1. Sheaves and cohomology

1.1. Sheaves. Let $X$ be a topological space. A sheaf $\mathcal{F}$ on $X$ associates to each open set $U$ an abelian group $\mathcal{F}(U)$, called the sections of $\mathcal{F}$ over $U$, along with a restriction map $r_{V,U}: \mathcal{F}(V) \to \mathcal{F}(U)$ for any open sets $U \subset V$ (for a section $\sigma \in \mathcal{F}(V)$, we often write $\sigma|_U$ to denote $r_{V,U}(\sigma)$), satisfying the following conditions:

1. For any open sets $U \subset V \subset W$, $r_{V,U} \circ r_{W,V} = r_{W,U}$;
2. For a collection of open sets $\{U_i\}_{i \in I}$ and sections $\alpha_i \in \mathcal{F}(U_i)$, if $\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$ for any $i, j \in I$, then there exists a unique $\alpha \in \mathcal{F}(\bigcup U_i)$ such that $\alpha|_{U_i} = \alpha_i$ for any $i$.

Remark 1.1. If $\mathcal{F}$ satisfies (1) only, we call it a presheaf. One can perform sheafification for a presheaf to make it become a sheaf. For many sheaves we consider, the restriction maps are the obvious ones by restricting functions from bigger subsets to smaller ones.

Exercise 1.2. Show that $\mathcal{F}(\emptyset)$ consists of exactly one element.

Example 1.3. Let $G$ be an abelian group. We have the sheaf of locally constant functions $\mathcal{G}$ on a topological space $X$, where $\mathcal{G}(U)$ is the group of locally constant maps $f: U \to G$ on a non-empty open set $U \subset X$, and $\mathcal{G}(\emptyset) = 0$.

Exercise 1.4. Show that for the sheaf $\mathcal{G}$ of locally constant functions, we have $\mathcal{G}(U) = G$ for any non-empty connected open set $U$.

Exercise 1.5. Suppose we define $\mathcal{G}(U) = G$ as the set of constant functions on a non-empty open set $U$ with the natural restriction maps. If $G$ contains at least two elements and if $X$ consists of two disjoint non-empty open subsets, show that $\mathcal{G}$ is not a sheaf.

Example 1.6. Let $X$ be a complex manifold and $U \subset X$ an open set.

1. Sheaf $\mathcal{O}$ of holomorphic functions:
   $\mathcal{O}(U) = \{\text{holomorphic functions on } U\}$.

The group law is given by addition.
(2) Sheaf $\mathcal{O}^*$ of nowhere zero holomorphic functions:

$$\mathcal{O}^*(U) = \{\text{holomorphic functions } f \text{ on } U : f(p) \neq 0 \text{ for every } p \in U\}.$$ 

The group law is given by multiplication.

(3) Sheaf $\mathcal{M}$ of meromorphic functions: strictly speaking, a meromorphic function is not a function, even we take $\infty$ into account. If $X$ is compact, we cannot take a meromorphic function as a quotient of two holomorphic functions, since any globally defined holomorphic function is a constant on $X$. Instead, we define $f \in \mathcal{M}(U)$ as local quotients of holomorphic functions compatible with each other. Namely, there exists an open covering $\{U_i\}$ of $X$ such that on each $U_i$, $f$ is given by $g_i/h_i$ for some $g_i, h_i \in \mathcal{O}(U_i)$ satisfying $g_i/h_i = g_j/h_j$, i.e. $g_i h_j = g_j h_i \in \mathcal{O}(U_i \cap U_j)$, hence these local quotients can be glued together over $U$.

(4) Sheaf $\mathcal{M}^*$ of meromorphic functions not identically zero: this is defined similarly as above and the group law is given by multiplication.

1.2. Maps between sheaves. Let $\mathcal{E}$ and $\mathcal{F}$ be two sheaves on a topological space $X$. A map $f: \mathcal{E} \to \mathcal{F}$ is a collection of group homomorphisms

$$\{f_U: \mathcal{E}(U) \to \mathcal{F}(U)\}$$

such that they commute with the restriction maps, i.e., for any open sets $U \subset V$ and $\sigma \in \mathcal{E}(V)$ we have

$$f_V(\sigma)|_U = f_U(\sigma|_U).$$

Define the sheaf of kernel $\ker(f)$ as

$$\ker(f)(U) = \{\ker(f_U: \mathcal{E}(U) \to \mathcal{F}(U))\}.$$ 

Exercise 1.7. Prove that in the above definition $\ker(f)$ is a sheaf.

Example 1.8. Let $X$ be a complex manifold. Define the exponential map

$$\exp: \mathcal{O} \to \mathcal{O}^*$$

by $\exp(h) = e^{2\pi i h}$ for any open set $U \subset X$ and section $h \in \mathcal{O}(U)$. It is easy to see that $\ker(\exp)$ is the locally constant sheaf $\mathbb{Z}$.

The sheaf of cokernel is harder to define. Naively, one would like to define $\text{coker}(f)(U) = \text{coker}(f_U: \mathcal{E}(U) \to \mathcal{F}(U))$, but this is problematic. For instance, consider the exponential map $\exp: \mathcal{O} \to \mathcal{O}^*$ on the punctured plane $\mathbb{C}\setminus\{0\}$. The section $z \in \mathcal{O}^*(\mathbb{C}\setminus\{0\})$ is not in the image of $f$, hence it would define a section in the cokernel. Nevertheless, restricted to any contractible open set $U \subset \mathbb{C}\setminus\{0\}$, $z$ lies in the image of $f$. Now cover $\mathbb{C}\setminus\{0\}$ by contractible open sets. By the gluing property of sheaves, $z$ would be zero everywhere, leading to a contradiction.

Instead, we define a section of $\text{coker}(f)(U)$ to be a collection of sections $\sigma_\alpha \in \mathcal{F}(U_\alpha)$ for an open covering $\{U_\alpha\}$ of $U$ such that for all $\alpha, \beta$ we have

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in f_{U_\alpha \cap U_\beta}(\mathcal{E}(U_\alpha \cap U_\beta)).$$

Here the definition still depends on the choice of an open covering. To get rid of this, we pass to the direct limit. More precisely, we further identify two collections $\{(U_\alpha, \sigma_\alpha)\}$ and $\{(V_\beta, \sigma_\beta)\}$ if for all $p \in U_\alpha \cap V_\beta$, there exists an open set $W$ satisfying $p \in W \subset U_\alpha \cap V_\beta$ such that

$$\sigma_\alpha|_W - \sigma_\beta|_W \in f_W(\mathcal{E}(W)).$$

This identification yields an equivalence relation and correspondingly we define $\text{coker}(f)(U)$ as the group of equivalence classes of the above sections.