EXTREMAL EFFECTIVE DIVISORS OF BRILL-NOETHER AND GIESEKER-PETRI TYPE IN $\overline{M}_{1,n}$

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Abstract. We show that certain divisors of Brill-Noether and Gieseker-Petri type span extremal rays of the effective cone in the moduli space of stable genus one curves with $n$ ordered marked points. In particular, they are different from the infinitely many extremal rays found in [CC].

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1. Introduction

Let $\overline{M}_{g,n}$ be the Deligne-Mumford moduli space of stable genus $g$ curves with $n$ ordered marked points. Denote by $\overline{\text{Eff}}(\overline{M}_{g,n})$ the cone of pseudoeffective divisors on $\overline{M}_{g,n}$. Understanding the structure of $\overline{\text{Eff}}(\overline{M}_{g,n})$ plays a central role in the study of the birational geometry of $\overline{M}_{g,n}$, see e.g. [HMu, Ha, EH, F, Lo, V, CT].

In [CC, Theorem 1.1], it was shown that there exist infinitely many extremal effective divisors in $\overline{M}_{1,n}$ for each $n \geq 3$. It provides the first (and the only) known example of $\overline{M}_{g,n}$ whose pseudoeffective cone is not finitely generated. Recall the definition of those divisors. Let $a = (a_1, \ldots, a_n)$ be a collection of $n$ integers satisfying that $\sum_{i=1}^n a_i = 0$, not all equal to zero. Define $D_a$ in $\overline{M}_{1,n}$ as the closure of the divisorial locus parameterizing smooth genus one curves with $n$ ordered marked points $(E; p_1, \ldots, p_n)$ such that $\sum_{i=1}^n a_i p_i \sim 0$ in $E$. For $n \geq 3$ and $\gcd(a_1, \ldots, a_n) = 1$, $D_a$ spans an extremal ray of $\overline{\text{Eff}}(\overline{M}_{1,n})$.

A natural question is whether the $D_a$ (and the boundary components) span all (rational) extremal rays of $\overline{\text{Eff}}(\overline{M}_{1,n})$. This is a meaningful question and one might expect an affirmative answer, as we explain in the following example. Consider the abelian surface $E \times E$, where $E$ is a general smooth elliptic curve with $p_0$ as the origin. Take $a_1, a_2 \in \mathbb{Z}$ such that they are relatively prime. Consider the locus

$$C = \{(p_1, p_2) \in E \times E \mid a_1 p_1 + a_2 p_2 \sim (a_1 + a_2) p_0\}.$$

We know that $C$ spans an extremal ray of $\overline{\text{Eff}}(E \times E)$. Moreover, all (rational) extremal rays of $\overline{\text{Eff}}(E \times E)$ are spanned by such $C$, see [K] II 4.16]. Note that $C$ is an analogue of $D_a$ with $a = (a_1, a_2, -a_1 - a_2)$ when we fix the moduli of a genus one curve with three marked points.

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Nevertheless, the main result of this paper shows that the above question has a negative answer.

**Theorem 1.1.** For every \( n \geq 6 \), there exist extremal effective divisors in \( \overline{M}_{1,n} \) that are different from the \( D_\alpha \)'s.

See Theorems 3.3, 4.4 and 5.4 for a more precise statement.

Let us explain our method. For a stable genus one curve with \( 2m \) marked points \( (E;p_1,\ldots,p_{2m}) \), identify \( p_{2i−1} \) and \( p_{2i} \) as a node for \( i = 1,\ldots,m \). We thus obtain an \( m \)-gonal curve of arithmetic genus \( m+1 \). It induces a gluing morphism \( \pi : \overline{M}_{1,2m} \to \overline{M}_{m+1} \). We will show that pulling back certain divisors of Brill-Noether and Gieseker-Petri type by \( \pi \) gives rise to extremal effective divisors different from the \( D_\alpha \)'s. To verify extremality, we exhibit a moving curve in (the main component of) the pullback divisor such that the curve has negative intersection number with the divisor. This idea was also used in [O] to construct extremal effective divisors in \( \overline{M}_{0,n} \) that are different from the hypertree divisors in [CT].

The paper is organized as follows. In Section 2, we review basic divisor theory of \( \overline{M}_{g,n} \) and carry out the calculation of pulling back divisor classes under the gluing map \( \pi : \overline{M}_{1,2m} \to \overline{M}_{m+1} \). In Section 3 we verify the extremality of the main component of the pullback of the Brill-Noether trigonal divisor. In Section 4 we study the pullback of Brill-Noether \( d \)-gonal divisors for general \( d \) and show that their main components are extremal. Finally, in Section 5 we study the pullback of the Gieseker-Petri divisor from \( \overline{M}_4 \) and show that its main component is extremal.

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### 2. Preliminaries on moduli spaces of curves

Denote by \( \lambda \) the first Chern class of the Hodge bundle on \( \overline{M}_{g,n} \). Let \( \Delta_{\text{irr}} \) be the locus in \( \overline{M}_{g,n} \) parameterizing curves with a non-separating node. For \( 0 \leq i \leq \lfloor g/2 \rfloor \), \( S \subset \{1,\ldots,n\} \) and \( 2i−2+|S| \geq 0 \), let \( \Delta_{i,S} \) denote the closure of the locus in \( \overline{M}_{g,n} \) that parameterizes nodal curves consisting of two components of genera \( i \) and \( g−i \), respectively, where the genus \( i \) component contains the marked points labeled by \( S \). Denote by \( \delta_* \) the divisor class of \( \Delta_* \) and let \( \delta \) be the class of the union of all boundary divisors on \( \overline{M}_{g,n} \). Let \( \psi_i \) be the first Chern class of the cotangent line bundle on \( \overline{M}_{g,n} \) associated to the \( i \)th marked point for \( 1 \leq i \leq n \). These divisor classes are defined on the moduli stack instead of the coarse moduli scheme, see e.g. [AC] [HMo] for more details.

In this paper, we focus on \( \overline{M}_{1,n} \) and \( \overline{M}_g \). The rational Picard group of \( \overline{M}_g \) is generated by \( \lambda,\delta_{\text{irr}},\delta_1,\ldots,\delta_{\lfloor g/2 \rfloor} \), and for \( g \geq 3 \) these divisor classes form a basis. The rational Picard group of \( \overline{M}_{1,n} \) has a basis given by \( \lambda \) and \( \delta_{0,S} \) for \( |S| \geq 2 \). The divisor classes \( \delta_{\text{irr}} \) and \( \psi_i \) on \( \overline{M}_{1,n} \) can be expressed as

\begin{align}
\delta_{\text{irr}} &= 12\lambda, \\
\psi_i &= \lambda + \sum_{i \in S} \delta_{0,S}.
\end{align}

Since \( \delta_{\text{irr}} \) and \( \lambda \) are proportional on \( \overline{M}_{1,n} \), we will use them interchangeably throughout the paper.

For a stable genus one curve with \( 2m \) marked point \( (E;p_1,\ldots,p_{2m}) \), identify \( p_{2i−1} \) and \( p_{2i} \) as a node for \( i = 1,\ldots,m \). We thus obtain a curve of arithmetic genus \( m+1 \) with \( m \) non-separating nodes. This induces a morphism

\[ \pi : \overline{M}_{1,2m} \to \overline{M}_{m+1}. \]

The image of \( \pi \) is contained in \( \Delta_{\text{irr}} \).
Let us calculate the pullback of divisor classes from $\mathcal{M}_{m+1}$ to $\mathcal{M}_{1,2m}$ via $\pi$. For $1 \leq i \leq m$, define $\Lambda_i = \{S \subset \{1, \ldots, 2m\} \mid S = \{2k_1 - 1, 2k_1, \ldots, 2k_i - 1, 2k_i\}, 1 \leq k_1 < \cdots < k_i \leq m\}$.

In other words, for $S \in \Lambda$, $S$ contains the labeling of $i$ pairs of marked points that are glued to $i$ nodes. Denote by $S^c$ the complement of $S$ in $\{1, \ldots, n\}$.

**Proposition 2.1.** Under the above setting, we have

$$\pi^* \lambda = \lambda,$$

$$\pi^* \delta_i = \sum_{S \in \Lambda_i} \delta_{0,S} + \sum_{S^c \in \Lambda_{i-1}} \delta_{0,S}, \quad 1 \leq i < \frac{m+1}{2},$$

$$\pi^* \delta_i = \sum_{S \in \Lambda_i} \delta_{0,S}, \quad i = \frac{m+1}{2} \text{ for odd } m,$$

$$\pi^* \delta = (12 - 2m)\lambda - \sum_{|S| \geq 2} (|S| - 1)\delta_{0,S}.$$

**Proof.** We can calculate $\pi^* \lambda$ by induction on the number of pairs of points that are glued to a node. Take a family $f : C \to B$ of stable curves of genus $g - 1$ with two disjoint sections $P_1$ and $P_2$. Consider the exact sequence

$$0 \to \Omega \to \Omega(P_1 + P_2) \to \mathcal{O}_{P_1} \oplus \mathcal{O}_{P_2} \to 0,$$

where $\Omega$ is the relative dualizing sheaf of the family $f : C \to B$. Applying $f_*$, we obtain that

$$0 \to f_*\Omega \to f_*(\Omega(P_1 + P_2)) \to \mathcal{O}_B \oplus \mathcal{O}_B \to \mathcal{O}_B \to 0.$$

It follows that

$$\pi^* \lambda = c_1(f_*(\Omega(P_1 + P_2))) = c_1(f_*\Omega) = \lambda.$$

If the image of $(E; p_1, \ldots, p_{2m}) \in \Delta_{0,S}$ under $\pi$ lies in $\Delta_i$, either the rational tail or the genus one tail becomes a component of arithmetic genus $i$ after the gluing process, which corresponds to the case $S \in \Lambda_i$ or $S^c \in \Lambda_{i-1}$, respectively. Note that for $1 \leq i \leq \frac{m+1}{2}$, $S \in \Lambda_i$ and $S^c \in \Lambda_{i-1}$ hold simultaneously if and only if $m$ is odd and $i = \frac{m+1}{2}$. The two equalities about $\pi^* \delta_i$ follow right away.

Finally, let $f : E \to B$ be a family of stable genus one curves with $2m$ sections $P_1, \ldots, P_{2m}$. If a node is obtained by identifying $P_{2k-1}$ and $P_{2k}$, its contribution in $\pi^* \delta$ is

$$f_*(P^2_{2k-1}) + f_*(P^2_{2k}) = -\psi_{2k-1} - \psi_{2k}.$$

If a fiber $E$ in $E$ is contained in $\delta_{0,S}$ or $\delta_{irr}$, a node of $E$ remains to be a node after the gluing process. This implies that

$$\pi^* \delta = \sum_{i=1}^{2m} (-\psi_i) + \delta = (12 - 2m)\lambda - \sum_{|S| \geq 2} (|S| - 1)\delta_{0,S},$$

where the relations (1) and (2) are used in the last step. 

\[ \square \]

3. Pulling Back the Trigonal Divisor in $\mathcal{M}_5$

Consider $\pi : \mathcal{M}_{1,5} \to \mathcal{M}_5$. Let $BN^1_3 \subset \mathcal{M}_5$ denote the Brill-Noether trigonal divisor whose general point parameterizes a curve admitting a triple cover of $\mathbb{P}^1$. By [HMu], it has divisor class

$$BN^1_3 = \begin{cases} 8\lambda - \delta_{irr} - 4\delta_1 - 6\delta_2 \\ 8\lambda - \delta - 3\delta_1 - 5\delta_2. \end{cases}$$
By Proposition 2.1 we obtain that
\[ \pi^*BN_3^1 = 4\Delta + \sum_{S \not\subseteq \Lambda} (|S| - 1)\delta_0,5 + 4\delta_0,\{1,...,8\} \]
\[ -2\sum_{k=1}^{4} \delta_0,\{2k-1,2k\} - 2\sum_{i<j} \delta_0,\{2i-1,2i,2j-1,2j\}. \]

Note that \(\pi^{-1}BN_3^1\) may contain some boundary components of \(\mathcal{M}_{1,8}\), see Remark 3.4. Denote by \(\mathcal{B}N_3^1\) the main component of \(\pi^{-1}BN_3^1\), i.e. \(\mathcal{B}N_3^1\) is actually the closure of \(\pi^{-1}BN_3^1 \cap \mathcal{M}_{1,8}\). (See Lemma 4.3 for the irreducibility of \((\pi^{-1}BN_3^1) \cap \mathcal{M}_{1,4d-4}\) for general \(d\).)

Let us characterize when \((E;p_1,\ldots,p_8)\) is contained in \(\mathcal{B}N_3^1\) for a smooth genus one curve \(E\) with eight distinct marked points.

**Proposition 3.1.** In the above setting, \((E;p_1,\ldots,p_8)\) is a general point of \(\mathcal{B}N_3^1\) if and only if \(E\) admits an embedding as a plane cubic in \(\mathbb{P}^2\) such that the four lines \(\overline{p_1p_2}, \overline{p_3p_4}, \overline{p_5p_6}\) and \(\overline{p_7p_8}\) are concurrent.

**Proof.** Identifying \(p_{2k-1}, p_{2k}\) in \(E\) for \(k = 1, 2, 3, 4\), we obtain a curve \(C\) of arithmetic genus five with four nodes. Suppose that \(C\) is a trigonal curve. Then the canonical model of \(C\) is contained in a cubic surface in \(\mathbb{P}^4\). Perform elementary transformations at the nodes of \(C\) and blow down the proper transform of the hyperplane section \(H\) spanned by the nodes. The cubic surface is transformed to \(\mathbb{P}^2\) and the image of \(C\) is the embedding of \(E\) as a plane cubic in \(\mathbb{P}^2\). The exceptional curve containing \(p_{2k-1}, p_{2k}\) is transformed to a line spanned by \(p_{2k-1}, p_{2k}\) for \(k = 1, 2, 3, 4\). The four lines are concurrent at a point \(v\) arising from the contraction image of \(H\).

Conversely, if \((E;p_1,\ldots,p_8)\) admits such a plane cubic configuration, projecting \(E\) from \(v\) to a line gives rise to a \(g_3^1\) on \(E\), which descends to a \(g_3^1\) on the 4-nodal curve \(\pi(E)\), because \(p_{2k-1}\) and \(p_{2k}\) map to the same image under the \(g_3^1\) for \(k = 1, 2, 3, 4\). \(\Box\)

Next we construct a curve \(B\) moving in \(\mathcal{B}N_3^1\) such that \(B \cdot \mathcal{B}N_3^1 < 0\). Fix four general concurrent lines \(L_1, \ldots, L_4\) in \(\mathbb{P}^2\) and fix two general points \(p_{2k-1}, p_k \in L_k\) for each \(k\). Consider the pencil \(B\) of plane cubics passing through the eight fixed points. Denote by \(p_{2k}\) the remaining intersection point of the cubics with \(L_k\) for each \(k\). See Figure 3 for this configuration.

Marking \(p_1,\ldots,p_8\), \(B\) can be viewed as a family of genus one curves with eight ordered marked points. By Proposition 3.1 \(B\) is a moving curve in \(\mathcal{B}N_3^1\).

**Lemma 3.2.** On \(\mathcal{M}_{1,8}\) we have the following intersection numbers:

\[ B \cdot \lambda = 1, \]
\[ B \cdot \delta_0,\{2k-1,2k\} = 1, \]
\[ B \cdot \delta_0,\{2,4,6,8\} = 1, \]
\[ B \cdot \delta_0,5 = 0, \quad S \neq \{2k-1,2k\}, \{2,4,6,8\}. \]

**Proof.** Since \(p_{2k}\) varies in \(L_k\), when it coincides with the fixed \(p_{2k-1}\), we obtain a curve parameterized in \(\delta_0,\{2k-1,2k\}\). There is no reducible cubic that can pass through the eight fixed points, hence every curve parameterized by \(B\) is an irreducible genus one curve. Moreover, since the locus of cuspidal cubics has codimension two in the total space of plane cubics, it implies that for a general configuration as in Figure 3 the pencil \(B\) does not contain cuspidal cubics. Finally, if the cubic passes through the common point \(v\) of the \(L_k\), then \(p_2, p_3, p_6, p_8\) coincide. Since they approach \(v\) in different directions, blowing up \(v\) results in a stable curve contained in \(\delta_0,\{2,4,6,8\}\). The desired intersection numbers follow right away. \(\Box\)
We claim that \( \widetilde{BN}_3 \) is an extremal effective divisor. Denote by \( f : \overline{M}_{1,n} \rightarrow \overline{M}_{1,m} \) the forgetful morphism forgetting the last \( n - m \) marked points.

**Theorem 3.3.** The divisor \( \widetilde{BN}_3 \) is an extremal divisor on \( \overline{M}_{1,8} \). For every \( n \geq 8 \), \( f^*\widetilde{BN}_3 \) spans an extremal ray of \( \text{Eff}(\overline{M}_{1,n}) \) that is different from the ray of \( D_a \).

**Proof.** Write \( \pi^*BN_3^1 = B\overline{N}_3^1 + B\overline{N}_3^1 \), where \( B\overline{N}_3^1 \) is the union of boundary divisors contained in \( \pi^{-1}BN_3^1 \). Using Lemma 3.2 and (3), a routine calculation shows that \( B \cdot \pi^*BN_3^1 = -1 < 0 \). Since \( B \) is not entirely contained in the boundary of \( \overline{M}_{1,8} \), it implies that \( B \cdot B\overline{N}_3^1 \geq 0 \), hence \( B \cdot B\overline{N}_3^1 < 0 \). By [CC] Lemma 4.1, we know that \( B\overline{N}_3^1 \) is extremal and rigid on \( \overline{M}_{1,8} \).

For \( n \geq 8 \), take \( n - 8 \) general very ample divisors \( H_i \) on \( \overline{M}_{1,n} \) and use them to cut out a curve \( B' \) in \( f^{-1}B \). Then the class of \( f_*B' \) is a positive multiple of \( B \). By the projection formula, \( B' \cdot f^*\widetilde{BN}_3 \overset{=}{=} (f_*B') \cdot \widetilde{BN}_3^1 < 0 \). Moreover, varying \( B \) in \( B\overline{N}_3^1 \) and \( H_i \) in \( \overline{M}_{1,n} \), it follows that \( B' \) is a moving curve in \( f^{-1}BN_3^1 \). Therefore, \( f^*BN_3^1 \) is extremal and rigid on \( \overline{M}_{1,n} \).

Recall that the divisor \( D_a = D_{-a} \) on \( \overline{M}_{1,n} \) parameterizes \( (E;p_1, \ldots, p_n) \) where \( \sum_{i=1}^n a_i p_i \sim 0 \) in \( E \), \( \sum_{i=1}^n a_i = 0 \) and \( \gcd(a_1, \ldots, a_n) = 1 \). Since \( D_a \) and \( f^{-1}B\overline{N}_3^1 \) are rigid, in order to prove that they span different extremal rays, it suffices to show that they have different supports. If \( f^{-1}BN_3^1 \) and \( D_a \) are set-theoretically the same, then \( a_i = 0 \) for \( i < 8 \) since there is no constraint imposed to \( p_i \) for \( i > 8 \) in the definition of \( B\overline{N}_3^1 \). Moreover, by the symmetry between the four pairs of nodes \( \{p_2k-1, p_{2k}\} \), we conclude that \( \{a_{2k-1}, a_{2k}\} = \{c, c\} \) or \( \{-c, -c\} \) for \( 1 \leq k \leq 4 \). It follows that \( c \leq 1 \) and without loss of generality, say, \( a_1 = a_2 = a_3 = a_4 = 1 \), \( a_5 = a_6 = a_7 = a_8 = -1 \) (up to reordering the four nodes). Nevertheless, the resulting relation \( p_1 + p_2 + p_3 + p_4 \sim p_5 + p_6 + p_7 + p_8 \) is not invariant under the symmetry between \( \{p_3, p_4\} \) and \( \{p_5, p_6\} \), leading to a contradiction. \( \square \)

**Remark 3.4.** We claim that \( B\overline{N}_3^1 \) is nonempty, i.e. \( \pi^{-1}BN_3^1 \) consists of the main component \( B\overline{N}_3^1 \) as well as some boundary divisors. For example, take a pencil of plane cubics and mark a base point as \( p_8 \). Attach it to \( \mathbb{P}^1 \) at another base point and mark seven general points in \( \mathbb{P}^1 \) as \( p_1, \ldots, p_7 \). We obtain a curve \( C \) moving in \( \delta_{0;\{1, \ldots, 7\}} \) with the following intersection numbers:

\[ C \cdot \lambda = 1, \]
It follows that $C \cdot \pi^* \Delta g$ is extremal. We thus obtain that the only nonzero interactions of $B \cdot \delta_{0,k}$ are both contained in $\Sigma_{d-4}$.

Consider the surface $S = E \times \mathbb{P}^1$, where $E$ is a smooth genus one curve. Let $\pi_0$ and $\pi_1$ be the projections to $\mathbb{P}^1$ and $E$, respectively. We have

$$\text{Pic}(S) \cong \mathbb{Z}[e] \oplus \pi_1^* \text{Pic}(E),$$

where $e = \pi_0^* O_{\mathbb{P}^1}(1)$ represents the genus one fiber class. Let $D$ be a divisor of degree $d$ on $E$. Projecting a curve in the linear system $|e + \pi_1^* D|$ via $\pi_0$, we obtain a degree $d$ cover of $\mathbb{P}^1$. Take $2d - 2$ general genus one fibers $E_1, \ldots, E_{2d-2}$ and fix a general point $p_{2k-1} \in E_k$ for $k = 1, \ldots, 2d - 2$. Since $\dim |e + \pi_1^* D| = 2d - 1$, we obtain a pencil of curves $C_b$ in $S$ that pass through all the fixed $p_{2k-1}$. Denote by $p_{2k}$ an intersection point of $C_b$ with $E_i$ other than $p_{2k-1}$. Then $(C_b, p_1, \ldots, p_{4d-4})$ is a genus one curve with $4d - 4$ marked points. Note that there are $d - 1$ choices for each $p_{2k}$. After a base change of degree $(d - 1)^{2d - 2}$, we obtain a one-dimensional family $B$ of curves in $\mathcal{M}_{1,2d-4}$. Since $p_{2k-1}$ and $p_{2k}$ are both contained in $E_k$, projecting to $\mathbb{P}^1$ realizes $\pi(E, p_1, \ldots, p_{4d-4})$ as a $d$-gonal curve.

Note that $C_b$ has the same $j$-invariant as that of $E$, because it admits a one-to-one map to $E$ via $\pi_1$. It implies that

$$B \cdot \lambda = B \cdot \delta_{\text{irr}} = 0.$$

Since $(e + \pi_1^* D)^2 = 2d$, besides $p_1, p_3, \ldots, p_{4d-5}$ there are two other base points $q_1$ and $q_2$ in the pencil. Hence it contains $2d$ singular curves before the base change, each of which consists of a genus zero fiber passing through $s = p_{2k-1}$ or $s = q_i$ union a (unique) genus one curve in $|e + \pi_1^* (D - s)|$ passing through the remaining base points.

**Example 4.1.** Consider $d = 3$ and $\pi : \mathcal{M}_{1,8} \to \mathcal{M}_5$. Make a base change of degree $2^4$, so that we can distinguish the two remaining intersection points of $E_k$ and $C_b$ besides $p_{2k-1}$ for $k = 1, 2, 3, 4$. See Figure 2 for the configuration.

If $p_{2k}$ coincides with $p_{2k-1}$, it contributes $2 \cdot 2^3 = 8$ to $B \cdot \delta_{0,\{2k-1,2k\}}$ (due to the base change). For a singular curve in $B$, if $s = q_i$ for $i = 1, 2$, it contributes 1 to $\delta_{0,S}$ where $S \subset \{2, 4, 6, 8\}$. If $s = p_{2k-1}$, then it contributes 2 to $\delta_{0,S \cup \{2k\}}$ for $S \subset \{2, 4, 6, 8\} \setminus \{2k\}$, where the number 2 comes from the choice of $p_{2k}$.

We thus obtain that the only nonzero intersections of $B$ with boundary divisors are as follows:

$$B \cdot \delta_{0,\{2k-1,2k\}} = 16,$$

$$B \cdot \delta_{0,S} = 2, \quad S \subset \{2, 4, 6, 8\},$$

$$B \cdot \delta_{0,S \cup \{2k\}} = 2, \quad S \subset \{2, 4, 6, 8\} \setminus \{2k\}.$$

It follows that

$$B \cdot \pi^* \Delta g = -128 + 2(1 \cdot 6 + 2 \cdot 4 + 3 \cdot 1) + 2 \cdot 4 \cdot (1 \cdot 3 + 2 \cdot 3 + 3 \cdot 1) = 2 > 0.$$
Nevertheless, for $d \geq 4$ the above intersection number turns out to be negative.

**Lemma 4.2.** In the above setting, $B \cdot \pi^* BN_1^d < 0$ for $d \geq 4$.

**Proof.** The calculation is similar to Example 4.1. Make a degree $(d - 1)^{2d-2}$ base change, so that we can distinguish the marked point $p_{2k}$ out of the remaining $d - 1$ intersection points of $E_k$ and $C_b$ for $k = 1, \ldots, 2d - 2$. If $p_{2k}$ coincides with $p_{2k-1}$, it contributes $(d - 1)(d - 1)^{2d-3} = (d - 1)^{2d-2}$ to $B \cdot \delta_{0;\{2k-1,2k\}}$. For a singular curve in $B$, if it passes through $s = q_i$ for $i = 1$ or 2, both of them contribute $(d - 1)^2$ to $\delta_0$ for $S \subset \{2, 4, \ldots, 4d - 4\}$, due to the choice of the marked points in $\{2, 4, \ldots, 4d - 4\} \setminus S$. If $s = p_{2k-1}$, it contributes $(d - 1)(d - 2)(d - 2)^{2d-3-|S|}$ to $\delta_{0;S|\{2k-1\}}$ for $S \subset \{2, 4, \ldots, 4d - 4\} \setminus \{2k\}$, due to the choice of $p_{2k}$ and the marked points in $\{2, 4, \ldots, 4d - 4\} \setminus \{2k\}$.

We thus obtain that

$$B \cdot \delta_{0;\{2k-1,2k\}} = (d - 1)^{2d-2},$$

$$B \cdot \delta_{0;S} = 2(d - 2)^{2d-2-|S|}, \quad S \subset \{2, 4, \ldots, 4d - 4\},$$

$$B \cdot \delta_{0;S|\{2k-1\}} = (d - 1)(d - 2)^{2d-3-|S|}, \quad S \subset \{2, 4, \ldots, 4d - 4\} \setminus \{2k\}.$$
respectively, by Proposition 2.1 and 4. It follows that
\[
\frac{1}{c} B \cdot \pi^* B \widetilde{N}_d^1 = \left( 2 - \frac{4}{3}d \right) (d - 2)/(d - 1)^{2d - 1} + \frac{d}{3} \left( \sum_{s=1}^{2d-3} \frac{2(d - s) (2d - 2)^{s-2} (s - 1)}{2d - 3} \right)
\]
\[
+ \frac{d}{3} (2d - 2) \left( \sum_{s=1}^{d-1} (d - 2)^{2d - 3 - s} (2d - 3)^{s} \right)
\]
\[
= \left( 2 - \frac{4}{3}d \right) (d - 2)/(d - 1)^{2d - 1} + \frac{2d}{3} \left( \frac{1}{2} d (2d - 1) (2d - 2)^{d - 2} \right)
\]
\[
= \frac{2}{3} \left( d^2 (d - 2)(2d - 2) - 2(d - 3)(d - 1)^{2d - 1} \right).
\]

It is easy to check that \( B \cdot \pi^* B \widetilde{N}_d^1 < 0 \) for \( d \geq 4 \). \( \square \)

Taking away possible boundary components, denote by \( \widetilde{BN}_d^1 \) the main component of \( \pi^{-1} B \widetilde{N}_d^1 \).

**Lemma 4.3.** The main component \( \widetilde{BN}_d^1 \) is irreducible.

**Proof.** Let \( U \subset \widetilde{BN}_d^1 \) be the open dense subset parameterizing simply branched degree \( d \), genus one covers \( f : E \to \mathbb{P}^1 \) with a general choice of \( m \) pairs of points \((p_1,p_2),\ldots,(p_{2m-1},p_{2m})\) in \( E \), where \( m = 2d - 2 \) and \( f(p_{2j-1}) = f(p_{2j}) = q_j \in \mathbb{P}^1 \). It suffices to show that \( U \) is irreducible.

Let \( b_1,\ldots,b_{2d} \in \mathbb{P}^1 \) be the set of branch points of \( f \). The branch data \( \phi_1,\ldots,\phi_{2d} \) associated to the branch points can be arranged as
\[
(1,2),(1,2),(1,2),(1,2),(2,3),(2,3),(2,3),(3,4),(3,4),\ldots, (d-1,d),(d-1,d),
\]
see e.g. [EEHS, page 100]. In other words, the monodromy induced by a closed, suitably oriented loop centering around \( b_i \) is the permutation \( \phi_i \in S_d \).

The choice of \( p_{2j-1} \) and \( p_{2j} \) amounts to choosing two distinct numbers \( a_{2j-1} \) and \( a_{2j} \) out of \( \{1,\ldots,d\} \), i.e. specifying two of the \( d \) sheets of \( f \) over \( q_j \). Without loss of generality, assume that \( a_{2j-1} < a_{2j} \). Vary \( q_j \) along the loops centering around the \( b_i \) with branch data \((a_{2j-1} - 1,a_{2j-1}),(a_{2j-1} - 2,a_{2j-1} - 1),\ldots, (1,2)\) successively. When \( q_j \) comes back to the original position, \( a_{2j-1} \) is transformed to 1. Next, vary \( q_j \) along the loops centering around the \( b_i \) with branch data \((a_{2j} - 1,a_{2j}),(a_{2j} - 2,a_{2j} - 1),\ldots, (2,3)\) successively. As a result, \( a_{2j} \) is transformed to 2. Finally, if we vary \( q_j \) along the loop around \( b_1 \) with branch datum \((1,2)\), the ordered pair \((1,2)\) is transformed to \((2,1)\). Meanwhile, the other pairs \((a_{2k-1},a_{2k})\) are unchanged since we did not vary \( q_k \) for \( k \neq j \). Carrying out this process for \( j = 1,\ldots,m \) one by one, eventually all the pairs \((a_{2j-1},a_{2j})\) can be transformed to \((1,2)\).

Let \( W \) be the open dense subset of the Hurwitz space of degree \( d \), genus one, simply branched covers of \( \mathbb{P}^1 \). The above process implies that the monodromy of \( U \to W \) is transitive, where the map is finite of degree \( (d(d-1))^m \) forgetting the \( p_i \). Since \( W \) is irreducible, see [Hu, EEHS], it follows that \( U \) is irreducible. \( \square \)

**Theorem 4.4.** For \( d \geq 4 \), \( \widetilde{BN}_d^1 \) spans an extremal ray of \( \overline{\text{Eff}}(\mathcal{M}_{1,4d-4}) \) and it is different from the ray of any \( D_a \).

**Proof.** By Lemma 4.2, \( B \cdot \pi^* B \widetilde{N}_d^1 < 0 \) and it is not entirely contained in the boundary of \( \mathcal{M}_{1,4d-4} \), hence \( B \cdot \widetilde{BN}_d^1 < 0 \). Moreover, by the construction of \( B \), it is a moving curve in \( B \widetilde{N}_d^1 \). It follows that \( \widetilde{BN}_d^1 \) spans an extremal ray of \( \overline{\text{Eff}}(\mathcal{M}_{1,4d-4}) \). The same argument as in the proof of Theorem 3.3 shows that \( \widetilde{BN}_d^1 \) and \( D_a \) span different extremal rays of \( \overline{\text{Eff}}(\mathcal{M}_{1,4d-4}) \). \( \square \)
5. Pulling back the Gieseker-Petri divisor in $\mathcal{M}_4$

Consider the gluing map $\pi : \mathcal{M}_{1,6} \to \mathcal{M}_4$. Let $GP$ denote the Gieseker-Petri divisor in $\mathcal{M}_4$. The divisor $GP$ has several geometric interpretations. Here we view it as the closure of the locus in $\mathcal{M}_4$ parameterizing genus four curves whose canonical images are contained in a quadric cone in $\mathbb{P}^3$. By [EH] Theorem 2, it has class

$$GP = 34\lambda - 4\delta_{\text{int}} - 14\delta_1 - 18\delta_2$$

$$= 34\lambda - 4\delta - 10\delta_1 - 14\delta_2.$$  

By Proposition 2.1 we obtain:

$$\pi^*GP = 10\lambda + 4 \sum_{S \notin \Delta_i}(|S| - 1)\delta_{0, S} + 10\delta_{0, \{1,2,3,4,5,6\}} - 6(\delta_{0, \{1,2\}} + \delta_{0, \{3,4\}} + \delta_{0, \{5,6\}}) - 2(\delta_{0, \{1,2,3,4\}} + \delta_{0, \{1,2,5,6\}} + \delta_{0, \{3,4,5,6\}}).$$

In principle, $\pi^{-1}GP$ may contain some boundary components of $\mathcal{M}_{1,6}$, as we saw in Remark 3.4. Denote by $\tilde{GP}$ the main component of $\pi^{-1}GP$, i.e. $\tilde{GP}$ is the closure of $(\pi^{-1}GP) \cap \mathcal{M}_{1,6}$.

The main result of this section is that $GP$ spans an extremal ray of $\text{Eff}(\mathcal{M}_{1,6})$. We prove it by constructing a moving curve $B \subset \tilde{GP}$ such that $B \cdot \tilde{GP} < 0$.

We begin by observing that the general element $(E; p_1, ..., p_6) \in \tilde{GP}$ is obtained as the (marked) normalization of a 3-nodal elliptic curve $C \subset \mathbb{P}^2$ where $\mathbb{P}^2$ is the second Hirzebruch surface. Let $\sigma$ be the directrix class on $\mathbb{P}^2$ and let $\tau$ be the section class satisfying $\tau \cdot \sigma = 0$. The class of $C$ is $3\tau$. We will essentially vary the 3-nodal curve $C$. However, it will also turn out to be necessary to “vary” the ambient surface $\mathbb{P}^2$. Let us first construct the variation of the ambient surface.

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$, with projections $\pi_1$ and $\pi_2$. Let $f_1$ and $f_2$ denote the classes of the fibers of $\pi_1$ and $\pi_2$, respectively. Let $O_Q(m, n)$ denote the line bundle associated to the divisor class $m f_1 + n f_2$. Let $V = O_Q \oplus O_Q(1, 2)$ be a rank two vector bundle on $Q$. Consider the $\mathbb{P}^1$-bundle

$$Y = \text{Proj}(V),$$

with its projection $p : Y \to Q$. Let $\phi : Y \to \mathbb{P}^1$ be the composite $\pi_1 \circ p$. Then $\phi$ is a non-trivial family of $\mathbb{P}^2$'s. The threefold $Y$ will essentially serve as the ambient space of the varying family of 3-nodal elliptic curves.

Let us understand the geometry of $Y$. The Picard group of $Y$ is $\mathbb{Z}[\zeta, f_1, f_2]$, where here, and in everything that follows, we suppress pullbacks $p^*$'s from notation. The projection $p$ has a distinguished section $\Sigma \subset Y$, which can be described as the union of the $(−2)$-curves in the $\mathbb{P}^2$'s. The class of $\Sigma$ is

$$\Sigma = \zeta - f_1 - 2f_2.$$ 

We also select a disjoint section $\Pi \subset Y$ having divisor class

$$\Pi = \zeta.$$  

In terms of the family $\phi : Y \to \mathbb{P}^1$, the section $\Pi$ provides a family of sections complementary to the directrices of the varying $\mathbb{P}^2$ fibers. The projection $p$ obviously restricts to an isomorphism on the section $\Pi$, so it makes sense to refer to the rulings of $\Pi$ by $f_1$ and $f_2$ as well.

We will now blow up $Y$ along the union of three disjoint curves. Restricted to each fiber $\mathbb{P}^2$, this amounts to blowing up three points corresponding to the nodes of the desired 3-nodal curve $C$.

Let $Z_1, Z_2$ and $Z_3$ be three disjoint curves of the ruling class $f_2$ on $\Pi$. Let $Z = Z_1 \cup Z_2 \cup Z_3$ and let $X = \text{Bl}_2 Y$ with the blow down map $\beta : X \to Y$. Denote by $\varphi : X \to \mathbb{P}^1$ the composite $\phi \circ \beta$. Then $\varphi$ is a family of surfaces $S_t$, each being the blow up of $\mathbb{P}^2$ at three points $z_1(t), z_2(t)$ and $z_3(t)$ for $t \in \mathbb{P}^1$. The Picard group of $X$ is $\mathbb{Z}[\zeta, f_1, f_2, e_1, e_2, e_3]$ where the $e_i$ are the classes of the respective exceptional divisors $E_i$. Let $e = e_1 + e_2 + e_3$ and $E = E_1 \cup E_2 \cup E_3$. Since $N_{Z/Y}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$, each exceptional divisor $E_i$ is isomorphic to the first Hirzebruch surface $\mathbb{F}_1$. 


Now we consider the divisor class $3\zeta + (a - 2)f_1 - 2e$ on $X$, where $a \gg 0$. Let $l$ be the class of a line and let $r$ be the class of a ruling on $E_i \cong \mathbb{F}_1$. Note that $\zeta|_{E_i} = f_1|_{E_i} = r$ and $e_i|_{E_i} = r - l$. Therefore,

$$3\zeta + (a - 2)f_1 - 2e|_{E_i} = 2l + (a - 1)r.$$

**Lemma 5.1.** Let $D$ be a general divisor with divisor class $3\zeta + (a - 2)f_1 - 2e$ on $X$, where $a \gg 0$.

(i) $D$ is a smooth surface in $X$. Under the map $\varphi : D \to \mathbb{P}^1$, the fibers have genus one and the singular fibers are at worst nodal.

(ii) The curves $C_i = D \cap E_i$ provide smooth 2-sections to the family $\varphi : D \to \mathbb{P}^1$ for $i = 1, 2, 3$. Moreover, the induced double cover $\varphi : C_i \to \mathbb{P}^1$ has 2a ramification points.

(iii) There are exactly $(a + 1)$ singular fibers in the family $\varphi : D \to \mathbb{P}^1$ having rational tails. Every rational tail meets $C_i$ once for each $i$.

**Proof.** Let $S$ be a fiber of $\varphi : X \to \mathbb{P}^1$. So $S = \text{Bl}_1 \mathbb{F}_2$. We first show that the restriction map

$$\rho : H^0(X, \mathcal{O}_X(D)) \to H^0(S, \mathcal{O}_S(D))$$

is surjective. By the long exact sequence, it is enough to show that

$$h^1(X, \mathcal{O}_X(D - S)) = 0.$$

By the Leray spectral sequence for $\varphi$, $h^1(X, \mathcal{O}_X(D - S)) = 0$ if

$$h^0(R^1\varphi_*\mathcal{O}_X(D - S)) = h^1(\varphi_*\mathcal{O}_X(D - S)) = 0.$$ 

The sheaf $R^1\varphi_*\mathcal{O}_X(D - S)$ vanishes, because the line bundle class $3\tau - 2e$ on $S = \text{Bl}_1 \mathbb{F}_2$ has no higher cohomology. Furthermore, by push-pull, we have

$$\varphi_*\mathcal{O}_X(D - S) \cong \varphi_*\mathcal{O}_X(3\zeta - 2e) \otimes O_{\mathbb{P}^1}(a - 3),$$

which vanishes higher cohomology for $a \gg 0$. Therefore, $\rho$ is surjective. The divisor class $3\tau - 2e$ on $\text{Bl}_1 \mathbb{F}_2$ is base point free. By adjunction, curves parameterized by the linear system $|3\tau - 2e|$ have genus one. Furthermore, the locus of curves with worse than nodal singularities is of codimension 2 in $|3\tau - 2e|$. Therefore, (i) follows from Bertini’s theorem.

In exactly the same way, one can show that the restriction map $H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{O}_{E_i}(D))$ is surjective when $a \gg 0$. The adjunction formula applied to the curve class $2l + (a - 1)r$ implies that $C_i$ has genus $a - 1$, thus proving (ii).

The proper transform $\Pi'$ of $\Pi$ in $X$ is isomorphic to $\Pi$ under the blow down map $\beta : X \to Y$. The intersection $D \cdot \Pi'$ has divisor class $(a + 1)f_1$ in $\Pi'$, where $f_1$ is the ruling class corresponding to $f_1$ under the isomorphism $\Pi' \cong \Pi$. It gives rise to $(a + 1)$ disjoint curves that are precisely the rational tails in the family $\varphi : D \to \mathbb{P}^1$. Hence (iii) follows right away. \hfill $\square$

Our next goal is to compute the degree of $\delta_{ir}$ restricted to $\varphi : D \to \mathbb{P}^1$, i.e. the number of nodal fibers in this family which do not have a rational tail. It is equal to $12\lambda$ by (1). Before doing so, we gather some relevant intersection products in the Chow ring of $X$.

**Lemma 5.2.** The following intersection numbers hold in the Chow ring of $X$:

\[f_2^2 = (\zeta - f_1 - 2f_2) \cdot \zeta = 0,\]
\[f_2 \cdot e_i = (\zeta - f_1) \cdot e_i = 0,\]
\[\zeta^3 = 4, \quad e_i^3 = -1,\]
\[e_i^2 \cdot f_1 = e_i^2 \cdot \zeta = -1,\]
\[\zeta^2 \cdot f_1 = 2, \quad \zeta^2 \cdot f_2 = \zeta \cdot f_1 \cdot f_2 = 1.\]

Furthermore, let $l_i$ and $r_i$ be the line and ruling classes of $E_i \cong \mathbb{F}_1$. Then $e_i^2 = r_i - l_i$ and $\zeta \cdot e_i = f_1 \cdot e_i = r_i$. 

Proof. We will prove only those which are not immediately clear. First, we see that
\[ \zeta^2 = (f_1 + 2f_2) \cdot \zeta \]
which follows from the general fact that
\[ \zeta^r = \sum_{i=1}^{r} (-1)^{i+1} c_i(V) \cdot \zeta^{r-i} \]
for any projective bundle \( \text{Proj} V \), where \( V \) is a rank \( r \) vector bundle and \( \zeta \) is the universal line bundle class of \( \text{Proj} V \). Here in our case \( \zeta \) is the class of \( \Pi \). From this, one easily derives the formula \( \zeta^3 = 4 \) as well as all other formulas involving only \( \zeta, f_1 \), and \( f_2 \).

We now deal with the intersections involving the exceptional divisor classes \( e_i \). If \( W \subset G \) is a smooth subscheme of a variety \( G \) and if \( \tilde{G} = \text{Bl}_W G \), then the exceptional divisor \( E \subset \tilde{G} \) is isomorphic to \( \text{Proj} (I_W/I_W^2) \), and \( \mathcal{O}_E(E) = \mathcal{O}_{E}(\mathcal{O}_{E}(\mathcal{O}_{E}(-1)) \) has class \( r - l \). In our setting, the conormal bundle of \( Z_i \) in \( Y \) is \( \mathcal{O}_{Z_i} \oplus \mathcal{O}_{Z_i}(-1) \). Therefore, \( E_i \) is abstractly isomorphic to \( \mathbb{P}_1 \), and \( \mathcal{O}_{E_i}(1) \) is the line bundle associated to the directrix \( \sigma \subset \mathbb{P}_1 \). The class of the directrix is also given by \( l_i - r_i \), and therefore \( e_i^2 = r_i - l_i \). Furthermore, since \( \zeta \cdot Z_i = f_1 \cdot Z_i = 1 \) in \( Y \), we conclude that \( \zeta \cdot e_i = f_1 \cdot e_i = r_i \).

The remaining intersection products follow from those explained above. \( \square \)

Lemma 5.3. For the family \( \varphi : D \to \mathbb{P}^1 \), \( \delta_{\text{irr}} = 12\lambda = 12(a - 1) \).

Proof. First observe that \( \lambda = \chi(\mathcal{O}_D) \). Secondly, by Noether's formula we know that
\[ 12\chi(\mathcal{O}_D) = K_D^2 + c_2(T_D). \]
Since the fibers of \( \varphi \) have genus one, the intersection number \( K_D^2 \) is \( -(a + 1) \), where each rational tail contributes \(-1 \).

So we need only to compute \( c_2(T_D) \). Using the exact sequence
\[ 0 \to T_D \to T_X|_D \to N_{D/X} \to 0, \]
we see that
\[
\begin{align*}
c_2(T_D) &= c_2(T_X|_D) - c_1(T_D) \cdot c_1(N_{D/X}) \\
&= c_2(T_X|_D) + K_D \cdot c_1(N_{D/X}) \\
&= c_2(T_X) \cdot D + (D + K_X) \cdot D^2.
\end{align*}
\]
Equality \( \blacksquare \) indicates that we need only to compute \( c_2(T_X) \), and then we can proceed by using the intersection formulas provided in Lemma 5.2. Consider the sequence
\[ 0 \to T_X \to \beta^* T_Y \to T_{E/Z}(E) \to 0, \]
from which we conclude
\[ c_2(T_X) = i_* [-\omega_{E/Z}] + E \cdot K_X + \beta^* c_2(T_Y) \]
where \( i : E \to X \) is the inclusion map. For \( \square \), we can use the Grothendieck-Riemann-Roch formula for the inclusion \( i \) to deduce
\[ c_1(T_{E/Z}(E)) = E, \]
\[ c_2(T_{E/Z}(E)) = i_* [c_1(\omega_{E/Z})] = r - 2f. \]
Using the exact sequence
\[ 0 \to \mathcal{O}_Y(\Sigma + \Pi) \to T_Y \to p^*(T_Q) \to 0, \]
we obtain
\[ c_2(T_Y) = -K_Y \cdot (\Sigma + \Pi) + 4f_1 \cdot f_2. \]
Finally, we can compute the quantity \( c_2(T_D) \) using the equality \( \blacksquare \) and the intersection numbers given in Lemma 5.2. The end result is
\[ c_2(T_D) = 13a - 11. \]
Therefore, $12\lambda = -(a + 1) + 13a - 11 = 12a - 12$ as claimed. \hfill\Box

There are three types of singular fibers in the genus one family $\varphi : D \to \mathbb{P}^1$:

(i) Irreducible nodal fibers.
(ii) Fibers having a rational tail. The rational tail meets each 2-section $C_i$ once.
(iii) Two $\mathbb{P}^1$’s joined at two nodes. One component will be the directrix $\sigma \subset Bl_3 \mathbb{F}_2$. The other component will intersect each $C_i$ twice.

In order to obtain a family of genus one curves with six ordered marked points, we need to perform a base change $T \to \mathbb{P}^1$ of degree $2^3 = 8$ to distinguish $p_{2k-1}$ and $p_{2k}$ that are glued as a node for $k = 1, 2, 3$. We name the sections so that the marked points $\{p_{2k-1}, p_{2k}\}$ correspond to the 2-section $C_k$. Then the curve $T$ carries over it a family of genus one curves with six ordered marked points.

Using Lemmas 5.1, 5.3 and the above analysis of singular fibers of $\varphi$, we obtain the following intersection numbers:

(i) $T \cdot \lambda = 8(a - 1)$.
(ii) $T \cdot \delta_{0;\{i,j,k\}} = a + 1$ for each triple $(i, j, k) \in \{1, 2\} \times \{3, 4\} \times \{5, 6\}$.
(iii) $T \cdot \delta_{0;\{2k-1,2k\}} = 4 \cdot (2a) = 8a$ for $k = 1, 2, 3$.

All intersections of $T$ with other boundary divisors of $\mathcal{M}_{1,6}$ are trivial. Plugging these numbers into the expression of $\pi^*(GP)$ in (5) yields:

$$T \cdot (\pi^*GP) = -16.$$

**Theorem 5.4.** The divisor $\overline{GP}$ is extremal in $\overline{Eff}(\mathcal{M}_{1,6})$. For $n \geq 6$, the pullback of $\overline{GP}$ to $\mathcal{M}_{1,n}$ spans an extremal ray of $\overline{Eff}(\mathcal{M}_{1,n})$, which is different from the rays spanned by $D_n$’s.

**Proof.** Since $T$ parameterizes curves in the main component $\overline{GP}$, we conclude that

$$T \cdot \overline{GP} \leq T \cdot (\pi^*GP) < 0.$$

Furthermore, the family $T$ is evidently moving in $\overline{GP}$, hence $\overline{GP}$ is an extremal divisor. Using the symmetry between the three pairs of marked points $\{p_{2k-1}, p_{2k}\}$ for $k = 1, 2, 3$, the same argument as in the proof of Theorem 5.3 shows the second part of the claim. \hfill\Box

**References**


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