GEOMETRY OF TEICHMÜLLER CURVES

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My research centers around the geometry of moduli spaces. One of my projects related to this meeting is to study the *algebro-geometric* properties of $\mathrm{SL}(2,\mathbb{R})$-submanifolds in the moduli space $\mathcal{H}$ of Abelian differentials.

Take a Riemann surface along with a holomorphic 1-form parameterized in $\mathcal{H}$. Its complex structure varies naturally with the 1-form via the $\mathrm{SL}(2,\mathbb{R})$ action. An *$SL(2,\mathbb{R})$-submanifold* is an orbit closure in $\mathcal{H}$ under this action. To name a few examples, if an orbit itself forms a closed complex curve, we call it a *Teichmüller curve*. The Hurwitz space parameterizing branched covers of tori and the strata in $\mathcal{H}$ parameterizing 1-forms with prescribed type of zeros are also $\mathrm{SL}(2,\mathbb{R})$-submanifolds.

In algebraic geometry it is often desirable to work with a compactified moduli space, like passing from the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces to its Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, i.e. we allow a slight degeneration of Riemann surfaces by pinching two points together. Here I would like to emphasize the significance of this viewpoint for the study of $\mathrm{SL}(2,\mathbb{R})$-submanifolds.

Take Teichmüller curves as illustration of the idea. One can associate three numbers: *the sum of Lyapunov exponents* L, *Siegel-Veech constant* c and *slope* s to a Teichmüller curve. The first two come from dynamics. Roughly speaking, Lyapunov exponents characterize the rate of separation of infinitesimally closed trajectories under the Teichmüller geodesic flow. The Siegel-Veech constant represents the average number of weighted horizonal cylinders in the orbit that generates the Teichmüller curve, where the Abelian differential defines a flat structure on the Riemann surface such that it decomposes into cylinders along a fixed direction and the weight of a cylinder is given by its height/length. The third one, slope, comes from algebraic geometry, by taking the quotient of the intersection of a Teichmüller curve with the boundary of $\overline{\mathcal{M}}_g$ and the intersection with the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_g$.

Although these three numbers seem unrelated, after the work of Kontsevich [K], Bouw-Möller [BM], Eskin-Kontsevich-Zorich [EKZ] (in much more generality) and myself [C], we now know a simple relation among them:

$$s = \frac{12c}{L} = \frac{12c}{c+\kappa},$$

where κ is a constant determined by the type of zeros of a generating Abelian differential. Namely, knowing any one of the three immediately tells the other two!

As an application, joint with Möller [CM] we show that for many strata of Abelian differentials in low genus the sum of Lyapunov exponents is *non-varying* for *all* Teichmüller curves in that stratum. Our idea is to prove that the slope is non-varying first, by exhibiting a geometrically defined *divisor* on $\overline{\mathcal{M}}_g$ that does not intersect Teichmüller curves. Then we can translate back to the dynamical side by the above relation.

Currently I am interested in generalizing the results of Teichmüller curves to quadratic differentials as well as higher dimensional $\mathrm{SL}(2, \mathbb{R})$-submanifolds.

References

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