widetilde{x}_2$, which is then a fixed point of g, so $g = \mathbb{1}$.

Given an action of a group G on a space Y, we can form a space Y/G in which each point y is identified with all its images $g(y)$ as g ranges over G. The points in Y/G are thus the *orbits* $Gy = \{g(y) \mid g \in G\}$ in Y, and Y/G is called the *orbit space* of the action. For example, a normal covering space $\widetilde{X} \to X$ has the orbit space $\widetilde{X}/G(\widetilde{X})$ equal to X.

Proposition 4.15. *If an action of G on Y satisfies ($*$), then*

- *The quotient map $p : Y \to Y/G$ by $p(y) = Gy$ is a normal covering space.*
- *If Y is path connected, then $G = G(p)$ is the group of deck transformations of p.*
- *If Y is path connected and locally path connected, then $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$.*

Proof. Let $U \subset Y$ be an open set as in condition $(*)$. The map p identifies all the disjoint homeomorphic sets $\{g(U) \mid g \in G\}$ to a single open set $p(U)$ in Y/G, so p restricted to $g(U)$ is a homeomorphism onto $p(U)$ for each $g \in G$, and hence p is a covering. Each element of G acts as a deck transformation of p. Since $g_2 g_1^{-1}$ takes $g_1(U)$ to $g_2(U)$, p is normal.

The deck transformation group $G(p)$ contains G as a subgroup. If Y is path connected, let f be any deck transformation of p. Then for a chosen $y \in Y$, y and $f(y)$ are in the same orbit, hence there is $g \in G$ such that $g(y) = f(y)$, hence $f = g$ since a deck transformation of a path connected covering space is determined by the image of a single point. It implies $G = G(p)$.

The last statement follows from the last part of the preceding proposition. □

Remark 4.16. Motivated by the above proposition, an action satisfying $(*)$ is called a *covering space action*. Given a covering space action G on a simply connected locally path connected space Y, the orbit space Y/G has fundamental group isomorphic to G. This provides a useful technique for computing fundamental groups, e.g., when we computed $\pi_1(S^1) \cong \mathbb{Z}$, since S^1 is the orbit space of $\mathbb{R}/\mathbb{Z}$ where the action is given by translations.

Example 4.17. The real projective space $\mathbb{RP}^n = S^n/\mathbb{Z}_2$, where the action on S^n is generated by $x \mapsto -x$, the antipodal map. This action is a covering space, since each open hemisphere in S^n is disjoint from its antipodal image. Moreover, S^n is simply connected for $n \geq 2$. It follows that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ for $n \geq 2$. A generator of $\pi_1(\mathbb{RP}^n)$ is any loop obtained by projecting a path in S^n connecting two antipodal points.

Example 4.18. Present a 5-fold cover of a genus 11 surface M_{11} over a genus 3 surface M_3, which can be realized as $M_{11}/\mathbb{Z}_5$ (draw a picture). In general, if there is a covering space $M_g \to M_h$, then $g = mn + 1$ and $h = m + 1$ for some integers m and n, which is a consequence of the Riemann-Hurwitz formula.

5. Simplicial and Singular Homology

5.1. **Δ-complexes.** Recall the standard *n-simplex*

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

Let $v_0, \ldots, v_n$ be the unit vectors along the coordinate axes. Then they correspond to *vertices* of Δ^n, so we also write $\Delta^n = [v_0, \ldots, v_n]$. For purposes of homology we consider the vertices of Δ^n being ordered. If we delete one of the vertices, the remaining n vertices span an $(n-1)$-simplex, called a *face* of Δ^n, and the ordering of the vertices is inherited from Δ^n. The union of all the faces of Δ^n is called the *boundary* of Δ^n, denoted by $\partial\Delta^n$ The *interior* of Δ^n is $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$.

A Δ-*complex* structure on a space X is a collection of maps $\delta_\alpha : \Delta^n \to X$, where n depends on α, such that:

- The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction.
- Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$.
- A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Among other things, the last condition rules out trivial cases like all points of X as individual vertices.

Example 5.1. Draw Δ-complex structure for the 2-torus and $\mathbb{RP}^2$, with arrows to indicate the ordering of the vertices.

5.2. **Simplicial Homology.** Given a Δ-complex X, let $\Delta_n(X)$ be the free abelian group generated by n-simplices of X. Elements of $\Delta_n(X)$ are called *n-chains*, written as finite formal sums $\sum_\alpha n_\alpha \sigma_\alpha$ or simply $\sum_\alpha n_\alpha e^n_\alpha$, where $n_\alpha \in \mathbb{Z}$ and $\sigma_\alpha : \Delta^n \to X$ is the defining map with e^n the interior of Δ^n.

The boundary of an n-simplex $[v_0, \ldots, v_n]$ consists of various $(n-1)$-simplices $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ by skipping a vertex v_i. Define

$$\partial[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

(draw the cases for $n = 1, 2$). Using this as a building block, we define the *boundary homomorphism* $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by specifying its values on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}.$$

Note that the right side lies in $\Delta_{n-1}(X)$, since each $\sigma_\alpha|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$ by definition is an $(n-1)$-simplex of X.

Lemma 5.2. *The composition* $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ *is zero.*

Proof. We have

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]}.$$

The two summations cancel, since after switching i and j in the second sum, it becomes the negative of the first. $\square$

In general, suppose we have a *chain complex*, i.e., a sequence of homomorphisms of abelian groups

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \partial_{n+1} = 0$ for each n, namely, $\operatorname{im} \partial_{n+1} \subset \ker \partial_n$. Then we define the *$n$-th homology group* to be the quotient group $H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$. Elements of $\ker \partial_n$ are called *cycles*, elements of $\operatorname{im} \partial_{n+1}$ are called *boundaries*, and elements of H_n are called *homology classes*. Two cycles representing the same homology class are called *homologous*. For the case $C_n = \Delta_n(X)$, the corresponding homology group is denoted by $H^\Delta_n(X)$ and called the *n-th simplicial homology group* of X.

Example 5.3. Let $X = S^1$ with one vertex v and one edge e (draw a picture). Then $\Delta_0(S^1) = \Delta_1(S^1) = \mathbb{Z}$ and the boundary map ∂_1 is zero since $\partial e = v - v$. Hence $H^\Delta_n(S^1) \cong \mathbb{Z}$ for $n = 0, 1$ and $H^\Delta_n(S^1) = 0$ for $n \geq 2$ since there are no higher dimensional simplices.

Example 5.4. Let X be the 2-torus T, with the Δ-complex structure given by one vertex v, three edges a, b, and c, and two 2-simplices U and L (draw a picture). As in the previous example, $\partial_1 = 0$ so $H^\Delta_0(T) \cong \mathbb{Z}$. Since $\partial U = b - c + a$ and $\partial L = a - c + b$ are the same, we have $H^\Delta_1(T) \cong \mathbb{Z}^{\oplus 3}/\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H^\Delta_2(T) \cong \mathbb{Z}$.

There are some obvious questions. Are the groups $H_n^{\Delta}(X)$ independent of the choice of Δ-complex structure on X? Do two homeomorphic spaces have the same homology groups? How about homotopy equivalent spaces? In order to better answer such questions, we need a more general setting as discussed in the next section.

5.3. **Singular Homology.** A *singular n-simplex* in a space X is just a continuous map $\sigma : \Delta^n \to X$. Let $C_n(X)$ be the free abelian group generated by the set of singular n-simplices in X. Elements of $C_n(X)$ are called *singular n-chains*, given by finite sums $\sum_i n_i\sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$. A boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined in the same way by

$$\partial_n(\sigma) = \sum_{i=0}^{n}(-1)^i\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}.$$

It induces the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$, sometimes simply denoted by ∂. Similarly as before, we have $\partial^2 = 0$, so we can define the *singular homology group* $H_n(X) = \ker\partial_n/\operatorname{im}\partial_{n+1}$. By definition, homeomorphic spaces have isomorphic singular homology groups.

Proposition 5.5. *Suppose X splits into its path connected components $\{X_\alpha\}$. Then there is an isomorphism $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$.*

Proof. Since a singular simplex always has path connected image, $C_n(X)$ splits as the direct sum of subgroups $C_n(X_\alpha)$, and the boundary maps preserve such decomposition, taking $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$, so do $\ker\partial_n$ and $\operatorname{im}\partial_{n+1}$. □

Proposition 5.6. *If X is path connected, then $H_0(X) \cong \mathbb{Z}$. Hence for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path connected component of X.*

Proof. Since $\partial_0 = 0$, we have $H_0(X) = C_0(X)/\operatorname{im}\partial_1$. Define a homomorphism $\varphi : C_0(X) \to \mathbb{Z}$ by $\varphi(\sum_i n_i\sigma_i) = \sum_i n_i$. Obvisouly φ is surjective. So it suffices to show that $\ker\varphi = \operatorname{im}\partial_1$.

Observe first that $\operatorname{im}\partial_1 \subset \ker\varphi$, since for a singular 1-simplex $\sigma : \Delta^1 \to X$, we have $\varphi\partial_1(\sigma) = \varphi(\sigma|_{[v_1]} - \sigma|_{[v_0]}) = 1-1 = 0$. For the converse, suppose $\varphi(\sum_i n_i\sigma_i) = 0$, so $\sum_i n_i = 0$. The images of σ_i's are simply points of X. Choose a path $\gamma_i : I \to X$ from a base point x_0 to $\sigma_i(v_0)$ and let σ_0 be the singular 0-simplex with image x_0. We can view γ_i as a singular 1-simplex: $[v_0, v_1] \to X$, and $\partial\gamma_i = \sigma_i - \sigma_0$. Hence $\partial(\sum_i n_i\gamma_i) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i$, which is a boundary. This shows that $\ker\varphi \subset \operatorname{im}\partial_1$. □

Proposition 5.7. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$.*

Proof. In this case there is a unique singular n-simplex σ_n for each n, and $\partial(\sigma_n) = \sum_{i=0}^{n}(-1)^i\sigma_{n-1}$, a sum of $n+1$ terms, which is 0 for n odd and σ_{n-1} for n even, $n \neq 0$. Thus we have the chain complex

$$\cdots \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with boundary maps alternatively isomorphisms and trivial maps, except for the last one. It is easy to see that the homology groups of this complex are trivial except for $H_0 \cong \mathbb{Z}$. □

It is often convenient to have a modified version of homology such that a point has trivial homology in all dimensions, including zero. Consider the chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varphi} \mathbb{Z} \to 0$$

where $\varphi(\sum_i n_i \sigma_i) = \sum_i n_i$ as in the above proof. Define the *reduced homology groups* $\widetilde{H}_n(X)$ to be the homology groups associated to this chain complex. Apparently $\widetilde{H}_n(X) \cong H_n(X)$ for all $n > 0$. Since $\varphi\partial_1 = 0$, $\operatorname{im}\partial_1 \subset \ker\varphi$, hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$, so $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.

5.4. **Homotopy Invariance.** We will prove that homotopy equivalent spaces have isomorphic homology groups.

For a map $f : X \to Y$, define a homomorphism $f_\sharp : C_n(X) \to C_n(Y)$ induced by composing each $\sigma : \Delta^n \to X$ with f, that is, $f_\sharp(\sigma) = f\sigma : \Delta^n \to Y$. Note that $f_\sharp$ commutes with ∂ since

$$\begin{aligned} f_\sharp\partial(\sigma) &= f_\sharp\left(\sum_i (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}\right) \\ &= \sum_i (-1)^i f\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} \\ &= \partial f_\sharp(\sigma). \end{aligned}$$

So we have a diagram between the chain complexes $C(X)$ and $C(Y)$ (draw a diagram) such that each square is commutative. By the relation $f_\sharp\partial = \partial f_\sharp$, it implies that $f_\sharp$ takes cycles to cycles and boundaries to boundaries. Hence $f_\sharp$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$. Such f in general is called a *chain map.*

Theorem 5.8. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.*

Proof. We want to divide $\Delta^n \times I$ into simplices. Let $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$, where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$. We can pass from $[v_0, \ldots, v_n]$ to $[w_0, \ldots, w_n]$ via a sequence of n-simplices: $[v_0, \ldots, v_n], [v_0, \ldots, v_{n-1}, w_n], [v_0, \ldots, v_{n-2}, w_{n-1}, w_n], \cdots, [w_0, \ldots, w_n]$ (draw a picture). The region between two nearby n-simplices is the $(n+1)$-simplex $[v_0, \ldots, v_i, w_i, \ldots, w_n]$, which has $[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$ as its lower face and $[v_0, \ldots, v_{i-1}, w_i, \ldots, w_n]$ as its higher face.

Given a homotopy $F : X \times I \to Y$ from f to g and $\sigma : \Delta^n \to X$, define the composition $F \circ (\sigma \times \mathbb{1}) : \Delta^n \times I \to X \times I \to Y$. Then we define the operators $P : C_n(X) \to C_{n+1}(Y)$ by

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbb{1})|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}.$$

We claim that

$$\partial P = g_\sharp - f_\sharp - P\partial,$$

where the left side represents the boundary of P, and the three terms on the right side represent the top $\Delta^n \times \{1\}$, the bottom $\Delta^n \times \{0\}$, and the sides $\partial\Delta^n \times I$ of P. More precisely,

$$\partial P(\sigma) = \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times \mathbb{1})|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]}$$

$$+\sum_{j\geq i}(-1)^i(-1)^{j+1}F\circ(\sigma\times\mathbb{1})|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,\ldots,w_n]}.$$

The terms with $i=j$ in the two sums cancel except for

$$F\circ(\sigma\times\mathbb{1})|_{[\hat{v}_0,w_0,\ldots,w_n]}=g\circ\sigma=g_\sharp(\sigma)$$

and

$$-F\circ(\sigma\times\mathbb{1})|_{[v_0,\ldots,v_n,\hat{w}_n]}=-f\circ\sigma=-f_\sharp(\sigma).$$

The terms with $i\neq j$ are exactly $-P\partial$, because

$$P\partial(\sigma)=\sum_{i<j}(-1)^i(-1)^jF\circ(\sigma\times\mathbb{1})|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,\ldots,w_n]}$$

$$+\sum_{i>j}(-1)^{i-1}(-1)^jF\circ(\sigma\times\mathbb{1})|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]}.$$

To conclude the proof, if $\alpha\in C_n(X)$ is a cycle, then $\partial\alpha=0$ and hence

$$g_\sharp(\alpha)-f_\sharp(\alpha)=\partial P(\alpha)+P\partial(\alpha)=\partial P(\alpha)$$

is a boundary. It implies that $g_\sharp(\alpha)$ and $f_\sharp(\alpha)$ determine the same homology class, so $g_*=f_*$ on the homology class of α. □

Remark 5.9. In general, if P satisfies the relation $\partial P+P\partial=g_\sharp-f_\sharp$, such P is called a *chain homotopy* between the chain maps $f_\sharp$ and $g_\sharp$. In particular, chain-homotopy maps induce the same homomorphism on homology.

Corollary 5.10. *If $f:X\to Y$ is a homotopy equivalence, then $f_*:H_n(X)\to H_n(Y)$ is an isomorphism for all n. In particular, if X is contractible, then $H_n(X)=0$ for all $n>0$.*

Proof. By assumption, there exists $g:Y\to X$ such that $f_*g_*=\mathbb{1}_*$ and $g_*f_*=\mathbb{1}_*$. The result thus follows. □

6. Exact Sequences and Excision

6.1. **Exact Sequences.** Recall that a sequence of homomorphisms

$$\cdots\to A_{n+1}\xrightarrow{\alpha_{n+1}}A_n\xrightarrow{\alpha_n}A_{n-1}\to\cdots$$

is called *exact* if $\ker\alpha_n=\operatorname{im}\alpha_{n+1}$ for each n. For a chain complex where $\alpha_n\alpha_{n+1}=0$, being exact means that the homology groups are trivial. In particular, an exact sequence $0\to A\to B\to C\to 0$ is called a *short exact sequence.*

Theorem 6.1. *If X is a space and A is a closed subspace that is a deformation retract of some neighborhood in X, then there is an exact sequence*

$$\cdots\to\widetilde{H}_n(A)\xrightarrow{i_*}\widetilde{H}_n(X)\xrightarrow{j_*}\widetilde{H}_n(X/A)\xrightarrow{\partial}\widetilde{H}_{n-1}(A)\xrightarrow{i_*}\cdots\to\widetilde{H}_0(X/A)\to 0$$

where $i:A\to X$ is the inclusion, $j:X\to X/A$ is the quotient map and ∂ will be constructed in the proof.

Pairs (X,A) satisfying the hypothesis of the theorem are called *good pairs.* For example, if X is a CW complex and A is a subcomplex, then (X,A) is a good pair.

Before proving the theorem, let us draw some consequences.

Corollary 6.2. $\widetilde{H}_n(S^n)\cong\mathbb{Z}$ *and* $\widetilde{H}_i(S^n)=0$ *for* $i\neq n$*.*

Proof. For $n > 0$, take $(X, A) = (D^n, S^{n-1})$ so $X/A = S^n$. Since D^n is contractible, the terms $\widetilde{H}_i(D^n)$ in the above long exact sequence are all zero. It implies that $\widetilde{H}_i(S^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and $\widetilde{H}_0(S^n) = 0$. Now the result follows by induction on n, starting from the known case of S^0. □

Corollary 6.3. *∂D^n is not a retract of D^n. Hence every map $f : D^n \to D^n$ has a fixed point.*

Proof. If $r : D^n \to \partial D^n$ is a retraction, then $ri = \mathbb{1}$ for the inclusion $i : \partial D^n \to D^n$. Then the composition

$$\widetilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \widetilde{H}_{n-1}(D^n) \xrightarrow{r_*} \widetilde{H}_{n-1}(\partial D^n)$$

is the identity map on $\widetilde{H}_{n-1}(\partial D^n) = \widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$, which contradicts that $\widetilde{H}_{n-1}(D^n) = 0$ for the middle term. The second statement follows from the same construction as in the proof of the Brouwer fixed point theorem. □

To prove the above theorem, we will indeed prove a more general exact sequence which holds for arbitrary pairs (X, A) but with the homolog groups of X/A replaced by *relative homology groups* $H_n(X, A)$, discussed in the next section.

6.2. **Relative Homology Groups.** Given a subspace $A \subset X$, let $C_n(X, A)$ be the quotient group $C_n(X)/C_n(A)$. In other words, chains in A are trivial in $C_n(X, A)$. It induces a quotient boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$ and hence a chain complex

$$\cdots \to C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \cdots$$

The homology groups of this chain complex are called the *relative homology groups* $H_n(X, A)$. The respective cycle and boundary elements are called *relative cycles* and *relative boundaries*.

Our goal is to show the following long exact sequence:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots \to H_0(X/A) \to 0.$$

The proof is entirely formal, using standard diagram chasing in homological algebra, see [H, p. 112] (draw a commutative diagram and explain ∂).

In particular, the boundary map $\partial : H_n(X, A) \to H_{n-1}(A)$ has a simple description. If a class $[\alpha] \in H_n(X, A)$ is represented by a relative cycle α, then $\partial[\alpha]$ is the class of the cycle $\partial\alpha$ in $H_{n-1}(A)$. Moreover, a completely analogous long exact sequence holds for reduced homology groups for a pair (X, A), where $H_n(X)$ and $H_n(A)$ are replaced by $\widetilde{H}_n(X)$ and $\widetilde{H}_n(A)$. This is done by adding the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0$ in dimension -1.

Example 6.4. In the long exact sequence of reduced homology groups for the pair $(D^n, \partial D^n = S^{n-1})$, the maps $H_i(D^n, \partial D^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for all $i > 0$ since the other terms $\widetilde{H}_i(D^n) = 0$ for all i. Thus we obtain $H_i(D^n, \partial D^n) \cong \mathbb{Z}$ for $i = n$ and 0 otherwise.

Along this circle of ideas, we have the following results.

Proposition 6.5. *If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$, then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$.*

Proof. Adapt the proof of the non-relative version. □

Proposition 6.6. *For a triple $B \subset A \subset X$, we have the long exact sequence*

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$$

Proof. It follows from the short exact sequences

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0.$$

□

6.3. **Excision.** In this section we discuss the following excision theorem and its applications.

Theorem 6.7. *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X - Z, A - Z) \to H_n(X, A)$ for all n.*

Equivalently, for subspaces $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $H_n(B, A \cap B) \to H_n(X, A)$ for all n.

The equivalence between the two versions is by setting $B = X - Z$ and $Z = X - B$. Then $A \cap B = A - Z$ and the condition that cl $Z \subset$ int A is equivalent to $X = \text{int } A \cap \text{int } B$ since $X - \text{int } B = \text{cl } Z$ (draw a picture).

The proof of the excision theorem relies on the following technical result, which allows homology groups to be computed using small singular simplices. Let $\mathcal{U} = \{U_j\}$ be a collection of subspaces of X whose interiors form an open cover of X. Let $C_n^{\mathcal{U}}(X)$ be the subgroup of $C_n(X)$ consisting of chains $\sum_i n_i \sigma_i$ such that each σ_i has image contained in some set U_j in the cover $\mathcal{U}$. Since $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$, the groups $C_n^{\mathcal{U}}(X)$ form a chain complex. Denote by $H_n^{\mathcal{U}}(X)$ the corresponding homology groups.

Proposition 6.8. *The inclusion $i : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence, namely, there exists a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ such that $i\rho$ and ρi are chain homotopic to the identity. Hence i induces an isomorphism $H_n^{\mathcal{U}}(X) \to H_n(X)$ for all n.*

Recall that a chain homotopy between two chain maps $f, g : A \to B$ is a map $P : A_n \to B_{n+1}$ such that $\partial P + P\partial = f - g$, which implies that f and g induce the same map on homology groups of A and B.

Outline of the proof. The upshot is a construction of $D : C_n(X) \to C_{n+1}(X)$ such that D is identically zero on $C_n^{\mathcal{U}}(X)$ and such that there exists a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ satisfying $\partial D + D\partial = \mathbb{1} - i\rho$, which implies also $\rho i = \mathbb{1}$. The proof uses essentially a barycentric subdivision of simplices, see the proof of [H, Proposition 2.21]. □

Proof of the excision theorem. We prove the second version where $X = A \cup B$. For the cover $\mathcal{U} = \{A, B\}$, we use the notation $C_n(A+B)$ for $C_n^{\mathcal{U}}(X)$, the sums of chains in A and chains in B. In the above we have $\partial D + D\partial = \mathbb{1} - i\rho$ and $\rho i = \mathbb{1}$. All the maps in these formulas take chains in A to A (and chains in B to B), hence they induce quotient maps when we factor out chains in A. These quotient maps also satisfy the same two formulas, so the inclusion $C_n(A+B)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$ induces an isomorphism on homology. The map $C_n(B)/C_n(A \cap B) \to C_n(A + B)/C_n(A)$ induced by inclusion is also an isomorphism, since both quotient groups are free generated by singular n-simplices in B that do not lie in A. Hence we obtain the desired isomorphism $H_n(B, A \cap B) \cong H_n(X, A)$ induced by inclusion. □

Recall that we have two similar long exact sequences of homology, where in the former the homology of the quotient is replaced by the relative homology in the latter. Let us show that the latter implies the former.

Proposition 6.9. *For good pairs (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms $q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$ for all n.*

Proof. By definition, suppose V is a neighborhood of A in X that deformation retracts onto A. We have the following diagram (draw a diagram). Since V deformation retracts onto A, we have a homotopy equivalence of pairs $(V, A) \simeq (A, A)$ and $H_n(A, A) = 0$, hence $H_n(V, A) = 0$ for all n. Then by the long exact sequence of the triple $A \subset V \subset X$, we obtain an isomorphism $H_n(X, A) \to H_n(X, V)$ for all n. Similarly $H_n(X/A, A/A) \to H_n(X/A, V/A)$ is also an isomorphism. The other two horizontal maps are also isomorphisms from excision. The right side vertical map q_* is an isomorphism since it restricts to a homeomorphism on the complement of A. Now by diagram chasing we conclude that the left side q_* is an isomorphism. □

Example 6.10. Let us find explicit cycles that generate $H_n(D^n, \partial D^n)$ and $\widetilde{H}_n(S^n)$. Replacing $(D^n, \partial D^n)$ by the equivalent pair $(\Delta^n, \partial\Delta^n)$, we will show by induction on n that the identity map $i_n : \Delta^n \to \Delta^n$, viewed as a singular n-simplex, generates $H_n(\Delta^n, \partial\Delta^n)$. It is clear that this is a cycle, since we are considering homology relative to $\partial\Delta^n$.

When $n = 0$, the claim obviously holds. For the induction step, let $\Lambda \subset \Delta^n$ be the union of all but one of the $(n-1)$-dimensional faces of Δ^n. Then there are isomorphisms

$$H_n(\Delta^n, \partial\Delta^n) \to H_{n-1}(\partial\Delta^n, \Lambda) \leftarrow H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

The first isomorphism follows from the boundary map in the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$, where the terms $H_i(\Delta^n, \Lambda)$ are zero since Δ^n deformation retracts onto Λ, hence $(\Delta^n, \Lambda) \simeq (\Lambda, \Lambda)$. The second isomorphism is induced by the inclusion $i : \Delta^{n-1} \to \partial\Delta^n$ as the face not contained in Λ. For $n = 1$, it is already an isomorphism at the chain level. When $n > 1$, $(\Delta^{n-1}, \partial\Delta^{n-1})$ is a good pair and i induces a homeomorphism of the quotients $\Delta^{n-1}/\partial\Delta^{n-1}$ and $\partial\Delta^n/\Lambda$.

The induction then follows since the cycle i_n is sent under the first isomorphism to the cycle ∂i_n which equals $\pm i_{n-1}$ in $C_{n-1}(\partial\Delta^n, \Lambda)$.

To find a cycle generating $\widetilde{H}_n(S^n)$, regard S^n as two n-simplices Δ^n_1 and Δ^n_2 with their boundaries identified in the obvious way, preserving the ordering of the vertices. The difference $\Delta^n_1 - \Delta^n_2$, viewed as a singular n-chain, is then a cycle, which we claim generates $\widetilde{H}_n(S^n)$. To see this, consider the isomorphisms

$$\widetilde{H}_n(S^n) \to H_n(S^n, \Delta^n_2) \leftarrow H_n(\Delta^n_1, \partial\Delta^n_1)$$

where the first isomorphism comes from the long exact sequence for the pair (S^n, Δ^n_2) and the second isomorphism is by passing to quotients as before. Under these isomorphisms, the cycle $\Delta^n_1 - \Delta^n_2$ in the first group corresponds to the cycle Δ^n_1 in the third group, which generates the group as we have shown above, hence $\Delta^n_1 - \Delta^n_2$ generates $\widetilde{H}_n(S^n)$.

The excision property also holds for subcomplexes of CW complexes.

Corollary 6.11. *If the CW complex X is the union of subcomplexes A and B, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A\cap B) \to H_n(X, A)$ for all n.*

Proof. Since CW pairs are good, the previous proposition implies that $H_n(X, A) \cong \widetilde{H}_n(X/A)$ and $H_n(B, A \cap B) \cong \widetilde{H}_n(B/(A \cap B))$. Notice that the quotient spaces $B/(A \cap B)$ and X/A are homeomorphic, hence the claim follows. □

Corollary 6.12. *Suppose a wedge sum $\vee_\alpha X_\alpha$ is formed at base points $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good. Then the inclusions $i_\alpha : X_\alpha \hookrightarrow \vee_\alpha X_\alpha$ induce an isomorphism $\sum_\alpha i_{\alpha*} : \sum_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(\vee_\alpha X_\alpha)$.*

Proof. Since $H_n(Y, y) \cong \widetilde{H}_n(Y)$ for any point y in a topological space Y, the claim follows from the previous proposition by setting $(X, A) = (\sqcup_\alpha X_\alpha, \sqcup_\alpha \{x_\alpha\})$. □

Theorem 6.13. *If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.*

Proof. For $x \in U$, we have $H_k(U, U - x) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - x)$ by excision. From the long exact sequence of reduced homology for the pair $(\mathbb{R}^m, \mathbb{R}^m - x)$ and $\widetilde{H}_k(\mathbb{R}^m) = 0$, we obtain $H_k(\mathbb{R}^m, \mathbb{R}^m - x) \cong \widetilde{H}_{k-1}(\mathbb{R}^m - x)$. Since $\mathbb{R}^m - x$ deformation retracts to S^{m-1}, we conclude that $H_k(\mathbb{R}^m, \mathbb{R}^m - x) \cong \mathbb{Z}$ for $k = m$ and 0 otherwise. By the same reason, $H_k(\mathbb{R}^n, \mathbb{R}^n - y) \cong \mathbb{Z}$ for $k = n$ and 0 otherwise. Since a homeomorphism $h : U \to V$ induces isomorphisms $H_k(U, U - x) \to H_k(V, V - h(x))$ for all k, we must have $m = n$. □

6.4. **Equivalence of Simplicial and Singular Homology.** We will show that for Δ-complexes the simplicial and singular homology groups are always isomorphic. It will be convenient to consider the relative case as well. Let X be a Δ-complex with $A \subset X$ a subcomplex, namely, A is the Δ-complex formed by any union of simplices of X. Define the relative simplicial homology groups $H_n^\Delta(X, A)$ in the same way, via the relative chains $\Delta_n(X, A) = \Delta_n(X)/\Delta_n(A)$, and it gives a long exact sequence of simplicial homology groups for the pair (X, A) by the same argument as for singular homology. There is a canonical homomorphism $H_n^\Delta(X, A) \to H_n(X, A)$ induced by the chain map $\Delta_n(X, A) \to C_n(X, A)$ sending each n-simplex of X to its defining map $\sigma : \Delta^n \to X$. The case $A = \emptyset$ is not excluded, in which the relative homology reduces to absolute homology.

Theorem 6.14. *The homomorphism $H_n^\Delta(X, A) \to H_n(X, A)$ is an isomorphism for all n and all Δ-complex pairs (X, A).*

Proof. For simplicity we prove the case when X is finite dimensional. First consider the case $A = \emptyset$. Let X^k be the k-skeleton of X, consisting of all simplices of dimension k or less. We have a commutative diagram of exact sequences (draw a diagram). The top and bottom horizontal sequences are from the respective long exact sequences for pairs.

Let us first show that $H_n^\Delta(X^k, X^{k-1}) \to H_n(X^k, X^{k-1})$ is an isomorphism for all n, hence the first and fourth vertical maps are isomorphisms. The simplicial chain group $\Delta_n(X^k, X^{k-1})$ is zero for $n \neq k$ and is a free abelian group generated by the k-simplices of X for $n = k$. Hence $H_n^\Delta(X^k, X^{k-1})$ has the same description. For the singular homology groups $H_n(X^k, X^{k-1})$, consider the map $\Phi : \sqcup_\alpha(\Delta_\alpha^k, \partial\Delta_\alpha^k) \to (X^k, X^{k-1})$ formed by the defining maps $\Delta^k \to X$ for all the k-simplices of X.

Since Φ induces a homeomorphism of quotient spaces between $\sqcup_\alpha \Delta^k_\alpha / \sqcup_\alpha \partial\Delta^k_\alpha$ and X^k/X^{k-1}, it induces isomorphisms on relative singular homology groups of the two pairs. Therefore, $H_n(X^k, X^{k-1})$ is zero for $n \neq k$, and for $n = k$ is free abelian with basis represented by the relative cycles of the defining maps of all the k-simplices of X, where we use the fact that $H_k(\Delta^k, \partial\Delta^k) \cong H_k(D^k, S^{k-1}) \cong \mathbb{Z}$ is generated by the identity map $\Delta^k \to \Delta^k$. We thus conclude that $H^\Delta_n(X, A) \to H_n(X, A)$ is an isomorphism.

By induction on k (and for $A = \emptyset$), we may assume the second and fifth vertical maps are isomorphisms. Now by the Five-Lemma, it follows that the middle vertical map is also an isomorphism (draw a diagram).

For the case $A \neq \emptyset$, it follows from the case $A = \emptyset$ by applying the Five-Lemma to the canonical map from the long exact sequence of simplicial homology groups for the pair (X, A) to the corresponding long exact sequence of singular homology groups. □

Remark 6.15. If X is a Δ-complex with finitely many n-simplices, then the simplicial chain group $\Delta_n(X)$ is finitely generated, hence $H_n(X) \cong H^\Delta_n(X)$ is a finitely generated abelian group. Write $H_n(X)$ as the direct sum of cyclic groups. The number of $\mathbb{Z}$ summands (i.e. the rank) is called the *nth Betti number* of X.

7. Computations and Applications

We move forward to consider some related concepts and calculations.

7.1. **Degree.** For a map $f : S^n \to S^n$ with $n > 0$, the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is a homomorphism from $\mathbb{Z}$ to itself, so it must be of the form $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$, depending on f only. Then we say that d is the *degree* of f, and denote it by $\deg f$. Below are some basic properties of degree:

(1) $\deg \mathbb{1} = 1$, since $\mathbb{1}_* = \mathbb{1}$.
(2) If f is not surjective, then $\deg f = 0$. To prove it, take $x_0 \in S^n - f(S^n)$, then f factors as a composition $S^n \to S^n - x_0 \hookrightarrow S^n$. Since $S^n - x_0$ is contractible, $H_n(S^n - x_0) = 0$, hence $f_* = 0$.
(3) If $f \simeq g$, then $\deg f = \deg g$, since $f_* = g_*$. Conversely, if $\deg f = \deg g$, then $f \simeq g$, which is Hopf's theorem and will proved later.
(4) $\deg fg = \deg f \deg g$, since $(fg)_* = f_* g_*$. In particular, if f is a homotopy equivalence, then there exists g such that $fg \simeq \mathbb{1}$, hence $\deg f \deg g = 1$, and $\deg f = \pm 1$.
(5) Suppose f is a reflection of S^n, fixing the points in a subsphere S^{n-1} and interchanging the two complementary hemispheres. Then $\deg f = -1$. To see this, give S^n a Δ-complex structure with these two hemispheres as its two n-simplices Δ^n_1 and Δ^n_2. We have seen that $\Delta^n_1 - \Delta^n_2$ generates $H_n(S^n)$. Since f interchanges Δ^n_1 and Δ^n_2, the claim follows.
(6) Let $-\mathbb{1}$ be the antipodal map defined by $x \mapsto -x$. Then it has degree $(-1)^{n+1}$, since it is a composition of $(n+1)$ reflections, each changing the sign of one coordinate in $\mathbb{R}^{n+1}$.
(7) If $f : S^n \to S^n$ has no fixed points, then $\deg f = (-1)^{n+1}$. To see this, note that the line segment from $f(x)$ to $-x$, defined by $t \mapsto (1-t)f(x) + t(-x)$ for $0 \leq t \leq 1$, does not pass through the origin. Hence

$$f_t(x) = \frac{(1-t)f(x) + t(-x)}{|(1-t)f(x) + t(-x)|}$$

defines a homotopy from f to the antipodal map $-\mathbb{1}$, which has degree $(-1)^{n+1}$.

Theorem 7.1. *S^n has a continuous field of nonzero tangent vectors iff n is odd.*

Proof. If n is odd, say $n = 2k-1$, define a vector field

$$v(x_1, x_2, \dots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}).$$

Since $|v(x) = 1|$ and $v(x)$ is orthogonal to x for all $x \in S^n$, v is a desired tangent vector field.

Conversely, suppose $x \mapsto v(x)$ is a tangent vector field on S^n. Regard $v(x)$ as a vector at the origin instead of at x, where tangency just means that x and $v(x)$ are orthogonal in $\mathbb{R}^{n+1}$. If $v(x) \neq 0$ for all x, we can normalize so that $|v(x)| = 1$ for all x by replacing $v(x)$ by $v(x)/|v(x)|$. Assuming that $|v(x)| = 1$, the vectors $(\cos t)x + (\sin t)v(x)$ lie in the unit circle in the plane spanned by x and $v(x)$. Varying t from 0 to π, we obtain a homotopy $f_t(x) = (\cos t)x + (\sin t)v(x)$ from the identity map $\mathbb{1}$ of S^n to the antipodal map $-\mathbb{1}$. It implies that $(-1)^{n+1} = \deg(-\mathbb{1}) = \deg \mathbb{1} = 1$, hence n must be odd. □

Recall the action of a group G on a space X. If for any nontrivial element of G the corresponding action has no fixed points, then we say that G acts *freely* on X. For example, the antipodal map $x \mapsto -x$ generates a free action of $\mathbb{Z}_2$ on S^n.

Proposition 7.2. *If n is even, then $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n.*

Proof. Since the degree of a homeomorphism of S^n must be ± 1, a group action G on S^n determines a degree function $d : G \to \{\pm 1\} \cong \mathbb{Z}_2$. Since $\deg f \deg g = \deg fg$, d is a homomorphism. If G acts freely, then d sends every nontrivial element of G to $(-1)^{n+1}$ by the above property (7). Thus when n is even, we conclude that d has trivial kernel, hence $G \subset \mathbb{Z}_2$, being either trivial or isomorphic to $\mathbb{Z}_2$. □

Next we introduce a technique for computing degrees in practice. Suppose $f : S^n \to S^n$ for $n > 0$ has the property that for some point $y \in S^n$, the preimage $f^{-1}(y)$ consists of only finitely many points, say, $x_1, \dots, x_m$. Let $U_1, \dots, U_m$ be disjoint neighborhoods of these points, mapped into a neighborhood V of y. Then $f(U_i - x_i) \subset V - y$ for each i. We have the following diagram (draw a diagram), where k_i and p_i are induced by inclusions. The upper two isomorphisms are from excision and the lower two isomorphisms are from exact sequences of pairs. Since $H_n(S^n) \cong \mathbb{Z}$, via this diagram the top homomorphism f_* can be regarded as multiplication by an integer, which is called the *local degree* of f at x_i and denoted by $\deg f|_{x_i}$. For example, if f is a homeomorphism, or more generally if f maps U_i homeomorphically onto V, then $\deg f|_{x_i} = \pm 1$.

Proposition 7.3. *In the above setting,* $\deg f = \sum_{i=1}^m \deg f|_{x_i}$.

Proof. By excision, the central term $H_n(S^n, S^n - f^{-1}(y)) \cong \oplus_{i=1}^m H_n(U_i, U_i - x_i) \cong \mathbb{Z}^{\oplus m}$, with k_i the inclusion of the i-th summand and p_i the projection onto the i-th summand. Identify the outer groups in the digram with $\mathbb{Z}$ as before (with 1 maps to 1). Commutativity of the lower triangle implies that $p_i \circ j(1) = 1$, hence $j(1) = (1, \dots, 1) = \sum_{i=1}^m k_i(1)$. Commutativity of the upper square implies that the middle f_* takes $k_i(1)$ to $\deg f|_{x_i}$, hence takes the sum $\sum_{i=1}^m k_i(1)$ to $\sum_{i=1}^m \deg f|_{x_i}$. Commutativity of the lower square then gives the desired formula $\deg f = \sum_{i=1}^m \deg f|_{x_i}$. □

Example 7.4. View S^1 as the unit circle in $\mathbb{C}$. Consider the map $f : S^1 \to S^1$ given by $f(z) = z^k$ for $k \in \mathbb{Z}$. We claim that $\deg f = k$. This is obvious when $k = 0$. The case $k < 0$ reduces to the case $k > 0$ by composing with $z \mapsto z^{-1}$, which is a reflection of degree -1. For the case $k > 0$, observe that for any $y \in S^1$, $f^{-1}(y)$ consists of k points $x_1, \ldots, x_k$, near each of which f is a local homeomorphism, stretching an arc by a factor of k. This local stretching can be eliminated by a deformation of f near x_i that does not change local degree, so the local degree of f at x_i is the same as for a rotation of S^1. Since a rotation is a homeomorphism and homotopic to the identity, its local degree at any point equals its global degree, which is 1. Hence $\deg f|_{x_i} = 1$ for all i and $\deg f = k$.

7.2. **Cellular Homology.** Cellular homology is an efficient tool for computing the homology of CW complexes. Before giving its definition, let us first introduce a few basic facts.

Lemma 7.5. *Suppose X is a CW complex. Then*

(1) *$H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis given by the n-cells of X.*
(2) *$H_k(X^n) = 0$ for $k > n$. In particular, $H_k(X) = 0$ for $k > \dim X$.*
(3) *The map $H_k(X^n) \to H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$ and surjective for $k = n$.*

Proof. Since (X^n, X^{n-1}) is a good pair, $H_k(X^n, X^{n-1}) \cong \widetilde{H}_k(X^n/X^{n-1})$. The quotient X^n/X^{n-1} is a wedge sum of n-spheres, one for each n-cell of X, hence $\widetilde{H}_k(X^n/X^{n-1})$ is isomorphic to the direct sum of $\widetilde{H}_k(S^n)$, one for each n-cell, which implies (1).

Now consider the long exact sequence of the pair (X^n, X^{n-1}):

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1})$$

For $k \neq n$, the last term is zero by (1), so the middle map is surjective. If $k \neq n-1$, then the first term is zero, so the middle map is injective. Next consider the inclusion-induced homomorphisms

$$H_k(X^0) \to H_k(X^1) \to \cdots \to H_k(X^{k-1}) \to H_k(X^k) \to H_k(X^{k+1}) \to \cdots$$

By the above conclusion, all of these maps are isomorphisms except that $H_k(X^{k-1}) \to H_k(X^k)$ may not be surjective and $H_k(X^k) \to H_k(X^{k+1})$ may not be injective. The first part of the sequence gives (2), since $H_k(X^0) = 0$ for $k > 0$. The last part of the sequence gives (3) (when X is finite dimensional). $\square$

Let X be a CW complex. By the above lemma, we have the following diagram (draw a diagram). The horizontal row is a chain complex:

$$\cdots \to H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

called the *cellular chain complex* of X, since $H_n(X^n, X^{n-1})$ is free with basis given by the n-cells of X. In other words, one can think of elements of $H_n(X^n, X^{n-1})$ as linear combinations of n-cells of X. The homology groups of this cellular chain complex are called *cellular homology groups* of X, denoted by $H_n^{CW}(X)$.

Theorem 7.6. *In the above setting, we have $H_n^{CW}(X) \cong H_n(X)$.*

Proof. From the above diagram, $H_n(X) \cong H_n(X^n)/\operatorname{im}\partial_{n+1}$. Since j_n is injective, it maps $\operatorname{im}\partial_{n+1}$ isomorphically onto $\operatorname{im}(j_n\partial_{n+1}) = \operatorname{im} d_{n+1}$, and maps $H_n(X^n)$ isomorphically onto $\operatorname{im} j_n = \ker\partial_n$. Since j_{n-1} is injective, we have $\ker\partial_n = \ker d_n$. Then j_n induces an isomorphism:

$$H_n(X^n)/\operatorname{im}\partial_{n+1} \cong \operatorname{im} j_n/\operatorname{im} d_{n+1} \cong \ker\partial_n/\operatorname{im} d_{n+1} \cong \ker d_n/\operatorname{im} d_{n+1} = H_n^{CW}(X),$$

thus completing the proof. □

Below are some immediate applications:

- If X is a CW complex with no n-cells, then $H_n(X) = 0$. This is because $H_n(X^n, X^{n-1}) = 0$, hence $H_n^{CW}(X) = 0$.
- More generally if X is a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements, which is because $H_n(X^n, X^{n-1})$ is free abelian on k generators, hence its subgroup $\ker d_n$ and quotient $\ker d_n/\operatorname{im} d_{n-1}$ must be generated by at most k elements.
- If X is a CW complex with no cells in adjacent dimensions, then $H_n(X)$ is free abelian with basis given by the n-cells of X. This is because the cellular boundary maps d_n are zero in this case.

Example 7.7. Consider S^n with the minimal CW structure with one 0-cell and one n-cell. Then $S^n \times S^n$ has a product CW structure with one 0-cell, two n-cells, and one $2n$-cell. If $n > 1$, there are no two cells in adjacent dimensions. We thus conclude that $H_k(S^n \times S^n)$ is $\mathbb{Z}$ for $k = 0$ and $2n$, is $\mathbb{Z} \oplus \mathbb{Z}$ for $k = n$, and is trivial for all other k.

Next we describe how to compute the cellular boundary maps d_n. When $n = 1$, the boundary map $d_1 : H_1(X^1, X^0) \to H_0(X^0)$ is the same as the simplicial boundary map $\partial : \Delta_1(X) \to \Delta_0(X)$. When X is connected and has only one 0-cell, d_1 must be 0, for otherwise $H_0(X)$ would not be $\mathbb{Z}$. When $n > 1$, we show that d_n can be computed in terms of degrees as follows.

Proposition 7.8. *In the above setting, $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$, where $d_{\alpha\beta}$ is the degree of the map $\Delta_{\alpha\beta}\colon S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$ that is the composition of the attaching map of e^n_α with the quotient map collapsing $X^{n-1} - e^{n-1}_\beta$ to a point.*

Here we identify the cells e^n_α and e^{n-1}_β with generators of the corresponding summands of the cellular chain groups. Since the attaching map of e^n_α has compact image, which meets only finitely many e^{n-1}_β, the summation in the formula contains only finitely many terms.

Proof. It follows from the following diagram (draw a diagram). □

Example 7.9. Let M_g be the closed orientable surface of genus g with its usual CW structure consisting of one 0-cell, $2g$ 1-cells, and one 2-cell attached by the product of commutators $[a_1, b_1]\cdots[a_g, b_g]$. The cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0.$$

As observed above, $d_1 = 0$. Since each a_i or b_i appears with its inverse in $[a_1, b_1]\cdots[a_g, b_g]$, the maps $\Delta_{\alpha\beta}$ are homotopic to constant maps, hence $d_2 = 0$. Therefore, the homology groups of M_g are the same as its cellular chain groups, namely, $\mathbb{Z}$ in dimension 0 and 2, and $\mathbb{Z}^{2g}$ in dimension 1.

Example 7.10. Let N_g be the closed nonorientable surface of genus g whose cell structure has one 0-cell, g 1-cells, and one 2-cell attached by the word $a_1^2 a_2^2 \cdots a_g^2$. The cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^g \xrightarrow{d_1} \mathbb{Z} \to 0.$$

Again we have $d_1 = 0$. Since each a_i appears twice in the same direction, each $\Delta_{\alpha\beta}$ is homotopic to the map $z \mapsto z^2$, of degree 2, hence $d_2(1) = (2, \dots, 2)$. Then d_2 is injective and hence $H_2(N_g) = 0$. If we use the basis

$$(1, 0, \dots, 0),\ (0, 1, 0, \dots, 0),\ \dots,\ (0, \dots, 0, 1, 0),\ (1, \dots, 1)$$

for $\mathbb{Z}^g$, then we see that $H_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

Example 7.11. Recall that $\mathbb{RP}^n$ has a CW structure with one cell e^k in each dimension $k \le n$, and the attaching map for e^k is the 2-sheeted covering $\varphi : S^{k-1} \to \mathbb{RP}^{k-1}$. To compute the boundary map d_k, we need to compute the degree of the composition

$$S^{k-1} \xrightarrow{\varphi} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1},$$

where q is the quotient map. Note that $q\varphi$ is a double cover, which is a homeomorphism restricted to each connected component of $S^{k-1} - S^{k-2}$, and these homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1} of degree $(-1)^k$. Hence we have

$$\deg q\varphi = \deg \mathbb{1} + \deg(-\mathbb{1}) = 1 + (-1)^k$$

by using the local degree formula. It follows that d_k is either 0 or multiplication by 2 according to k being odd or even. The cellular chain complex for $\mathbb{RP}^n$ is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \quad n \text{ even}$$

and

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \quad n \text{ odd}.$$

It follows that $H_k(\mathbb{RP}^n) = \mathbb{Z}$ for $k = 0$ and for $k = n$ odd, $\mathbb{Z}_2$ for k odd and $0 < k < n$, and 0 otherwise.

7.3. **Euler Characteristic.** For a finite CW complex X, the *Euler characteristic* $\chi(X)$ is defined to be $\sum_n (-1)^n c_n$, where c_n is the number of n-cells of X. The following result shows that $\chi(X)$ can be defined purely in terms of homology, hence depends only on the homotopy type of X. In particular, $\chi(X)$ is independent of the choice of CW structure on X.

Theorem 7.12. $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X) = \sum_n (-1)^n b_n(X)$, *where* b_n *is the* n*-th Betti number of* X.

In the proof of the theorem, we will use the simple fact that for a short exact sequence of finitely generated abelian groups $0 \to A \to B \to C \to 0$, we have $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

Proof. This is purely algebraic. Let

$$0 \to C_k \xrightarrow{d_k} C_{k-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

be a chain complex of finitely generated abelian groups, with cycles $Z_n = \ker d_n$, boundaries $B_n = \operatorname{im} d_{n+1}$, and homology $H_n = Z_n/B_n$. Then we have short exact sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$ and $0 \to B_n \to Z_n \to H_n \to 0$, hence

$$\operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1},$$

$$\operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n.$$

It follows that $\operatorname{rank} C_n = \operatorname{rank} B_n + \operatorname{rank} B_{n-1} + \operatorname{rank} H_n$. Taking the alternating sum, we thus obtain that

$$\sum_n (-1)^n \operatorname{rank} C_n = \sum_n (-1)^n \operatorname{rank} H_n.$$

Applying it to the chain complex with $C_n = H_n(X^n, X^{n-1})$ gives the desired formula. $\square$

Example 7.13. The closed orientable surface M_g of genus g has $\chi(M_g) = 1 - 2g + 1 = 2 - 2g$, hence all M_g are distinguished from each other by their Euler characteristics. Similarly the closed nonorientable surface N_g of genus g has $\chi(N_g) = 1 - g + 1 = 2 - g$, which distinguishes all N_g.

7.4. **Mayer-Vietoris Sequences.** Let X be the union of two subspaces A and B. The *Mayer-Vietoris sequence* is the following long exact sequence of homology:

$$\cdots \to H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{d} H_{n-1}(A \cap B) \to \cdots H_0(X) \to 0.$$

We briefly explain why the Mayer-Vietoris sequence holds. Let $C_n(A+B)$ be the subgroup of $C_n(X)$ consisting of chains that are sums of chains in A and chains in B. Note that ∂ takes $C_n(A+B)$ to $C_{n-1}(A+B)$, so the $C_n(A+B)$'s form a chain complex. Moreover, the inclusion $C_n(A+B) \hookrightarrow C_n(X)$ induces an isomorphism on homology, by barycentric subdivision (see the proof of Excision). Now the Mayer-Vietoris sequence follows from the long exact sequence of homology of the short exact sequence of chain complexes:

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0,$$

where $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$.

The boundary map $d : H_n(X) \to H_{n-1}(A \cap B)$ can be made explicit. For a cycle in $C_n(X)$, choose a homology class representative $z = x + y$ in $C_n(A + B)$ (by barycentric subdivision), where x and y do not need to by cycles individually. Then $d(z) \in H_{n-1}(A \cap B)$ can be represented by the cycle $\partial x = -\partial y$, well-defined since $\partial(x + y) = 0$, which follows from the boundary map definition in the long exact sequence of homology.

There is also a formally identical Mayer-Vietoris sequence for reduced homology, obtained by augmenting the above short exact sequence by adding the degree maps in the obvious way (draw a diagram).

Example 7.14. Let $S^n = A \cup B$ where A and B are northern and southern hemispheres, so $A \cap B = S^{n-1}$. Then in the reduced Mayer-Vietoris sequence

$$\cdots \to \widetilde{H}_k(S^{n-1}) \to \widetilde{H}_k(A) \oplus \widetilde{H}_k(B) \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}) \to \cdots$$

the terms $\widetilde{H}_k(A) \oplus \widetilde{H}_k(B)$ are zero, so we obtain $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$. We thus get another way of calculating the homology of S^n by induction.

Example 7.15. Consider the Klein bottle K as the union of two Möbus bands A and B glued together by identifying their boundary circles (print out some figures). Then A, B, and $A \cap B$ are homotopy equivalent to circles, so the nontrivial part of the reduced Mayer-Vietoris sequence is

$$0 \to H_2(K) \to H_1(A \cap B) \cong \mathbb{Z} \xrightarrow{\Phi} H_1(A) \oplus H_1(B) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_1(K) \to 0.$$

The map Φ is given by $1 \mapsto (2, -2)$, since the boundary circle of a Möbius band wraps twice around the core circle. Since Φ is injective, we obtain $H_2(K) = 0$. Moreover, $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. All the higher homology groups of K are zero from the earlier part of the sequence.

7.5. **Homology with Coefficients.** So far we have considered chains $\sum_i n_i \sigma_i$ where σ_i are singular n-simplices and $n_i \in \mathbb{Z}$. If we take n_i as an element in a fixed abelian group G, we obtain the chain group $C_n(X; G)$. The groups $C_n(X; G)$ still form a chain complex, and the corresponding homology groups are called *homology groups with coefficients in G*. The relative and reduced homology groups with G-coefficients are defined similarly.

All general theories hold similarly for homology with G-coefficients, such as the equivalence between the simplicial, singular, cellular homology theories, Excision and M-V sequence, etc. Moreover, if X is a point, then $H_n(X; G) = G$ for $n = 0$ and 0 for $n > 0$. For S^k, $\widetilde{H}_n(S^k; G) = G$ for $n = k$ and 0 otherwise.

Example 7.16. Consider the homology of $\mathbb{RP}^n$ with coefficients in a field F. Recall the cellular chain complex is

$$\cdots \xrightarrow{2} F \xrightarrow{0} F \xrightarrow{2} F \xrightarrow{0} F \to 0.$$

If $\mathrm{ch}\, F = 2$, e.g., if $F = \mathbb{Z}_2$, then $H_k(\mathbb{RP}^n) \cong F$ for all $0 \leq k \leq n$, giving a more uniform answer than the $\mathbb{Z}$-coefficients. If $\mathrm{ch}\, F \neq 2$, then $F \xrightarrow{2} F$ is a isomorphism, hence $H_k(\mathbb{RP}^n) \cong F$ for $k = 0$ and for $k = n$ odd, and is zero otherwise.

Remark 7.17. There is a general algebraic formula expressing homology with arbitrary coefficients in terms of homology with $\mathbb{Z}$ coefficients, see [H, §3.A].

Some concluding remarks.

Remark 7.18. By regarding loops as singular 1-cycles, we obtain a homomorphism $h : \pi_1(X, x_0) \to H_1(X)$. If X is path-connected, then h is surjective and $\ker h$ is the commutator subgroup of $\pi_1(X)$, and in this case h induces an isomorphism from the abelianization of $\pi_1(X)$ onto $H_1(X)$. See [H, §2.A] for a proof.

Remark 7.19. A formal viewpoint of homology theory using axioms, categories, and functors can be found in [H, 2.3].

8. Cohomology Groups

To obtain the cohomology groups $H^n(X; G)$ with coefficients in a group G, the idea is to replace the singular chain group $C_n(X)$ by the dual groups $\mathrm{Hom}(C_n, G)$ and replace the boundary maps by their dual maps.

8.1. **The Universal Coefficients Theorem.** First we sort out the algebra needed in such a process. Consider a general chain complex C of free abelian groups

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$$

Let $C_n^* = \operatorname{Hom}(C_n, G)$ be the dual *cochain group*, which is the group of homomorphisms $C_n \to G$. For a homomorphism $\alpha : A \to B$, the dual $\alpha^* : \operatorname{Hom}(B,G) \to \operatorname{Hom}(A,G)$ is defined by $\alpha^*(\phi) = \phi \circ \alpha$. Dual homomorphisms satisfy $(\alpha\beta)^* = \beta^*\alpha^*$, $\mathbb{1}^* = \mathbb{1}$, and $0^* = 0$. Let $\delta = \partial^* : C_{n-1}^* \to C_n^*$ be the dual *coboundary map*, which goes in the opposite direction. Since $\partial\partial = 0$, it follows that $\delta\delta = 0$. We thus define the *cohomology group* $H^n(C;G)$ to be $\ker\delta / \operatorname{im}\delta$ at C_n^* in the chain complex

$$\cdots \leftarrow C_{n+1}^* \xleftarrow{\delta} C_n^* \xleftarrow{\delta} C_{n-1}^* \leftarrow \cdots$$

We will show that the $H^n(C;G)$'s are determined by G and $H_k(C)$'s. In general, $H^n(C;G)$ is not isomorphic to $\operatorname{Hom}(H_n(C), G)$. However, there is a map

$$h : H^n(C;G) \to \operatorname{Hom}(H_n(C), G)$$

defined as follows. Let Z_n and B_n be the groups of cycles and boundaries in C_n. Then a class in $H^n(C;G)$ is represented by a homomorphism $\phi : C_n \to G$ such that $\delta\phi = 0$, i.e., $\phi\partial = 0$. In other words, ϕ vanishes on B_n. The restriction $\phi_0|_{Z_n}$ induces a quotient homomorphism $\overline{\phi}_0 : Z_n/B_n \to G$, which is an element of $\operatorname{Hom}(H_n(C), G)$. Moreover if $\phi \in \operatorname{im}\delta$, say $\phi = \delta\psi = \psi\partial$, then ϕ is zero on Z_n, hence $\phi_0 = 0$ and $\overline{\phi}_0 = 0$. Therefore, the quotient map $h : H^n(C;G) \to \operatorname{Hom}(H_n(C), G)$ by $\phi \mapsto \overline{\phi}_0$ is well-defined.

Lemma 8.1. *In the above setting, h is surjective.*

Proof. Since B_{n-1} is free (as a subgroup of the free abelian group C_{n-1}), the short exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0$$

splits. Hence there is a projection homomorphism $p : C_n \to Z_n$ such that $p|_{Z_n} = \mathbb{1}$. Then given any homomorphism $\phi_0 : Z_n \to G$, one can extend it to a homomorphism $\phi = \phi_0 p : C_n \to G$. In particular, if ϕ_0 vanishes on $B_n \subset Z_n$, then ϕ still vanishes on B_n. In other words, it extends homomorphisms $H_n(C) \to G$ to elements of $\ker\delta$. We thus obtain a homomorphism $\operatorname{Hom}(H_n(C), G) \to \ker\delta$ given by $\phi_0 \mapsto \phi_0 p$. Composing it with the quotient map $\ker\delta \to H^n(C;G)$ gives a homomorphism $\operatorname{Hom}(H_n(C), G) \to H^n(C;G)$. If we compose this with h, we thus obtain the identity map on $\operatorname{Hom}(H_n(C), G)$, since by definition h simply undoes the effect of extending homomorphisms via p. It follows that h is surjective. In fact the proof shows that we have a split short exact sequence

$$0 \to \ker h \to H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \to 0$$

since h has a lift. $\square$

Next we analyze $\ker h$. Consider the following diagram of short exact sequences (draw a diagram). Since the rows are split, dualizing it we obtain another diagram of short exact sequences (draw a diagram), where the rows are still split, since $\operatorname{Hom}(A \oplus B, G) \cong \operatorname{Hom}(A, G) \oplus \operatorname{Hom}(B, G)$. Recall that such a short exact sequence of chain complexes induces a long exact sequence of homology groups. In the dual diagram the boundary maps in the first and third complexes Z_n^* and B_n^* are zero, hence their homology groups are the same as the chain groups. Hence the associated long exact sequence of the dual short exact sequence has the form

$$\cdots B_n^* \leftarrow Z_n^* \leftarrow H^n(C;G) \leftarrow B_{n-1}^* \leftarrow Z_{n-1}^* \leftarrow \cdots$$

The maps $Z_n^* \to B_n^*$ are the dual maps i_n^* of the inclusions $i_n : B_n \to Z_n$ (by a diagram chasing).

Break the above long exact sequence into short exact sequences as follows:

$$0 \leftarrow \ker i_n^* \leftarrow H^n(C;G) \leftarrow \operatorname{coker} i_{n-1}^* \leftarrow 0.$$

Note that elements of $\ker i_n^*$ are homomorphisms $Z_n \to G$ that vanish on B_n, which are the same as homomorphisms $Z_n/B_n \to G$, hence we can identify $\ker i_n^*$ with $\operatorname{Hom}(H_n(C), G)$, and then the map $H^n(C;G) \to \ker i_n^*$ becomes the map h defined before. We thus rewrite the above short exact sequence as

$$0 \to \operatorname{coker} i_{n-1}^* \to H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \to 0.$$

Finally we will show that $\operatorname{coker} i_{n-1}^*$ depends only on $H_{n-1}(C)$ and G in a natural way. Notice that in general taking the dual of a short exact sequence does not yield a short exact sequence. For example, if we dualize $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$ by applying $\operatorname{Hom}(-, \mathbb{Z})$, we get $0 \leftarrow \mathbb{Z} \xleftarrow{n} \mathbb{Z} \leftarrow 0 \leftarrow 0$ which fails to be exact at the left-hand $\mathbb{Z}$. On the other hand, it is easy to check that the other part of the exactness holds. Namely, if $A \to B \to C \to 0$ is exact, then applying $\operatorname{Hom}(-, G)$ give an exact sequence $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$.

In our situation the short exact sequence

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0$$

is a free resolution of $H_{n-1}(C)$. In general a *free resolution* of an abelian group H is an exact sequence

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where each F_i is free. If we apply $\operatorname{Hom}(-, G)$, we may lose exactness, but at least we get a cochain complex of the form

$$\cdots \leftarrow F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \leftarrow 0.$$

We temporarily denote by $H^n(F;G)$ for the homology group $\ker f_{n+1}^* / \operatorname{im} f_n^*$ of this dual complex. Then the group $\operatorname{coker} i_{n-1}^*$ in this setting is $H^1(F;G)$, where F is the above short exact sequence as a free resolution. Then the following lemma shows that $\operatorname{coker} i_{n-1}^*$ depends only on $H_{n-1}(C)$ and G.

Lemma 8.2. *For any two free resolutions F and F' of H, there are canonical isomorphisms $H^n(F;G) \cong H^n(F';G)$ for all n.*

We remark that every abelian group H has a free resolution of the form $0 \to F_1 \to F_0 \to H \to 0$ with $F_i = 0$ for $i > 1$, obtained as follows. Choose a set of generators for H and let F_0 be a free abelian group generated by these generators. Then we have a surjective homomorphism $f_0 : F_0 \to H$. The kernel of f_0 is free, since it is a subgroup of a free abelian group, so let F_1 be the kernel with $f_1 : F_1 \to F_0$ the inclusion. For this free resolution we obviously have $H^n(F;G) = 0$ for $n > 1$, so this holds for all free resolution of H by the above lemma. Thus the only interesting group is $H^1(F;G)$, which depends only on H and G. Denote by $\operatorname{Ext}(H, G) = H^1(F;G)$. We summarize the above discussion in the following result.

Theorem 8.3 (Universal Coefficient Theorem for Cohomology)**.** *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex* $\mathrm{Hom}(C_n, G)$ *are determined by the split exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0.$$

Here are some basic properties of Ext:

- $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$. This is because the direct sum of free resolutions of H and H' is a free resolution for $H \oplus H'$.
- $\mathrm{Ext}(H, G) = 0$ if H is free. This follows from the free resolution $0 \to H \to H \to 0$.
- $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$. This is by dualizing the free resolution $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_n \to 0$ to produce an exact sequence as follows (draw a diagram).

If H is finitely generated, by the standard decomposition theorem, these properties imply that $\mathrm{Ext}(H, \mathbb{Z})$ is isomorphic to the torsion subgroup of H, and moreover, $\mathrm{Hom}(H, \mathbb{Z})$ is isomorphic to the free part of H.

Corollary 8.4. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$, then $H^n(C;\mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$.*

Another corollary can be obtain by using the five-lemma and some naturality property.

Corollary 8.5. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G.*

8.2. **Cohomology of Spaces.** Given a space X and an abelian group G, define the group $C^n(X;G)$ of *singular n-cochains with coefficients in G* to be $\mathrm{Hom}(C_n(X), G)$. An element $\varphi \in C^n(X;G)$ is thus a function from singular n-simplices $\sigma : \Delta^n \to X$ to G.

The *coboundary map* $\delta : C^n(X;G) \to C^{n+1}(X;G)$ is the dual ∂^*. More precisely for a singular $(n+1)$-simplex $\sigma : \Delta^{n+1} \to X$, we have

$$\delta\varphi(\sigma) = \varphi(\partial\sigma) = \sum_i (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{n+1}]}).$$

Define the *cohomology group $H^n(X;G)$ with coefficients in G* to be $\ker\delta/\operatorname{im}\delta$ at $C^n(X;G)$ in the cochain complex

$$\cdots \leftarrow C^{n+1}(X;G) \xleftarrow{\delta} C^n(X;G) \xleftarrow{\delta} C^{n-1}(X;G) \leftarrow \cdots \leftarrow C^0(X;G) \leftarrow 0.$$

Elements of $\ker\delta$ are *cocycles* and elements of $\operatorname{im}\delta$ are *coboundaries*. A cochain φ is a cocyle if $\delta\varphi = \varphi\partial = 0$, that is, φ vanishes on boundaries.

By the universal coefficient theorem, we have a split short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X;G) \to \mathrm{Hom}(H_n(X), G) \to 0.$$

When $n = 0$, there is no Ext term, hence $H^0(X;G) \cong \mathrm{Hom}(H_0(X), G)$. For $n = 1$, since $H_0(X)$ is free, $\mathrm{Ext}(H_0(X), G) = 0$, hence $H^1(X;G) \cong \mathrm{Hom}(H_1(X), G)$.

For a field F, let $C_n(X;F)$ be the free F-module generated by singular n-simplices in X, and let $H_n(X;F)$ be the homology group of the corresponding chain complex. The dual complex $\mathrm{Hom}_F(C_n(X;F), F)$ (of F-module homomorphisms) is the same as $\mathrm{Hom}(C_n(X), F)$, since both parameterize functions from singular n-simplices to F. Hence the homology groups of the dual complex $\mathrm{Hom}_F(C_n(X;F), F)$ are the cohomology groups $H^n(X;F)$. Since F-modules are always free, the Ext_F terms vanish, and we obtain

$$H^n(X;F) \cong \mathrm{Hom}_F(H_n(X;F), F)$$

by using a generalized universal coefficient theorem for modules over F:

$$0 \to \mathrm{Ext}_F(H_{n-1}(X;F), F) = 0 \to H^n(X;F) \to \mathrm{Hom}_F(H_n(X;F), F) \to 0.$$

In other words, with field coefficients cohomology is the dual of homology.

By dualizing the augmented chain complex $\cdots \to C_0(X) \to \mathbb{Z} \to 0$, we define the *reduced cohomology groups* $\widetilde{H}^n(X;G)$. It is easy to see that $\widetilde{H}^n(X;G) \cong H^n(X;G)$ for $n > 0$ and $\widetilde{H}^0(X;G) \cong \mathrm{Hom}(\widetilde{H}_0(X), G)$ by the universal coefficient theorem, parameterizing all functions $X \to G$ that are constant on path-connected components modulo the functions that are constant on all of X.

For a pair (X, A), dualizing the short exact sequence

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0,$$

we obtain

$$0 \leftarrow C^n(A;G) \leftarrow C^n(X;G) \leftarrow C^n(X, A;G) \leftarrow 0$$

which remains exact, where $C^n(X, A;G) = \mathrm{Hom}(C_n(X, A), G)$. The associated long exact sequence is of the form

$$\cdots \to H^n(X, A;G) \to H^n(X;G) \to H^n(A;G) \to H^{n+1}(X, A;G) \to \cdots$$

Similarly there is a long exact sequence of reduced cohomology for (X, A), where $\widetilde{H}^n(X, A;G) = H^n(X, A;G)$ for all n, as in homology. Taking A to be a point x_0, the exact sequence gives $\widetilde{H}^n(X;G) \cong H^n(X, x_0;G)$.

For a map $f : X \to Y$, let $f_\sharp : C_n(X) \to C_n(Y)$ be the induced chain maps. Then the dual $f^\sharp : C^n(Y;G) \to C^n(X;G)$ are the *cochain maps*, which induce homomorphisms $f^* : H^n(Y;G) \to H^n(X;G)$. If f and g are homotopy equivalent, then $f^* = g^*$ as in the case of homology.

For subspaces $Z \subset A \subset X$ with the closure of Z contained in the interior of A, the Excision Theorem holds, namely, the inclusion $i : (X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $i^* : H^n(X, A;G) \to H^n(X - Z, A - Z;G)$ for all n.

For a CW pair (X, A), using the simplicial cochain groups $\mathrm{Hom}(\Delta_n(X, A), G)$ one can define the *simplicial cohomology* $H^n_\Delta(X, A;G)$. We have isomorphisms $H^n(X, A;G) \cong H^n_\Delta(X, A;G)$.

For a CW complex X, using the cellular cochain complex $H^n(X^n, X^{n-1})$ one can define the *cellular cohomology*. The cellular cochain complex $H^n(X^n, X^{n-1})$ is the dual of the cellular chain complex, by applying $\mathrm{Hom}(-, G)$, and the cellular cohomology groups are isomorphic to the singular cohomology groups.

If X is the union of A and B, we have the *Mayer-Vietoris Sequence*

$$\cdots \to H^n(X;G) \to H^n(A;G) \oplus H^n(B;G) \to H^n(A \cap B;G) \to H^{n+1}(X;G) \to \cdots$$

9. Cup Product

We consider cohomology with coefficients in a ring R, such as $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, the *cup product* $\varphi \smile \psi \in C^{k+\ell}(X;R)$ is the cochain whose value on $\sigma : \Delta^{k+\ell} \to X$ is given by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,v_{k+\ell}]}).$$

Lemma 9.1. *For $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, we have*

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi.$$

Proof. For $\sigma : \Delta^{k+\ell+1} \to X$, we have

$$(\delta\varphi \smile \psi)(\sigma) = \sum_{i=0}^{k+1}(-1)^i \varphi(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+\ell+1}]}),$$

$$(-1)^k(\varphi \smile \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1}(-1)^i \varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,\hat{v}_i,\ldots,v_{k+\ell+1}]}).$$

Adding these two, the last term of the first sum cancels the first term of the second sum, and the remaining terms are exactly $\delta(\varphi \smile \psi)(\sigma) = (\varphi \smile \psi)(\partial\sigma)$, since $\partial\sigma = \sum_{i=0}^{k+\ell+1}(-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_{k+\ell+1}]}$. □

The lemma implies that the cup product of two cocycles is again a cocycle. Moreover, the cup product of a cocycle and a coboundary in either order is again a coboundary, since $\varphi \smile \delta\psi = \pm\delta(\varphi \smile \psi)$ if $\delta\varphi = 0$, and $\delta\varphi \smile \psi = \delta(\varphi \smile \psi)$ if $\delta\psi = 0$. It follows that there is an induced cup product

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R),$$

which is associative and distributive, since these properties hold at the level of cochains. If R has an identity element 1, then the cup product has an identity element given by the class $1 \in H^0(X;R)$, defined by the 0-cocycle assigning the value 1 on each singular 0-simplex.

Remark 9.2. There are also relative cup products

$$H^k(X;R) \times H^\ell(X,A;R) \xrightarrow{\smile} H^{k+\ell}(X,A;R),$$
$$H^k(X,A;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X,A;R),$$
$$H^k(X,A;R) \times H^\ell(X,A;R) \xrightarrow{\smile} H^{k+\ell}(X,A;R),$$

since if φ or ψ vanishes on chains in A, so does $\varphi \smile \psi$.

Proposition 9.3. *For a map $f : X \to Y$, the induced maps $f^* : H^n(Y;R) \to H^n(X;R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$, and similarly in the relative case.*

Proof. This follows from the cochain formula

$$\begin{aligned}
(f^\sharp\varphi \smile f^\sharp\psi)(\sigma) &= f^\sharp\varphi(\sigma|_{[v_0,\ldots,v_k]})f^\sharp\psi(\sigma|_{[v_k,\ldots,v_{k+\ell}]}) \\
&= \varphi(f\sigma|_{[v_0,\ldots,v_k]})\psi(f\sigma|_{[v_k,\ldots,v_{k+\ell}]}) \\
&= (\varphi \smile \psi)(f\sigma) \\
&= f^\sharp(\varphi \smile \psi)(\sigma).
\end{aligned}$$

□

Theorem 9.4. *For $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$, if R is commutative, then $\alpha \smile \beta = (-1)^{k\ell}\beta \smile \alpha$.*

Proof. See the proof of [H, Theorem 3.11]. □

Corollary 9.5. *For $\alpha \in H^k(X, A; R)$ with k odd, $2(\alpha \smile \alpha) = 0$. In particular if $H^k(X, A; R)$ has no elements of order two, then $\alpha \smile \alpha = 0$.*

A cup product for simplicial cohomology can be defined by the same formula, so the canonical isomorphism between simplicial and singular cohomology respects cup products. Here are two examples of calculating cup products using simplicial cohomology.

Example 9.6. Let M be the closed orientable surface of genus $g \geq 1$ with the following Δ-complex structure (draw a picture). We analyze the cup product $H^1(M) \times H^1(M) \to H^2(M)$. Taking $\mathbb{Z}$ coefficients, a basis for $H_1(M)$ is formed by the edges a_i and b_i, as we showed, say by using cellular homology. By the universal coefficient theorem or the cellular cohomology, we have $H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbb{Z})$. Let α_i be the dual to a_i that assigns 1 to a_i and 0 to the other basis elements, and similarly β_i the dual to b_i.

To represent α_i by a simplicial cocycle φ_i, we need to choose values for φ_i on the edges from the central vertex such that $\delta\varphi_i = 0$. This condition has a geometric interpretation in terms of curves transverse to the edges of M. Consider the arc labelled by α_i, which represents a loop in M meeting a_i in one point and disjoint from all the other basis elements. Define φ_i to have value 1 on edges meeting α_i and value 0 on all other edges. Namely, φ_i counts the intersection number of each edge with α_i. Similarly we define ψ_i counting intersections with the arc β_i, which represents β_i.

Note that the ordering of the vertices of each 2-simplex is compatible with the indicated orientations of its edges. Recall that for 1-cochain φ and ψ, $\varphi \smile \psi(\sigma) = \varphi(\sigma|_{[v_0,v_1]})\psi(\sigma|_{[v_1,v_2]})$. We see for example that $\varphi_1 \smile \psi_1$ takes value 0 on all 2-simplices except the one with edge b_1 in the lower right part of the figure, where it takes value 1. Let c be the 2-chain formed by the sum of the 2-simplices with the signs indicated in the figure. One checks that $\partial c = 0$, hence c is a 2-cycle, and represents a nonzero element of $H_2(M)$. Then $\varphi_1 \smile \psi_1$ takes value 1 on c, as a generator of $\mathbb{Z}$, which implies that c represents a generator of $H_2(M) \cong \mathbb{Z}$ and $\varphi_1 \smile \psi_1$ represents the dual generator γ of $H^2(M) \cong \mathbb{Z}$. Therefore, $\alpha_1 \smile \beta_1 = \gamma$.

One similarly checks that $\alpha_i \smile \beta_j = -\beta_i \smile \alpha_j = \gamma$ if $i = j$ and 0 if $i \neq j$, $\alpha_i \smile \alpha_j = 0$, and $\beta_i \smile \beta_j = 0$. These relations completely determine the cup product $H^1(M) \times H^1(M) \to H^2(M)$.

Example 9.7. Let N be the closed nonorientable surface of genus g, with the following Δ-complex structure (draw a picture). Recall that $H_0(N_g) \cong \mathbb{Z}$, $H_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$, and $H_2(N_g) = 0$. For simplicity let us consider $\mathbb{Z}_2$ coefficients instead, where $H^1(N_g; \mathbb{Z}_2) \cong \mathbb{Z}_2^g$ and $H^2(N_g, \mathbb{Z}_2) \cong \mathbb{Z}_2$ by the universal coefficient theorem. The edges a_i give a basis for $H_1(N_g; \mathbb{Z}_2)$. The dual basis elements $\alpha_i \in H^1(N_g; \mathbb{Z}_2)$ can be represented by cocycles with values given by counting intersections with the arcs labeled by α_i in the figure. Then one checks that $\alpha_i \smile \alpha_i$ is the nonzero element of $H^2(N_g, \mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\alpha_i \smile \alpha_j = 0$ for $i \neq j$. In particular when $g = 1$, we have $N = \mathbb{RP}^2$, and the cup product of a generator of $H^1(\mathbb{RP}^2; \mathbb{Z}_2)$ with itself is a generator of $H^2(\mathbb{RP}^2; \mathbb{Z}_2)$.

9.1. **Cohomology Ring.** Cup product induces a ring structure on the cohomology groups of a space X. Define $H^*(X;R)$ to be the direct sum of the groups $H^n(X;R)$. Let us simply write $\alpha\beta$ for $\alpha \smile \beta$. Define $(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i\beta_j$, which makes $H^*(X;R)$ into a *graded ring*, with identity if R has an identity. Similarly $H^*(X,A;R)$ is a ring under the relative cup product. Taking scalar multiplication by elements of R into account, the cohomology ring can also be regarded as an R-algebra.

Example 9.8. Recall the cup product structure on $\mathbb{RP}^2$. Elements of $H^*(\mathbb{RP}^2;\mathbb{Z}_2)$ are polynomials $a_0 + a_1\alpha + a_2\alpha^2$ with $a_i \in \mathbb{Z}_2$, where α generates $H^1(\mathbb{RP}^2;\mathbb{Z}_2)$ and α^2 generates $H^2(\mathbb{RP}^2;\mathbb{Z}_2)$. So $H^*(\mathbb{RP}^2;\mathbb{Z}_2)$ can be identified with the quotient of the polynomial ring $\mathbb{Z}_2[\alpha]/(\alpha^3)$.

Example 9.9. For the torus $T = S^1 \times S^1$, $H^*(T;\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha,\beta]$, an example of the exterior algebra, where $\alpha^2 = \beta^2 = 0$ and $\alpha\beta = -\beta\alpha$.

Example 9.10. The isomorphism $H^*(\bigsqcup_\alpha X_\alpha;R) \to \prod_\alpha H^*(X_\alpha;R)$ induced by the inclusions $i : X_\alpha \hookrightarrow \bigsqcup_\alpha X_\alpha$ is a ring isomorphism. Similarly the isomorphism $\widetilde{H}^*(\bigvee_\alpha X_\alpha;R) \cong \prod_\alpha \widetilde{H}^*(X_\alpha;R)$ is a ring isomorphism, where we identify reduced cohomology with cohomology relative to a point and use relative cup products, and moreover we assume that the base points $x_\alpha \in X_\alpha$ are deformation retracts of neighborhoods such that the claimed isomorphism holds. This product ring structure can sometimes be used to rule out splittings of a space as a wedge sum up to homotopy equivalence.

9.2. **Künneth Formula.** There is a connection between cup product and product spaces. In order to describe it, we need to use *tensor product*, so let us review the definition and basic properties of tensor products.

For abelian groups A and B, the tensor product $A\otimes B$ is defined to be the abelian group with generators $a\otimes b$ for $a \in A$, $b \in B$, with relations $(a+a')\otimes b = a\otimes b + a'\otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$. The zero element is $0 \otimes 0 = 0 \otimes b = a \otimes 0$, and $-(a \otimes b) = (-a) \otimes b = a \otimes (-b)$. Moreover, we have

- $A \otimes B \cong B \otimes A$.
- $(\bigoplus_i A_i) \otimes B = \bigoplus_i (A_i \otimes B)$.
- $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.
- $\mathbb{Z} \otimes A \cong A$.
- $\mathbb{Z}_n \otimes A \cong A/nA$.
- A pair of homomorphisms $f : A \to A'$ and $g : B \to B'$ induces a homomorphism $f \otimes g : A \otimes B \to A' \otimes B'$ via $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.
- A bilinear map $\varphi : A \times B \to C$ induces a homomorphism $A \otimes B \to C$ by $a \times b \mapsto \varphi(a,b)$.

One can generalize tensor product to modules over a commutative ring R, by defining $A \otimes_R B$ for R-modules A and B to be the quotient of $A \otimes B$ by imposing one more relation $ra \otimes b = a \otimes rb$ for $r \in R$, $a \in A$, and $b \in B$. Then it ensures that $A \otimes_R B$ is again an R-module. In special cases, $A \otimes_R B = A \otimes B$, e.g., when R is $\mathbb{Z}_m$ or $\mathbb{Q}$, but in general $A \otimes_R B \neq A \otimes B$. The above properties hold similarly, where $R \otimes_R A \cong A$ is induced by $r \otimes a \mapsto ra$.

Now for two spaces X, Y and a commutative ring R, define the *cross product* to be the map

$$\mu : H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R)$$

given by $\mu(a\otimes b) = a\times b := p_1^*(a) \smile p_2^*(b)$, where p_1 and p_2 are the projections of $X\times Y$ onto X and Y. Since cup product is distributive, the cross product is bilinear on the tensor product, hence it is a (group) homomorphism on the R-tensor product. We further define the multiplication on the left side by $(a\otimes b)(c\otimes d) = (-1)^{|b||c|}ac\otimes bd$, where $|x|$ denotes the (co)dimension of the class x. Then we have

$$\begin{aligned}
\mu((a\otimes b)(c\otimes d)) &= (-1)^{|b||c|}\mu(ac\otimes bd)\\
&= (-1)^{|b||c|}(a\smile c)\times(b\smile d)\\
&= (-1)^{|b||c|}p_1^*(a\smile c)\smile p_2^*(b\smile d)\\
&= (-1)^{|b||c|}p_1^*(a)\smile p_1^*(c)\smile p_2^*(b)\smile p_2^*(d)\\
&= p_1^*(a)\smile p_2^*(b)\smile p_1^*(c)\smile p_2^*(d)\\
&= (a\times b)(c\times d)\\
&= \mu(a\otimes b)\mu(c\otimes d).
\end{aligned}$$

Therefore, μ is a ring homomorphism.

Theorem 9.11 (Künneth Formula)**.** *The cross product $H^*(X;R)\otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X\times Y;R)$ is a ring isomorphism if $H^k(Y;R)$ is a finitely generated free R-module for all k.*

Example 9.12. For k_i odd, the Künneth Formula gives an isomorphism

$$H^*(S^{k_1}\times\cdots\times S^{k_n};\mathbb{Z}) \cong H^*(S^{k_1};\mathbb{Z})\otimes\cdots\otimes H^*(S^{k_n};\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_1,\ldots,\alpha_n],$$

the exterior algebra where α_i is in dimension k_i and $\alpha_i\alpha_j = -\alpha_j\alpha_i$ and $\alpha_i^2 = 0$. When some k_i are even, the left side would be the tensor product of an exterior algebra for the odd-dimensional spheres and the truncated polynomial rings $\mathbb{Z}[\alpha]/(\alpha^2)$ for the even-dimensional spheres. Note that as rings $\Lambda_{\mathbb{Z}}[\alpha] \cong \mathbb{Z}[\alpha]/(\alpha)^2$, but as graded rings α is odd dimensional in the former and even dimensional in the latter, so they make a difference.

10. Poincaré Duality

We discuss a beautiful duality between homology and cohomology. Recall that a *real manifold of dimension* n is a Hausdorff space such that each point has an open neighborhood homeomorphic to $\mathbb{R}^n$.

10.1. **Orientations.** For an n-manifold M and $x\in M$, note that an open neighborhood of x is homeomorphic to $\mathbb{R}^n$. Then we have

$$\begin{aligned}
H_n(M, M-x) &\cong H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \text{ by excision}\\
&\cong \widetilde{H}_{n-1}(\mathbb{R}^n - 0) \text{ since } \mathbb{R}^n \text{ is contractible}\\
&\cong \widetilde{H}_{n-1}(S^{n-1}) \text{ since } \mathbb{R}^n - 0 \text{ is homeomorphic to } S^{n-1},
\end{aligned}$$

where S^{n-1} is a sphere centered at x. We say that a *local orientation* of M at x is a choice of a generator μ_x of the local homology $H_n(M|x) := H_n(M, M-x)$. An *orientation* of M is a function $x\mapsto\mu_x$ assigning to each $x\in M$ a local orientation $\mu_x\in H_n(M|x)$, such that the local consistency condition holds:

- For each x there is an (open ball) neighborhood B such that the local orientations μ_y at all $y\in B$ are the images of one generator μ_B of $H_n(M|B \cong H_n(\mathbb{R}^n, \mathbb{R}^n - B)$ under the natural maps $H_n(M|B)\to H_n(M|y)$.

If an orientation exists for M, then M is called *orientable*.

Example 10.1. $\mathbb{R}^n$ is orientable. One can choose an orientation of $\mathbb{R}^n$ at a point x that determines an orientation at every other point y via the canonical isomorphism $H_n(\mathbb{R}^n|x) \cong H_n(\mathbb{R}^n|B) \cong H_n(\mathbb{R}^n|y)$, where B is a ball containing both x and y.

Every manifold M has an orientable two-sheeted covering space $\widetilde{M}$. For instance, $\mathbb{RP}^2$ is covered by S^2. In general, let

$$\widetilde{M} = \{\mu_x \mid x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x\}.$$

The map $\mu_x \mapsto x$ defines a two-to-one surjection $\widetilde{M} \to M$. Given any open ball $B \subset M$ and a generator $\mu_B \in H_n(M|B)$, let $U(\mu_B)$ be the set of all $\mu_x \in \widetilde{M}$ such that $x \in B$ and μ_x is the image of μ_B under the natural map $H_n(M|B) \to H_n(M|x)$. One checks that the sets $U(\mu_B)$ form a basis for a topology on $\widetilde{M}$ and that the projection $\widetilde{M} \to M$ is a covering space. Moreover, at each $\mu_x \in \widetilde{M}$ there is a canonical local orientation given by $\widetilde{\mu}_x \in H_n(\widetilde{M}|\mu_x)$ corresponding to μ_x under the isomorphisms $H_n(\widetilde{M}|\mu_x) \cong H_n(U(\mu_B)|\mu_x) \cong H_n(B|x)$. By construction these local orientations satisfy the local consistency condition, hence $\widetilde{M}$ is orientable.

Proposition 10.2. *If M is connected, then M is orientable iff $\widetilde{M}$ has two connected components. In particular, M is orientable if it is simply connected, or more generally, if $\pi_1(M)$ has no subgroup of index two.*

Proof. If M is connected, then $\widetilde{M}$ has either one or two components, since it is a two-sheeted covering of M. If $\widetilde{M}$ has two components, then each of them maps to M homeomorphically, so M is also orientable, being one of the components of the orientable $\widetilde{M}$. Conversely if M is orientable, then it has exactly two orientations since it is connected, and each of these orientations defines a component of $\widetilde{M}$. The last statement follows from that connected two-sheeted covering spaces of M correspond to index-two subgroups of $\pi_1(M)$. □

The covering space $\widetilde{M}$ can be embedded in a larger covering space $M_{\mathbb{Z}}$ consisting of all elements $\alpha_x \in H_n(M|x)$ for $x \in M$. A topology of $M_{\mathbb{Z}}$ is given via the basis of sets $U(\alpha_B)$ consisting of α_x with $x \in B$ being the image of an element $\alpha_B \in H_n(M|B)$ under the map $H_n(M|B) \to H_n(M|x)$. Then $M_{\mathbb{Z}} \to M$ is an infinite-sheeted cover since α_x ranges over $H_n(M|x) \cong \mathbb{Z}$. Restricting α_x to be zero, we get a copy M_0 of M in $M_{\mathbb{Z}}$. The rest of $M_{\mathbb{Z}}$ consists of infinitely many copies M_k of $\widetilde{M}$ for $k = 1, 2, \ldots$, where M_k consists of those α_x being k times either generator ± 1 of $H_n(M|x)$. A continuous map $M \to M_{\mathbb{Z}}$ of the form $x \mapsto \alpha_x \in H_n(M|x)$ is called a *section*. Then an orientation of M is the same as a section $x \mapsto \mu_x$ such that μ_x is a generator of $H_n(M|x)$ for all x.

Let R be a commutative ring with identity. In general, an R-orientation of M assigns to each $x \in M$ a generator of $H_n(M|x; R) \cong R$, subject to the same local consistency condition, where a generator of R is just a unit u (i.e., an invertible element of R since R has identity). The definition of $M_{\mathbb{Z}}$ generalizes to a covering space $M_R \to M$, and an R-orientation is a section of this covering whose value at each $x \in M$ is a generator of $H_n(M|x; R)$. Each $r \in R$ determines a subcovering space M_r of M_R consisting of $\pm r \in H_n(M|x; R) \cong H_n(M|x) \otimes R \cong R$. If r has order 2 in R, then $r = -r$, and M_r is just a copy of M. Otherwise M_r is isomorphic to the two-sheeted cover $\widetilde{M}$. Therefore, M_R is the union of these M_r's, which are disjoint except for the identification $M_r = M_{-r}$.

In particular, a ($\mathbb{Z}$-) orientable manifold is R-orientable for all R, while a ($\mathbb{Z}$) nonorientable manifold is R-orientable iff R contains a unit of order 2, which is equivalent to having $2 = 0$ in R. Consequently every manifold is $\mathbb{Z}_2$-orientable. In practice, it means that the two most important cases are $R = \mathbb{Z}$ and $R = \mathbb{Z}_2$ for the study of orientation.

The orientability of a closed manifold is reflected in its homology as follows.

Theorem 10.3. *Let M be a closed, connected n-manifold. Then we have*

(1) If M is R-orientable, the map $H_n(M;R) \to H_n(M|x;R) \cong R$ is an isomorphism for all $x \in M$.
(2) If M is not R-orientable, the map $H_n(M;R) \to H_n(M|x;R) \cong R$ is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.
(3) $H_i(M;R) = 0$ for $i > n$.

In particular, $H_n(M;\mathbb{Z})$ is $\mathbb{Z}$ or 0, depending on whether M is orientable or not, and in either case $H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$.

An element of $H_n(M;R)$ whose image in $H_n(M|x;R)$ is a generator for all x is called a *fundamental class* for M with coefficients in R. By the theorem, a fundamental class exists if M is closed and R-orientable. Conversely if a fundamental class μ exists, let μ_x be its image in $H_n(M|x;R)$. Then the function $x \mapsto \mu_x$ is an R-orientation since the map $H_n(M;R) \to H_n(M|x;R)$ factors through $H_n(M|B;R)$ for an open ball B in M containing x. Moreover, since μ_x can only be nonzero for x in the image of a cycle representing μ and this image is compact, hence M must be compact as well. In this sense a fundamental class is sometimes also called an *orientation class* for M.

The theorem follows from the following more general statement.

Lemma 10.4. *Let M be a manifold of dimension n, and $A \subset M$ be a compact subset. Then*

(a) If $x \mapsto \alpha_x$ is a section of the covering $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M|A;R)$ whose image in $H_n(M|x;R)$ is α_x for all $x \in A$.
(b) $H_i(M|A;R) = 0$ for $i > n$.

Proof of Theorem 10.3. Choose $A = M$ and then (3) follows from (b). Let $\Gamma_R(M)$ be the set of sections of $M_R \to M$. Since the sum of two sections is a section and a scalar multiple of a section is a section, we conclude that $\Gamma_R(M)$ is an R-module. There is a homomorphism $H_n(M;R) \to \Gamma_R(M)$ sending a class α to the section $x \mapsto \alpha_x$, where α_x is the image of α under the map $H_n(M;R) \to H_n(M|x;R)$. Then (a) says that this homomorphism is an isomorphism. If M is connected, each section of $M_R \to M$ is uniquely determined by its value at one point, and (1), (2) follow from the structure of M_R. □

Proof outline of Lemma 10.4. First show that if the lemma is true for compact sets A, B, and $A \cap B$, then it is true for $A \cup B$, by using the Mayer-Vietoris sequence

$$0 \to H_n(M|A \cup B) \to H_n(M|A) \oplus H_n(M|B) \to H_n(M|A \cap B)$$

where the zero on the left is from the assumption that $H_{n+1}(M|A \cap B) = 0$.

Next we reduce to the case $M = \mathbb{R}^n$. Write the compact set A as a union of finitely many compact sets $A_1, \dots, A_m$ such that each A_i is contained in an open

$\mathbb{R}^n \subset M$. Apply the previous step to $A_1 \cup \cdots \cup A_{m-1}$ and A_m. By induction it reduces to the case $m = 1$, and then excision allows to replace M by the neighborhood $\mathbb{R}^n \subset M$.

Finally do the case $M = \mathbb{R}^n$. □

Remark 10.5. For a closed n-manifold with a Δ-complex structure, one may construct a fundamental class more explicitly. Consider the case with $\mathbb{Z}$ coefficients. In simplicial homology, a fundamental class μ is represented by some linear combination $\sum_i k_i \sigma_i$ of the n-simplices σ_i of M. The condition that μ maps to a generator of $H_n(M|x;\mathbb{Z})$ imposed to points x in the interior of σ_i means that each k_i must be ± 1. Moreover, the choice of k_i must make μ a cycle. It implies that if σ_i and σ_j share a common $(n-1)$-dimensional face, then k_i determines k_j, and vice versa. Analyzing this in more detail, one can show that a choice of signs for k_i making $\sum_i k_i \sigma_i$ a cycle is possible iff M is orientable, and then μ defines a fundamental class. With $\mathbb{Z}_2$ coefficients there is no issue of signs, hence such a cycle $\sum_i \sigma_i$ always defines a fundamental class.

One can also obtain some information about $H_{n-1}(M)$ from the above theorem. First we need the following universal coefficient theorem for homology, proved in [H, Theorem 3A.3].

Theorem 10.6. *If C is a chain complex of free abelian groups, then there are short exact sequences*

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to \mathrm{Tor}(H_{n-1}(C),G) \to 0$$

for all n and G, and these sequences split (though not canonically).

Corollary 10.7. *If M is a closed, connected n-manifold, the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}_2$ if M is nonorientable.*

Proof. Let Tor be the torsion part of $H_{n-1}(M;\mathbb{Z})$. If M is orientable and Tor is not trivial, then $H_n(M;\mathbb{Z}_p)$ would be larger than $\mathbb{Z}_p$ for some p by the universal coefficient theorem for homology, contradicting that $H_n(M;\mathbb{Z}_p) \cong \mathbb{Z}_p$. If M is nonorientable, then $H_n(M;\mathbb{Z}_m)$ is either $\mathbb{Z}_2$ or 0, depending on whether m is even or odd. It forces Tor to be $\mathbb{Z}_2$ by the universal coefficient theorem for homology. □

For M a noncompact manifold, we have the following result.

Proposition 10.8. *If M is a connected noncompact n-manifold, then $H_i(M;R) = 0$ for $i \geq n$.*

Proof. Let z be a cycle representing an element of $H_i(M;R)$. Then z has compact image in M. There is an open set $U \subset M$ containing the image of z and having compact closure $\overline{U} \subset M$. Let $V = M - \overline{U}$. Part of the long exact sequence of the triple $(M, U \cup V, V)$ fits into the following diagram (draw a diagram).

When $i > n$, $H_{i+1}(M, U \cup V; R)$ and $H_i(M, V; R)$ are zero by the previous lemma, since $U \cup V$ and V are the complements of compact sets in M. Hence we conclude that $H_i(U;R) = 0$, so z is a boundary in U and therefore a boundary in M, and then $H_i(M;R) = 0$.

When $i = n$, the class $[z] \in H_n(M;R)$ defines a section $x \mapsto [z]_x$ of the covering $M_R \to M$. Since M is connected, this section is determined by its value at a single point, so $[z]_x$ will be zero for all x if it is zero for one x, which it must be since z has compact image and M is noncompact (i.e., take x outside of the image of

z). By the previous lemma, z then represents zero in $H_n(M,V;R)$, hence also zero in $H_n(U;R)$, since $H_{n+1}(M,U\cup V;R)=0$ by the lemma. It follows that $[z]=0$ in $H_n(M;R)$, and hence $H_n(M;R)=0$, since $[z]$ was an arbitrary element of this group. □

10.2. **The Duality Theorem.** Poincaré duality asserts that for an R-orientable closed n-manifold, there is a naturally defined map $H^k(M;R)\to H_{n-k}(M;R)$ which is an isomorphism. The definition of this map is in terms of *cap product*, which is closely related to cup product.

For a space X and coefficient ring R, define an R-bilinear cap product $\frown$: $C_k(X;R)\times C^\ell(X;R)\to C_{k-\ell}(X;R)$ for $k\geq\ell$ induced by

$$\sigma\frown\varphi=\varphi(\sigma|_{[v_0,\dots,v_\ell]})\sigma|_{[v_\ell,\dots,v_k]}$$

for $\sigma:\Delta^k\to X$ and $\varphi\in C^\ell(X;R)$. As in the case of cup product, one can check that

$$\partial(\sigma\frown\varphi)=(-1)^\ell(\partial\sigma\frown\varphi-\sigma\frown\delta\varphi).$$

It follows that $\sigma\frown\varphi$ is a cycle for a cycle σ and a cocycle φ. Moreover, if $\partial\sigma=0$, then $\partial(\sigma\frown\varphi)=\pm(\sigma\frown\delta\varphi)$, so the cap product of a cycle and a coboundary is a boundary. If $\delta\varphi=0$, then $\partial(\sigma\frown\varphi)=\pm(\partial\sigma\frown\varphi)$, so the cap product of a boundary and a cocycle is a boundary. Hence there is an induced cap product

$$H_k(X;R)\times H^\ell(X;R)\xrightarrow{\frown}H_{k-\ell}(X;R)$$

which is obviously R-bilinear. Similarly one checks that cap product has relative forms.

Given a map $f:X\to Y$, consider the following diagram (draw a diagram). The diagram does not commute directly, but commutativity makes sense as in the formula

$$f_*(\sigma)\frown\varphi=f_*(\sigma\frown f^*(\varphi)).$$

This formula can be checked by

$$\begin{aligned}(f\sigma)\frown\varphi &= \varphi((f\sigma)|_{[v_0,\dots,v_\ell]})(f\sigma)|_{[v_\ell,\dots,v_k]}\\ &= (f^*\varphi)(\sigma|_{[v_0,\dots,v_\ell]})(f\sigma)|_{[v_\ell,\dots,v_k]}\\ &= (f^*\varphi)(\sigma|_{[v_0,\dots,v_\ell]})(f_*\sigma)|_{[v_\ell,\dots,v_k]}\\ &= f_*(\sigma\frown f^*(\varphi)).\end{aligned}$$

Now we can state Poincaré duality for closed manifolds.

Theorem 10.9 (Poincaré Duality)**.** *If M is a closed R-orientable n-manifold with fundamental class $[M]\in H_n(M;R)$, then the map $D:H^k(M;R)\to H_{n-k}(M;R)$ defined by $D(\alpha)=[M]\frown\alpha$ is an isomorphism for all k.*

Recall that a fundamental class for M is an element of $H_n(M;R)$ whose image in $H_n(M|x;R)\cong R$ is a generator for all $x\in M$.

Example 10.10. Let M be the closed orientable surface of genus g with the following Δ-complex structure (draw a picture). We compute cap products in $\mathbb{Z}$ coefficients using simplicial homology and cohomology. Recall that a fundamental class $[M]$ generating $H_2(M)$ can be represented by the 2-cycle formed by the sum of all 2-simplices with the signs indicated. The edges a_i,b_i form a basis for $H_1(M)$, and the cohomology classes α_i, β_i form a basis for $H^1(M) \cong \text{Hom}(H_1(M), \mathbb{Z})$, which are defined using intersections with the corresponding arcs. By definition, we have

$$[M] \frown \alpha_i = \alpha_i([M]|_{[v_0,v_1]})[M]|_{[v_1,v_2]} = b_i,$$
$$[M] \frown \beta_i = \beta_i([M]|_{[v_0,v_1]})[M]|_{[v_1,v_2]} = -a_i,$$

since in both cases there is only one 2-simplex $[v_0, v_1, v_2]$ in $[M]$ where α_i or β_i is nonzero on the edge $[v_0, v_1]$. Then b_i is the Poincaré dual of α_i and $-a_i$ is the Poincaré dual of β_i. Geometrically, such duality is reflected in the fact that the loops α_i and b_i are homotopic, as are the loops β_i and a_i.

Example 10.11. Let N be the closed nonorientable surface of genus g with the following Δ-complex structure (draw a picture). Under $\mathbb{Z}_2$-coefficients N is orientable, and the fundamental class $[N]$ can be represented by the sum of all 2-simplices. The edges a_i form a basis for $H_1(N;\mathbb{Z}_2)$ and the classes α_i form a basis for $H^1(N;\mathbb{Z}_2)$. Then one checks that $[N] \frown \alpha_i = a_i$, so a_i is the Poincaré dual of α_i. Geometrically, the loops a_i and α_i are homotopic.

10.3. **Cohomology with Compact Supports.** The proof of Poincaré duality will be given by an inductive argument using Mayer-Vietoris sequences. The induction step requires a version of Poincaré duality for open subsets of M, so we need a different kind of cohomology.

Let $C^i_c(X;G)$ be the subgroup of $C^i(X;G)$ consisting of cochains $\varphi : C_i(X) \to G$ for which there exists a compact set $K \subset X$ such that φ is zero on all chains in $X - K$. Note that then $\delta\varphi$ is also zero on chains in $X - K$, hence $\delta\varphi \in C^{i+1}_c(X;G)$. Then the $C^i_c(X;G)$ for varying i form a subcomplex of the singular cochain complex of X. The corresponding cohomology groups $H^i_c(X;G)$ are called *cohomology with compact supports*.

There is a useful alternative definition of $H^i_c(X;G)$ in terms of direct limits. The cochain group $C^i_c(X;G)$ is the union of subgroups $C^i(X, X - K;G)$ as K ranges over compact subsets of X. Each inclusion $K_1 \hookrightarrow K_2$ induces inclusions $C^i(X, X - K_1;G) \hookrightarrow C^i(X, X-K_2;G)$ for all i, so there are induced maps $H^i(X, X-K_1;G) \to H^i(X, X - K_2;G)$. One may hope that $H^i_c(X;G)$ can be described in terms of the system of groups $H^i(X, X - K;G)$ for varying K.

In general, suppose $\{G_\alpha\}$ is a system of abelian groups with some *partially ordered* index set I, such that for each $\alpha, \beta \in I$, there exists $\gamma \in I$ with $\alpha \le \gamma$ and $\beta \le \gamma$. Such an I is called a *directed set*. Suppose also that for each $\alpha \le \beta$ there is a homomorphism $f_{\alpha\beta} : G_\alpha \to G_\beta$ such that $f_{\alpha\alpha} = \mathbb{1}$ for all α, and if $\alpha \le \beta \le \gamma$ then $f_{\alpha\gamma} = f_{\beta\gamma} f_{\alpha\beta}$. Such a system is called a *directed system* of groups. Define an equivalence relation on the set $\sqcup_\alpha G_\alpha$ by $a \sim b$ for $a \in G_\alpha$ and $b \in G_\beta$, if $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$ for some γ. One checks that this relation is reflexive, symmetric, and transitive. Now define the *direct limit* $\varinjlim G_\alpha$ to be the set of equivalence classes, which is also an abelian group whose group structure is induced from those G_α. A simple consequence is that if a subset $J \subset I$ has the property that for each $\alpha \in I$ there exists a $\beta \in J$ with $\alpha \le \beta$, then $\varinjlim G_\alpha$ is the same as $\varinjlim G_\beta$ using the index subset J. In particular, if I has a maximal element γ that is bigger than or equal to all α, then $\varinjlim G_\alpha = G_\gamma$.

Suppose a space X is the union of subspaces X_α forming a direct set with respect to inclusion. Then the groups $H_i(X_\alpha;G)$ for fixed i and G form a direct system. The maps $H_i(X_\alpha;G) \to H_i(X;G)$ induce a homomorphism $\varinjlim H_i(X_\alpha;G) \to H_i(X;G)$.

Proposition 10.12. *If X is the union of a directed set of subspaces X_α with the property that each compact set in X is contained in some X_α, then the map $\varinjlim H_i(X_\alpha; G) \to H_i(X; G)$ is an isomorphism for all i and G.*

Proof. For surjectivity, represent a cycle in X by a finite sum of singular simplices. The union of the images of these singular simplices is compact in X, hence lies in some X_α. For injectivity, if a cycle in some X_α is a boundary in X, compactness implies that it is a boundary in some $X_\beta \supset X_\alpha$, hence represents zero in $\varinjlim H_i(X_\alpha; G)$. □

The compact subsets $K \subset X$ form a directed set under inclusion since the union of two compact sets is compact. To each compact set $K \subset X$ we associate the group $H^i(X, X - K; G)$, with a fixed i and G. The resulting limit group $\varinjlim H^i(X, X - K; G)$ is then equal to $H^i_c(X; G)$, since each element of this limit group is represented by a cocycle in $C^i(X, X - K; G)$ for some compact set K, and such a cocycle is zero in the limit group iff it is the coboundary of a cochain in $C^{i-1}(X, X - L; G)$ for some compact set $L \supset K$.

Note that if X is compact, then $H^i_c(X; G) = H^i(X; G)$, since there is a unique maximal compact set $K = X \subset X$. This is also obvious from the original definition since $C^i_c(X; G) = C^i(X; G)$ if X is compact.

Example 10.13. Consider $X = \mathbb{R}^n$. To compute $\varinjlim H^i(\mathbb{R}^n, \mathbb{R}^n - K; G)$ it suffices to let K range over balls B_k of integer radius k centered at the origin since every compact set is contained in such a ball. Note that $H^i(\mathbb{R}^n, \mathbb{R}^n - B_k; G)$ is nonzero only for $i = n$, and in that case it is G and the maps $H^i(\mathbb{R}^n, \mathbb{R}^n - B_k; G) \to H^i(\mathbb{R}^n, \mathbb{R}^n - B_{k+1}; G)$ are isomorphisms for all k. We thus conclude that $H^i_c(\mathbb{R}^n; G) = 0$ for $i \neq n$ and $H^n_c(\mathbb{R}^n; G) \cong G$, which is not the same as the singular cohomology of $\mathbb{R}^n$.

Remark 10.14. Since $\mathbb{R}^n$ is contractible, this example shows that cohomology with compact supports is not invariant up to homotopy equivalence. This seems to cause some trouble, e.g., the constant map from $\mathbb{R}^n$ to a point does not induce a map on cohomology with compact supports. The maps which do induce maps on H^*_c are the *proper* maps, for which the inverse image of each compact set is compact.

The group $H^i(X, X - K; G)$ for K compact depends only on a neighborhood of K in X by excision (assuming that X is Hausdorff so that K is closed). For simplicity we will write $H^i(X|K; G)$ for this group. One can think of cohomology with compact supports as the limit of these local cohomology groups at compact subsets.

10.4. **Duality for Noncompact Manifolds.** For M an R-orientable n-manifold, possibly noncompact, we can define a duality map $D_M : H^k_c(M; R) \to H_{n-k}(M; R)$ by a limiting process as follows. For compact sets $K \subset L \subset M$, there is a diagram (draw a diagram). We have seen that there are unique elements $\mu_K \in H_n(M|K; R)$ and $\mu_L \in H_n(M|L; R)$ restricting to a given orientation of M at each point of K and L, respectively. Then we have $i_*(\mu_L) = \mu_K$. Consequently $\mu_K \frown \varphi = i_*(\mu_L) \frown \varphi = \mu_L \frown i^*(\varphi)$ for all $\varphi \in H^k(M|K; R)$. Letting K vary over compact sets in M, the homomorphisms $H^k(M|K; R) \to H_{n-k}(M; R)$ given by $\varphi \mapsto \mu_K \frown \varphi$ induce in the direct limit a homomorphism $D_M : H^k_c(M; R) \to H_{n-k}(M; R)$.

Since $H_c^*(M;R) = H^*(M;R)$ if M is compact, the following theorem generalizes Poincaré duality for closed manifolds.

Theorem 10.15. *The duality map $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$ is an isomorphism for all k whenever M is an R-oriented n-manifold.*

The proof relies on the following lemma, which generalizes Mayer-Vietoris sequences into the current context. From now on the coefficient ring R will be fixed, so for simplicity we skip it from the notation

Lemma 10.16. *If M is the union of two open sets U and V, then there is a diagram of Mayer-Vietoris sequences, commutative up to sign: (draw a diagram).*

Proof. The proof is similar to the proof of Mayer-Vietoris sequences, plus using barycentric subdivision as in the proof of Excision. See the proof of [H, Lemma 3.36] for details. □

Proof of Theorem 10.15. We make two observations first.
(A) If M is the union of open sets U and V, and if D_U, D_V, and $D_{U\cap V}$ are isomorphisms, then so is D_M. This follows from the preceding lemma by applying the five-lemma.
(B) If M is the union of a sequence of open sets $U_1 \subset U_2 \subset \cdots$ and each duality map $D_{U_i} : H_c^k(U_i) \to H_{n-k}(M)$ is an isomorphism, then so is D_M. To show this, notice first that $H_c^k(U_i)$ is the limit of $H^k(M|K)$ as K ranges over compact subsets of U_i by excision. Hence there are natural maps $H_c^k(U_i) \to H_c^k(U_{i+1})$ since the target is a limit over a larger collection of K's. Then we can form $\varinjlim H_c^k(U_i)$, which is isomorphic to $H_c^k(M)$, since the compact sets in M are just the compact sets in all the U_i's. We have seen before that $H_{n-k}(M) \cong \varinjlim H_{n-k}(U_i)$. The map D_M is thus the limit of the isomorphisms D_{U_i}, hence is an isomorphism.

Now the proof breaks into the following steps.

(1) Consider $M = \mathbb{R}^n$ first. Regard $\mathbb{R}^n$ as the interior of Δ^n. Then by excision the map D_M can be identified with the map $H^k(\Delta^n, \partial\Delta^n) \to H_{n-k}(\Delta^n)$ given by cap product with a unit times the generator $[\Delta^n] \in H_n(\Delta^n, \partial\Delta^n)$, where $[\Delta^n]$ is defined by the identity map of Δ^n as a relative cycle. The only nontrivial case is $k = n$. In this case a generator of $H^n(\Delta^n, \partial\Delta^n) \cong \mathrm{Hom}(H_n(\Delta^n, \partial\Delta^n), R)$ is represented by a cocycle φ taking the value 1 on Δ^n, so $\Delta^n \frown \varphi$ is the last vertex v_n of Δ^n, representing a generator of $H_0(\Delta^n)$. Therefore, the cap product map is an isomorphism.
(2) Next consider M as an arbitrary open set in $\mathbb{R}^n$. Write M as a countable union of open balls U_i, and let $V_i = \bigcup_{j<i} U_j$. Both V_i and $U_i \cap V_i$ are unions of (at most) $i-1$ bounded convex open sets (homeomorphic to $\mathbb{R}^n$), so by induction on the number of such sets in a cover we may assume that D_{V_i} and $D_{U_i\cap V_i}$ are isomorphisms. By (1), D_{U_i} is an isomorphism since U_i is homeomorphic to $\mathbb{R}^n$. Hence $D_{U_i\cup V_i}$ is an isomorphism by (A). Since M is the increasing union of the V_i's and each D_{V_i} is an isomorphism, so is D_M by (B).
(3) If M is a finite or countably infinite union of open sets U_i homeomorphic to $\mathbb{R}^n$, the proof follows from the same argument in (2), with "open balls" replaced by "open sets in $\mathbb{R}^n$". Thus the proof is finished for closed manifolds, as well as all noncompact manifolds one actually encounters in practice.
(4) To handle a completely general noncompact manifold M, one needs to use Zorn's lemma. Consider the collection of open sets $U \subset M$ for which D_U

are isomorphisms. This collection is partially ordered by inclusion, and the union of every totally ordered subcollection is again in the collection by (B). Zorn's lemma then implies that there exists a maximal open set U for which the theorem holds. If $U \neq M$, choose $x \in M - U$ and an open neighborhood V of x homeomorphic to $\mathbb{R}^n$. Then the theorem holds for V by (1), for $U \cap V$ by (2), and for U by assumption. So the theorem holds for $U \cup V$ by (A), contradicting the maximality of U.

□

Corollary 10.17. *A closed manifold of odd dimension has Euler characteristic zero.*

Proof. Let M be a closed n-manifold. If M is orientable, then by duality

$$\operatorname{rank} H_i(M;\mathbb{Z}) = \operatorname{rank} H^{n-i}(M;\mathbb{Z}),$$

which equals $\operatorname{rank} H_{n-i}(M;\mathbb{Z})$ by the universal coefficient theorem. Thus if n is odd, the terms of $\sum_i (-1)^i \operatorname{rank} H_i(M;\mathbb{Z})$ cancel in pairs.

If M is not orientable, we apply the same argument using $\mathbb{Z}_2$ coefficients, with $\operatorname{rank} H_i(M;\mathbb{Z})$ replaced by $\dim H_i(M;\mathbb{Z}_2)$ (as a vector space over $\mathbb{Z}_2$), to conclude that $\sum_i (-1)^i \dim H_i(M;\mathbb{Z}_2) = 0$. It remains to check that this alternating sum equals $\sum_i (-1)^i \operatorname{rank} H_i(M;\mathbb{Z})$. This can be done by using $H_i(M;\mathbb{Z}_2) \cong H^i(M;\mathbb{Z}_2)$ and applying the universal coefficient theorem as follows. Each $\mathbb{Z}$ summand of $H_i(M;\mathbb{Z})$ gives a $\mathbb{Z}_2$ summand of $H^i(M;\mathbb{Z}_2)$. Each $\mathbb{Z}_m$ summand of $H_i(M;\mathbb{Z})$ with m even gives a $\mathbb{Z}_2$ summand of $H^i(M;\mathbb{Z}_2)$ and a $\mathbb{Z}_2$ summand of $H^{i+1}(M;\mathbb{Z}_2)$, whose contributions to $\sum_i (-1)^i \dim H_i(M;\mathbb{Z}_2)$ cancel. The remaining $\mathbb{Z}_m$ summands of $H_i(M;\mathbb{Z})$ with m odd contribute nothing to $H^*(M;\mathbb{Z}_2)$. □

10.5. **Connection with Cup Product.** Cup and cap product are related by the formula

$$\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$$

where $\alpha \in C_{k+\ell}(X;R)$, $\varphi \in C^k(X;R)$, and $\psi \in C^\ell(X;R)$. This holds since for a singular $(k+\ell)$-simplex $\sigma : \Delta^{k+\ell} \to X$ we have

$$\begin{aligned} \psi(\sigma \frown \varphi) &= \psi(\varphi(\sigma|_{[v_0,\ldots,v_k]})\sigma|_{[v_k,\ldots,v_{k+\ell}]}) \\ &= \varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,v_{k+\ell}]}) \\ &= (\varphi \smile \psi)(\sigma). \end{aligned}$$

This formula induces the following commutative diagram (draw a diagram), where the horizontal maps h were introduced in the universal coefficient theorem for cohomology. When h are isomorphisms (e.g., when R is a field or the homology groups of X are free with $R = \mathbb{Z}$), then the map $\varphi \smile$ is the dual of $\frown \varphi$. Thus in this case cup and cap product determine each other. In general if there is torsion in homology, then cap and cup product are not always equivalent.

For a closed R-orientable n-manifold M, we have the cup product pairing

$$H^k(M;R) \times H^{n-k}(M;R) \to R, \quad (\varphi,\psi) \mapsto (\varphi \smile \psi)[M].$$

In general, a bilinear pairing $A \times B \to R$ is called *nonsingular* if the maps $A \to \operatorname{Hom}_R(B,R)$ and $B \to \operatorname{Hom}_R(A,R)$, obtained by viewing the pairing as a function of each variable separately, are both isomorphisms.

Proposition 10.18. *The cup product pairing is nonsingular for closed R-orientable manifolds when R is a field, or when $R = \mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out.*

Proof. Consider the composition

$$H^{n-k}(M;\mathbb{R}) \xrightarrow{h} \mathrm{Hom}_R(H_{n-k}(M;R),R) \xrightarrow{D^*} \mathrm{Hom}_R(H^k(M;R),R)$$

where h is evaluation of cochains on chains, and D^* is the Hom-dual of the Poincaré duality map $D : H^k(M;R) \to H_{n-k}(M;R)$. Then $D^* \circ h$ sends $\psi \in H^{n-k}(M;R)$ to the homomorphism $\varphi \mapsto \psi([M] \frown \varphi) = (\varphi \smile \psi)[M]$. For field coefficients or for integer coefficients with torsion factored out, h is an isomorphism. Then nonsingularity of the pairing in one of its variables is equivalent to D being an isomorphism, which holds by Poincaré duality. Nonsingularity in the other variable follows by swapping k and $n-k$. □

Corollary 10.19. *If M is a closed connected orientable n-manifold, then an element $\alpha \in H^k(M;\mathbb{Z})$ generates an infinite cyclic summand of $H^k(M;\mathbb{Z})$ iff there exists an element $\beta \in H^{n-k}(M;\mathbb{Z})$ such that $\alpha \smile \beta$ is a generator of $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$. With coefficients in a field this holds for any $\alpha \neq 0$.*

Proof. For α to generate a $\mathbb{Z}$ summand of $H^k(M;\mathbb{Z})$ is equivalent to the existence of a homomorphism $\varphi : H^k(M;\mathbb{Z}) \to \mathbb{Z}$ with $\varphi(\alpha) = \pm 1$. By the nonsingularity of the cup product pairing, φ can be realized by taking cup product with an element $\beta \in H^{n-k}(M;\mathbb{Z})$ and evaluating on $[M]$, so $\varphi(\alpha) = \pm 1$ is equivalent to having a β with $\alpha \smile \beta$ generating $H^n(M;\mathbb{Z})$. The case of field coefficients is similar, since every nonzero element can generate the field. □

Example 10.20. The inclusion $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ (as a hyperplane) induces an isomorphism on H^i for $i \leq 2n-2$, so by induction on n, one can show that $H^{2i}(\mathbb{CP}^n;\mathbb{Z})$ is generated by α^i for $i < n$, where $\alpha \in H^2(\mathbb{CP}^n;\mathbb{Z})$ represents a hyperplane class and generates $H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$. By the above corollary, there is an integer m such that the product $\alpha \smile (m\alpha^{n-1}) = m\alpha^n$ generates $H^{2n}(\mathbb{CP}^n;\mathbb{Z})$, and apparently this can only happen if $m = \pm 1$. Therefore, we conclude that the cohomology ring $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$. Similarly one can show that $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, where α generates $H^1(\mathbb{RP}^n;\mathbb{Z}_2)$.

References

[H] Allen Hatcher, Algebraic Topology.