# Orbit closures in Teichmüller dynamics and extremal cycles in algebraic geometry

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(joint work with Izzet Coskun)

Let $X$ be a normal projective variety. Let $D = \sum_{i=1}^n a_i Z_i$ be a divisor, i.e. a linear combination of codimension-one subvarieties $Z_i \subset X$. Define an equivalence relation "$\equiv$" for divisors, called *numerical equivalence*: $D \equiv D'$ if $D \cdot C = D' \cdot C$ for every curve $C \subset X$, where $D \cdot C$ is the intersection number of $D$ and $C$. Define the *effective cone* of $X$ by

$$\mathrm{Eff}(X) = \{D = \sum_{i=1}^n a_i Z_i \mid a_i \geq 0\}/\equiv .$$

By definition, $\mathrm{Eff}(X)$ has a convex structure. If the class of a divisor $D$ spans a one-dimensional face of $\mathrm{Eff}(X)$, we call $D$ an *extremal effective divisor*. Understanding $\mathrm{Eff}(X)$ amounts to finding out all the extremal effective divisors on $X$.

The effective cone governs the birational geometry of $X$. For instance, the canonical divisor class of $X$ is contained in the interior of $\mathrm{Eff}(X)$ if and only if $X$ is of *general type*, which is a higher dimensional analogue of curves of genus $\geq 2$. On the other hand, $\mathrm{Eff}(X)$ may fail to be closed or finite polyhedral, see [K, II 4.16] for an example of the latter.

Here we focus on the case when $X$ is the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable genus $g$ curves with $n$ ordered marked points. Since the 1980s, motivated by the problem of determining the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$, many authors have constructed families of effective divisors on $\overline{\mathcal{M}}_{g,n}$. For example, Harris, Mumford and Eisenbud [HM, H, EH], using Brill-Noether and Gieseker-Petri divisors showed that $\overline{\mathcal{M}}_g$ is of general type for $g > 23$.

Although we know many examples of effective divisors on $\overline{\mathcal{M}}_{g,n}$, the structure of $\mathrm{Eff}(\overline{\mathcal{M}}_{g,n})$ remains mysterious in general. In particular, for a long time it was not known whether there exist $g$ and $n$ such that $\mathrm{Eff}(\overline{\mathcal{M}}_{g,n})$ is not finitely generated.

In [CC], we study the case of genus one. By exhibiting infinitely many extremal effective divisors on $\overline{\mathcal{M}}_{1,n}$ for every $n \geq 3$, we are able to show that $\mathrm{Eff}(\overline{\mathcal{M}}_{1,n})$ is not finitely generated. The construction of those extremal divisors is motivated by the strata of quadratic differentials in genus one.

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a collection of $n$ integers satisfying $\sum_{i=1}^n a_i = 0$, not all equal to zero. Consider the stratum of quadratic differentials $\mathcal{Q}(\mathbf{a})$ parameterizing quadratic differentials $q$ on smooth genus one curves $E$ such that

$$(q)_0 - (q)_\infty = \sum_{i=1}^n a_i p_i,$$

where $p_1, \ldots, p_n \in E$ are distinct. We do not require $q$ to have simple poles only. Denote by $D_{\mathbf{a}}$ the closure of the projection of $\mathcal{Q}(\mathbf{a})$ in $\overline{\mathcal{M}}_{1,n}$. Then $D_{\mathbf{a}}$ is an effective divisor on $\overline{\mathcal{M}}_{1,n}$.

Assume that $n \geq 3$ and that $\gcd(a_1, \ldots, a_n) = 1$. The main result of [CC] says that $D_{\mathbf{a}}$ is an extremal effective divisor on $\overline{\mathcal{M}}_{1,n}$. Moreover, these $D_{\mathbf{a}}$ provide infinitely many extremal divisors, and hence $\mathrm{Eff}(\overline{\mathcal{M}}_{1,n})$ is not finitely generated.

Here we present the outline of a proof that works for the nonvarying strata $\mathcal{Q}(k, -1^k)$ and $\mathcal{Q}(k, 1, -1^{k+1})$ in the sense of [CM] (see [CC] for a different proof that works in general). Take a Teichmüller curve $C$ in such a nonvarying stratum $\mathcal{Q}(\mathbf{a})$. Since we know the sum of Lyapunov exponents of $C$ by [CM, Section 8], one can check that $C \cdot D_{\mathbf{a}} < 0$ in this case. Since the union of Teichmüller curves forms a (Zariski) dense subset in $D_{\mathbf{a}}$, it follows that $D_{\mathbf{a}}$ is extremal.

As a concluding remark, $\mathrm{SL}(2, \mathbb{R})$-orbit closures in the strata of abelian and quadratic differentials provide a number of algebraic cycles in $\mathrm{Eff}(\overline{\mathcal{M}}_{g,n})$, based on the recent breakthrough [EM] and [F]. These cycles are insufficiently studied from the viewpoint of algebraic geometry. It would be interesting to figure out their intersection-theoretic properties and extremality in the cone of effective cycles.

## References


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