Moduli Spaces of Curves and Teichmüller Dynamics

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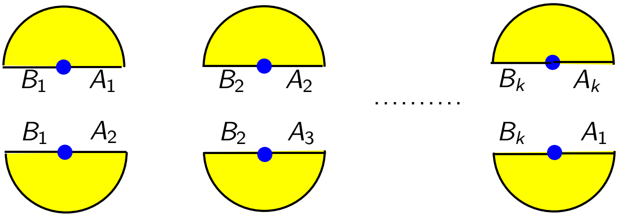

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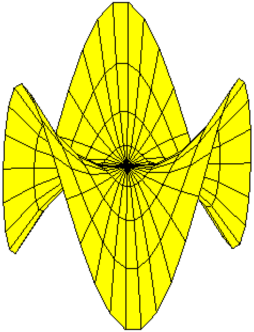

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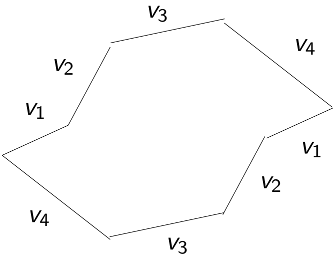

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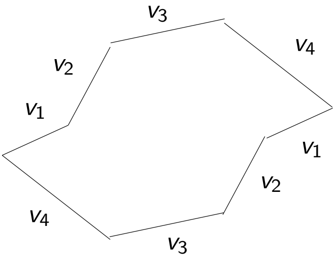

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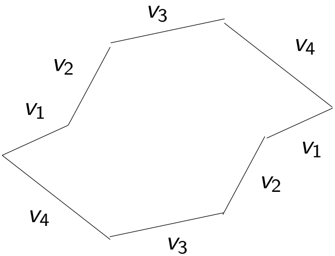

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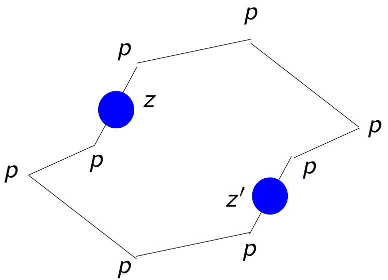

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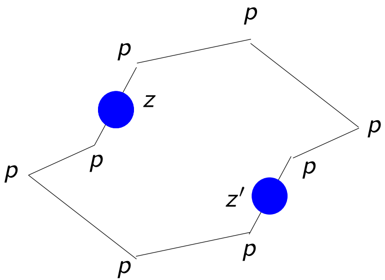

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- In summary, ω is a holomorphic one-form with a unique zero of order 2 on a Riemann surface of genus two.

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- In summary, ω corresponds to a translation surface with n saddle points, each having angle $(m_i + 1) \cdot (2\pi)$.

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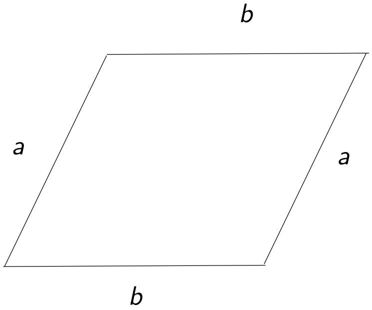

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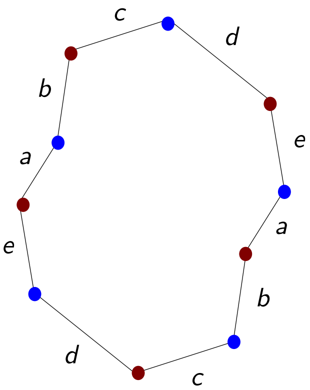

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- $\mathcal{H} := \bigcup_{\mu \vdash 2g-2} \mathcal{H}(\mu)$ is a vector bundle of rank g, called the *Hodge bundle*, on the *moduli space* $\mathcal{M}_g$ *of genus* g *Riemann surfaces*.

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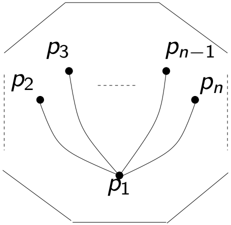

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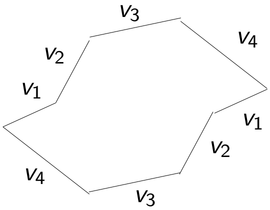

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- [Kontsevich-Zorich, 2003] classified connected components for all $\mathcal{H}(\mu)$.

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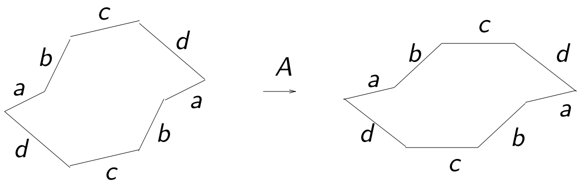

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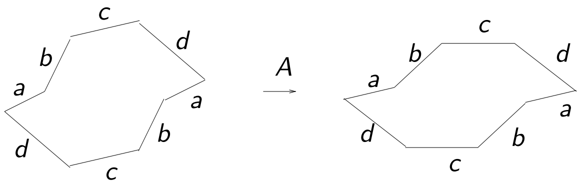

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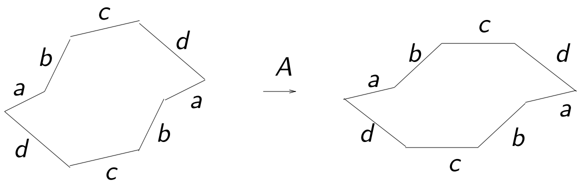

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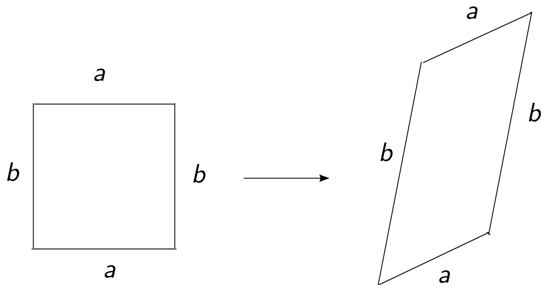

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- [Eskin-Mirzakhani-Mohammadi, 2013] showed that *any* $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure is a *linear submanifold*.

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- [Filip, 2014] showed that all $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures are algebraic varieties defined over $\overline{\mathbb{Q}}$.

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- Such (X, ω) are called *square-tiled surfaces* (or *origami*).

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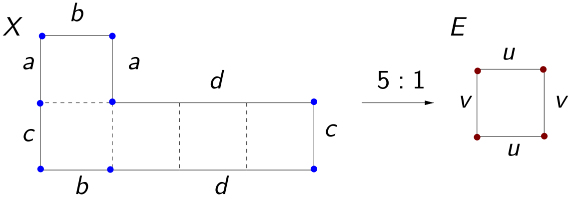

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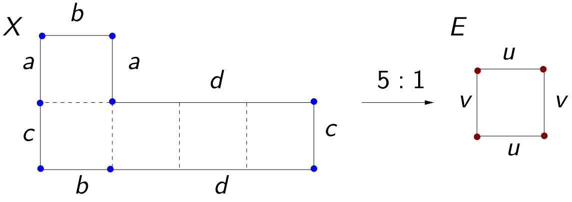

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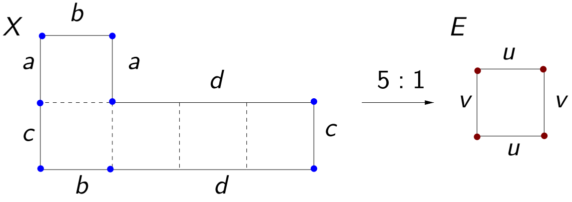

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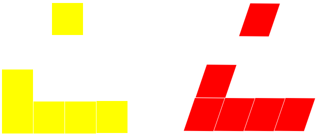

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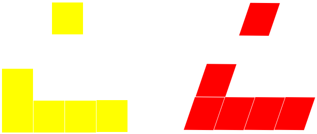

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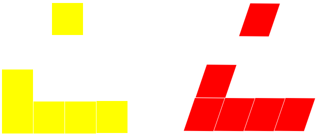

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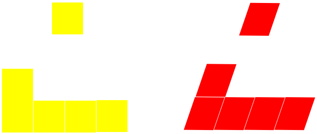

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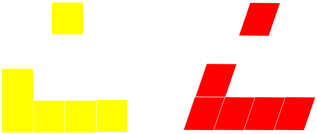

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- Hurwitz spaces are algebraic varieties $\Longrightarrow$ The $GL_2^+(\mathbb{R})$-orbit of a square-tiled surface produces a Teichmüller curve.
- Varying d, we get *infinitely many* Teichmüller curves in $\mathcal{H}(\mu)$.

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- A complete classification of Teichmüller curves is still missing.

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Figure: An L-shaped billiard table

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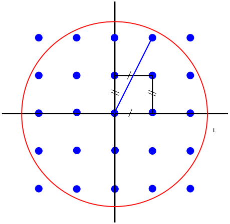

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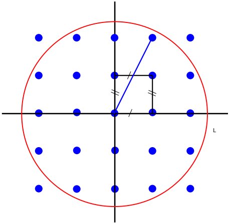

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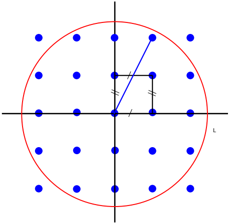

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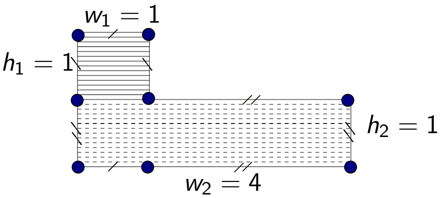

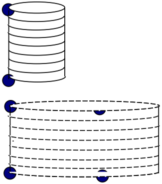

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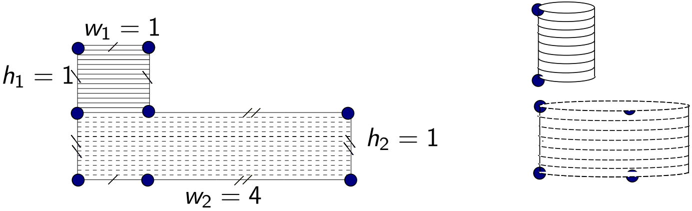

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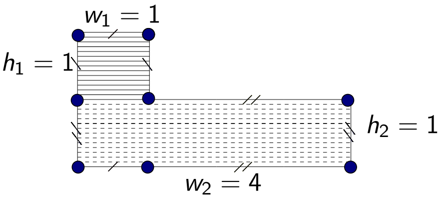

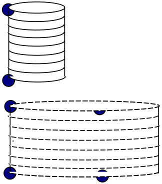

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- $\text{Area(cyl)} = w \cdot h$.

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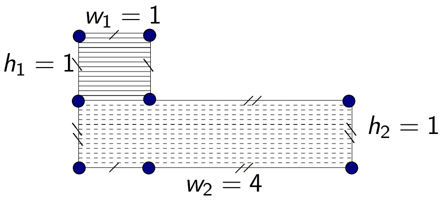
Example



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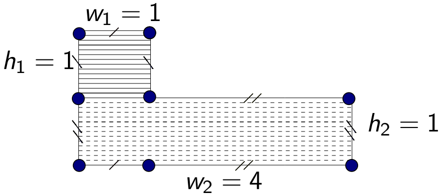

The square-tiled surface above contributes

$$\frac{h_1}{w_1} + \frac{h_2}{w_2} = \frac{1}{1} + \frac{1}{4}$$

to the sum M.

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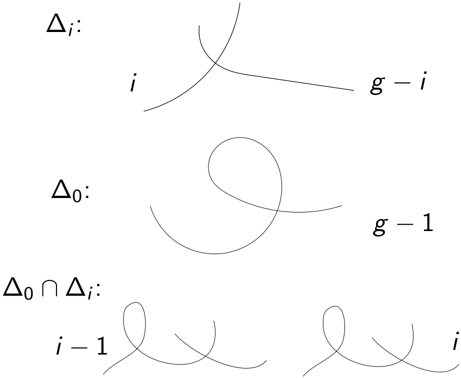

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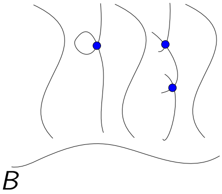

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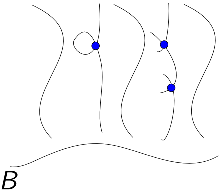

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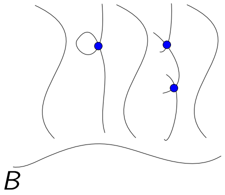

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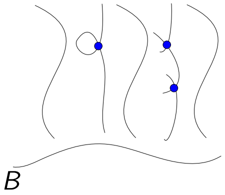

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- Slope measures the variation of complex structures with respect to the number of nodes in a one-dimensional family of curves.

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Theorem (C., 2011; C.-Möller, 2012)

The area Siegel-Veech constant and slope of a Teichmüller curve in $\mathcal{H}(\mu)$ determine each other:

$$s = \frac{12c}{c+\kappa_\mu}.$$

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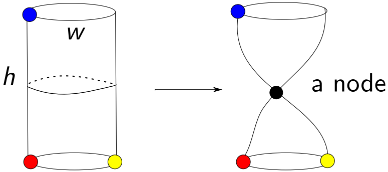

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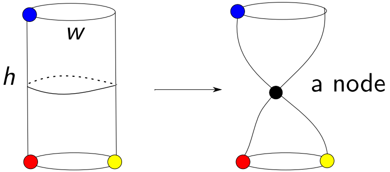

- On $\overline{\mathcal{M}}_g$, we have $\lambda \equiv \dfrac{\kappa + \Delta}{12}$.

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Corollary

For a Teichmüller curve, any one of the three numbers s, c and L determine the other two.

Kontsevich-Zorich's conjecture

- By computer experiments, Kontsevich-Zorich (2003) observed that for many strata $\mathcal{H}(\mu)$ in *low* genus, *all* Teichmüller curves in the same stratum (component) have *non-varying* area Siegel-Veech constant.

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- This phenomenon ends for $g \geq 6$ except for *hyperelliptic strata*.

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Kontsevich-Zorich's conjecture holds, i.e., for each stratum in the above list, all Teichmüller curves it contains have the same area Siegel-Veech constant, hence they have the same sum of Lyapunov exponents.

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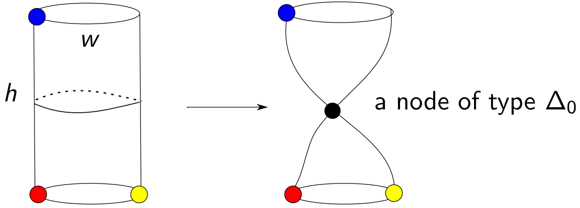

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- By the slope and Siegel-Veech formula, area Siegel-Veech constants are the same for *all* Teichmüller curves in $\mathcal{H}(3,1)$.

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Thank you!