EXTREMAL EFFECTIVE DIVISORS OF BRILL-NOETHER AND GIESEKER-PETRI TYPE IN $\overline{\mathcal{M}}_{1,n}$

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Abstract. We show that certain divisors of Brill-Noether and Gieseker-Petri type span extremal rays of the effective cone in the moduli space of stable genus one curves with n ordered marked points. In particular, they are different from the infinitely many extremal rays found in [CC].

Contents

1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford moduli space of stable genus g curves with n ordered marked points. Denote by $\overline{\text{Eff}}(\overline{\mathcal{M}}_{g,n})$ the cone of pseudoeffective divisors on $\overline{\mathcal{M}}_{g,n}$. Understanding the structure of $\overline{\text{Eff}}(\overline{\mathcal{M}}_{g,n})$ plays a central role in the study of the birational geometry of $\overline{\mathcal{M}}_{g,n}$, see e.g. [HMu, Ha, EH, F, Lo, V, CT].

In [CC, Theorem 1.1], it was shown that there exist infinitely many extremal effective divisors in $\overline{\mathcal{M}}_{1,n}$ for each $n \geq 3$. It provides the first (and the only) known example of $\overline{\mathcal{M}}_{g,n}$ whose pseudoeffective cone is *not* finitely generated. Recall the definition of those divisors. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a collection of n integers satisfying that $\sum_{i=1}^n a_i = 0$, not all equal to zero. Define $D_{\mathbf{a}}$ in $\overline{\mathcal{M}}_{1,n}$ as the closure of the divisorial locus parameterizing smooth genus one curves with n ordered marked points $(E; p_1, \ldots, p_n)$ such that $\sum_{i=1}^n a_i p_i \sim 0$ in E. For $n \geq 3$ and $\gcd(a_1, \ldots, a_n) = 1$, $D_{\mathbf{a}}$ spans an extremal ray of $\overline{\text{Eff}}(\overline{\mathcal{M}}_{1,n})$.

A natural question is *whether the $D_{\mathbf{a}}$ (and the boundary components) span all (rational) extremal rays of* $\overline{\text{Eff}}(\overline{\mathcal{M}}_{1,n})$. This is a meaningful question and one might expect an affirmative answer, as we explain in the following example. Consider the abelian surface $E \times E$, where E is a general smooth elliptic curve with p_0 as the origin. Take $a_1, a_2 \in \mathbb{Z}$ such that they are relatively prime. Consider the locus

$$C = \{(p_1, p_2) \in E \times E \mid a_1 p_1 + a_2 p_2 \sim (a_1 + a_2) p_0\}.$$

We know that C spans an extremal ray of $\overline{\text{Eff}}(E \times E)$. Moreover, all (rational) extremal rays of $\overline{\text{Eff}}(E \times E)$ are spanned by such C, see [K, II 4.16]. Note that C is an analogue of $D_{\mathbf{a}}$ with $\mathbf{a} = (a_1, a_2, -a_1 - a_2)$ when we fix the moduli of a genus one curve with three marked points.

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Nevertheless, the main result of this paper shows that the above question has a negative answer.

Theorem 1.1. *For every $n \geq 6$, there exist extremal effective divisors in $\overline{\mathcal{M}}_{1,n}$ that are different from the $D_{\mathbf{a}}$'s.*

See Theorems 3.3, 4.4 and 5.4 for a more precise statement.

Let us explain our method. For a stable genus one curve with $2m$ marked points $(E; p_1, \ldots, p_{2m})$, identify p_{2i-1} and p_{2i} as a node for $i = 1, \ldots, m$. We thus obtain an m-nodal curve of arithmetic genus $m+1$. It induces a gluing morphism $\pi : \overline{\mathcal{M}}_{1,2m} \to \overline{\mathcal{M}}_{m+1}$. We will show that pulling back certain divisors of Brill-Noether and Gieseker-Petri type by π gives rise to extremal effective divisors different from the $D_{\mathbf{a}}$'s. To verify extremality, we exhibit a moving curve in (the main component of) the pullback divisor such that the curve has negative intersection number with the divisor. This idea was also used in [O] to construct extremal effective divisors in $\overline{\mathcal{M}}_{0,n}$ that are different from the hypertree divisors in [CT].

The paper is organized as follows. In Section 2, we review basic divisor theory of $\overline{\mathcal{M}}_{g,n}$ and carry out the calculation of pulling back divisor classes under the gluing map $\pi : \overline{\mathcal{M}}_{1,2m} \to \overline{\mathcal{M}}_{m+1}$. In Section 3, we verify the extremality of the main component of the pullback of the Brill-Noether trigonal divisor. In Section 4, we study the pullback of Brill-Noether d-gonal divisors for general d and show that their main components are extremal. Finally, in Section 5 we study the pullback of the Gieseker-Petri divisor from $\overline{\mathcal{M}}_4$ and show that its main component is extremal.

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2. Preliminaries on moduli spaces of curves

Denote by λ the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$. Let Δ_{irr} be the locus in $\overline{\mathcal{M}}_{g,n}$ parameterizing curves with a non-separating node. For $0 \leq i \leq [g/2]$, $S \subset \{1, \ldots, n\}$ and $2i-2+|S| \geq 0$, let $\Delta_{i;S}$ denote the closure of the locus in $\overline{\mathcal{M}}_{g,n}$ that parameterizes nodal curves consisting of two components of genera i and $g-i$, respectively, where the genus i component contains the marked points labeled by S. Denote by $\delta_{\bullet}$ the divisor class of $\Delta_{\bullet}$ and let δ be the class of the union of all boundary divisors on $\overline{\mathcal{M}}_{g,n}$. Let ψ_i be the first Chern class of the cotangent line bundle on $\overline{\mathcal{M}}_{g,n}$ associated to the ith marked point for $1 \leq i \leq n$. These divisor classes are defined on the moduli stack instead of the coarse moduli scheme, see e.g. [AC, HMo] for more details.

In this paper, we focus on $\overline{\mathcal{M}}_{1,n}$ and $\overline{\mathcal{M}}_g$. The rational Picard group of $\overline{\mathcal{M}}_g$ is generated by $\lambda, \delta_{\mathrm{irr}}, \delta_1, \ldots, \delta_{[g/2]}$, and for $g \geq 3$ these divisor classes form a basis. The rational Picard group of $\overline{\mathcal{M}}_{1,n}$ has a basis given by λ and $\delta_{0;S}$ for $|S| \geq 2$. The divisor classes δ_{irr} and ψ_i on $\overline{\mathcal{M}}_{1,n}$ can be expressed as

$$\delta_{\mathrm{irr}} = 12\lambda, \tag{1}$$

$$\psi_i = \lambda + \sum_{i \in S} \delta_{0;S}. \tag{2}$$

Since δ_{irr} and λ are proportional on $\overline{\mathcal{M}}_{1,n}$, we will use them interchangeably throughout the paper.

For a stable genus one curve with $2m$ marked point $(E; p_1, \ldots, p_{2m})$, identify p_{2i-1} and p_{2i} as a node for $i = 1, \ldots, m$. We thus obtain a curve of arithmetic genus $m+1$ with m non-separating nodes. This induces a morphism

$$\pi : \overline{\mathcal{M}}_{1,2m} \to \overline{\mathcal{M}}_{m+1}.$$

The image of π is contained in Δ_{irr}.

Let us calculate the pullback of divisor classes from $\overline{\mathcal{M}}_{m+1}$ to $\overline{\mathcal{M}}_{1,2m}$ via π. For $1 \leq i \leq m$, define

$$\Lambda_i = \{S \subset \{1, \ldots, 2m\} \mid S = \{2k_1 - 1, 2k_1, \ldots, 2k_i - 1, 2k_i\}, 1 \leq k_1 < \cdots < k_i \leq m\}.$$

In other words, for $S \in \Lambda$, S contains the labeling of i pairs of marked points that are glued to i nodes. Denote by S^c the complement of S in $\{1, \ldots, n\}$.

Proposition 2.1. *Under the above setting, we have*

$$\pi^*\lambda = \lambda,$$

$$\pi^*\delta_i = \sum_{S\in\Lambda_i} \delta_{0;S} + \sum_{S^c\in\Lambda_{i-1}} \delta_{0;S}, \quad 1 \leq i < \frac{m+1}{2},$$

$$\pi^*\delta_i = \sum_{S\in\Lambda_i} \delta_{0;S}, \quad i = \frac{m+1}{2} \text{ for odd } m,$$

$$\pi^*\delta = (12 - 2m)\lambda - \sum_{|S|\geq 2} (|S| - 1)\delta_{0;S}.$$

Proof. We can calculate $\pi^*\lambda$ by induction on the number of pairs of points that are glued to a node. Take a family $f : \mathcal{C} \to B$ of stable curves of genus $g-1$ with two disjoint sections P_1 and P_2. Consider the exact sequence

$$0 \to \Omega \to \Omega(P_1 + P_2) \to \mathcal{O}_{P_1} \oplus \mathcal{O}_{P_2} \to 0,$$

where Ω is the relative dualizing sheaf of the family $f : \mathcal{C} \to B$. Applying f_*, we obtain that

$$0 \to f_*\Omega \to f_*(\Omega(P_1 + P_2)) \to \mathcal{O}_B \oplus \mathcal{O}_B \to \mathcal{O}_B \to 0.$$

It follows that

$$\pi^*\lambda = c_1(f_*(\Omega(P_1 + P_2))) = c_1(f_*\Omega) = \lambda.$$

If the image of $(E; p_1, \ldots, p_{2m}) \in \Delta_{0;S}$ under π lies in Δ_i, either the rational tail or the genus one tail becomes a component of arithmetic genus i after the gluing process, which corresponds to the case $S \in \Lambda_i$ or $S^c \in \Lambda_{i-1}$, respectively. Note that for $1 \leq i \leq \frac{m+1}{2}$, $S \in \Lambda_i$ and $S^c \in \Lambda_{i-1}$ hold simultaneously if and only if m is odd and $i = \frac{m+1}{2}$. The two equalities about $\pi^*\delta_i$ follow right away.

Finally, let $f : \mathcal{E} \to B$ be a family of stable genus one curves with $2m$ sections $P_1, \ldots, P_{2m}$. If a node is obtained by identifying P_{2k-1} and P_{2k}, its contribution in $\pi^*\delta$ is

$$f_*(P_{2k-1}^2) + f_*(P_{2k}^2) = -\psi_{2k-1} - \psi_{2k}.$$

If a fiber E in $\mathcal{E}$ is contained in $\delta_{0;S}$ or δ_{irr}, a node of E remains to be a node after the gluing process. This implies that

$$\pi^*\delta = \sum_{i=1}^{2m}(-\psi_i) + \delta = (12 - 2m)\lambda - \sum_{|S|\geq 2} (|S| - 1)\delta_{0;S},$$

where the relations (1) and (2) are used in the last step. □

3. Pulling back the trigonal divisor in $\overline{\mathcal{M}}_5$

Consider $\pi : \overline{\mathcal{M}}_{1,8} \to \overline{\mathcal{M}}_5$. Let $BN_3^1 \subset \overline{\mathcal{M}}_5$ denote the Brill-Noether trigonal divisor whose general point parameterizes a curve admitting a triple cover of $\mathbb{P}^1$. By [HMu], it has divisor class

$$\begin{aligned} BN_3^1 &= 8\lambda - \delta_{\text{irr}} - 4\delta_1 - 6\delta_2 \\ &= 8\lambda - \delta - 3\delta_1 - 5\delta_2. \end{aligned}$$

By Proposition 2.1, we obtain that

$$\pi^* BN_3^1 = 4\lambda + \sum_{S \not\in \cup \Lambda_i} (|S|-1)\delta_{0;S} + 4\delta_{0;\{1,\ldots,8\}} -2\sum_{k=1}^{4} \delta_{0;\{2k-1,2k\}} - 2\sum_{i<j} \delta_{0;\{2i-1,2i,2j-1,2j\}}. \tag{3}$$

Note that $\pi^{-1}BN_3^1$ may contain some boundary components of $\overline{\mathcal{M}}_{1,8}$, see Remark 3.4. Denote by $\widetilde{BN}_3^1$ the main component of $\pi^{-1}BN_3^1$, i.e. $\widetilde{BN}_3^1$ is actually the closure of $(\pi^{-1}BN_3^1) \cap \mathcal{M}_{1,8}$. (See Lemma 4.3 for the irreducibility of $(\pi^{-1}BN_d^1) \cap \mathcal{M}_{1,4d-4}$ for general d.)

Let us characterize when $(E; p_1, \ldots, p_8)$ is contained in $\widetilde{BN}_3^1$ for a smooth genus one curve E with eight distinct marked points.

Proposition 3.1. *In the above setting, $(E; p_1, \ldots, p_8)$ is a general point of $\widetilde{BN}_3^1$ if and only if E admits an embedding as a plane cubic in $\mathbb{P}^2$ such that the four lines $\overline{p_1p_2}$, $\overline{p_3p_4}$, $\overline{p_5p_6}$ and $\overline{p_7p_8}$ are concurrent.*

Proof. Identifying p_{2k-1}, p_{2k} in E for $k = 1, 2, 3, 4$, we obtain a curve C of arithmetic genus five with four nodes. Suppose that C is a trigonal curve. Then the canonical model of C is contained in a cubic surface in $\mathbb{P}^4$. Perform elementary transformations at the nodes of C and blow down the proper transform of the hyperplane section H spanned by the nodes. The cubic surface is transformed to $\mathbb{P}^2$ and the image of C is the embedding of E as a plane cubic in $\mathbb{P}^2$. The exceptional curve containing p_{2k-1}, p_{2k} is transformed to a line spanned by p_{2k-1}, p_{2k} for $k = 1, 2, 3, 4$. The four lines are concurrent at a point v arising from the contraction image of H.

Conversely, if $(E; p_1, \ldots, p_8)$ admits such a plane cubic configuration, projecting E from v to a line gives rise to a g_3^1 on E, which descends to a g_3^1 on the 4-nodal curve $\pi(E)$, because p_{2k-1} and p_{2k} map to the same image under the g_3^1 for $k = 1, 2, 3, 4$. □

Next we construct a curve B moving in $\widetilde{BN}_3^1$ such that $B \cdot \widetilde{BN}_3^1 < 0$. Fix four general concurrent lines $L_1, \ldots, L_4$ in $\mathbb{P}^2$ and fix two general points $p_{2k-1}, r_k \in L_k$ for each k. Consider the pencil B of plane cubics passing through the eight fixed points. Denote by p_{2k} the remaining intersection point of the cubics with L_k for each k. See Figure 1 for this configuration.

Marking $p_1, \ldots, p_8$, B can be viewed as a family of genus one curves with eight ordered marked points. By Proposition 3.1, B is a moving curve in $\widetilde{BN}_3^1$.

Lemma 3.2. *On $\overline{\mathcal{M}}_{1,8}$ we have the following intersection numbers:*

$$B \cdot \lambda = 1,$$

$$B \cdot \delta_{0;\{2k-1,2k\}} = 1,$$

$$B \cdot \delta_{0;\{2,4,6,8\}} = 1,$$

$$B \cdot \delta_{0;S} = 0, \quad S \neq \{2k-1, 2k\}, \{2,4,6,8\}.$$

Proof. Since p_{2k} varies in L_k, when it coincides with the fixed p_{2k-1}, we obtain a curve parameterized in $\delta_{0;\{2k-1,2k\}}$. There is no reducible cubic that can pass through the eight fixed points, hence every curve parameterized by B is an irreducible genus one curve. Moreover, since the locus of cuspidal cubics has codimension two in the total space of plane cubics, it implies that for a general configuration as in Figure 1, the pencil B does not contain cuspidal cubics. Finally, if the cubic passes through the common point v of the L_k, then p_2, p_4, p_6, p_8 coincide. Since they approach v in different directions, blowing up v results in a stable curve contained in $\delta_{0;\{2,4,6,8\}}$. The desired intersection numbers follow right away. □

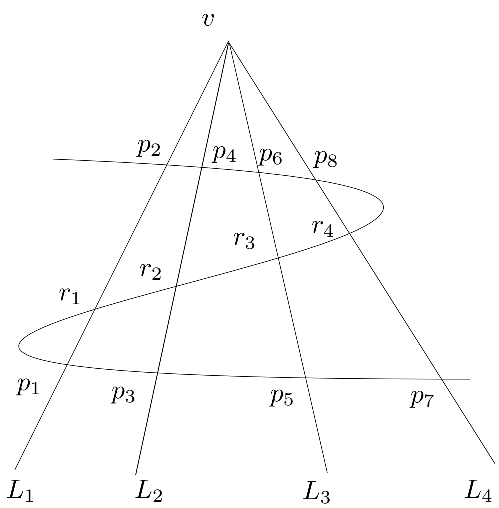

FIGURE 1. A plane cubic marked at $p_1, \ldots, p_8$

Now we can show that $\widetilde{BN}^1_3$ is an extremal effective divisor. Denote by $f : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,m}$ the forgetful morphism forgetting the last $n - m$ marked points.

Theorem 3.3. *The divisor $\widetilde{BN}^1_3$ is an extremal divisor on $\overline{\mathcal{M}}_{1,8}$. For every $n \geq 8$, $f^*\widetilde{BN}^1_8$ spans an extremal ray of $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n})$ that is different from the ray of $D_{\mathbf{a}}$.*

Proof. Write $\pi^* BN^1_3 = \widetilde{BN}^1_3 + \overline{BN}^1_3$, where $\overline{BN}^1_3$ is the union of boundary divisors contained in $\pi^{-1}BN^1_3$. Using Lemma 3.2 and (3), a routine calculation shows that $B \cdot \pi^* BN^1_3 = -1 < 0$. Since B is not entirely contained in the boundary of $\overline{\mathcal{M}}_{1,8}$, it implies that $B \cdot \overline{BN}^1_3 \geq 0$, hence $B \cdot \widetilde{BN}^1_3 < 0$. By [CC, Lemma 4.1], we know that $\widetilde{BN}^1_3$ is extremal and rigid on $\overline{\mathcal{M}}_{1,8}$.

For $n \geq 8$, take $n - 8$ general very ample divisors H_i on $\overline{\mathcal{M}}_{1,n}$ and use them to cut out a curve B' in $f^{-1}B$. Then the class of f_*B' is a positive multiple of B. By the projection formula, $B' \cdot f^*\widetilde{BN}^1_3 = (f_*B') \cdot \widetilde{BN}^1_3 < 0$. Moreover, varying B in $\widetilde{BN}^1_3$ and H_i in $\overline{\mathcal{M}}_{1,n}$, it follows that B' is a moving curve in $f^{-1}\widetilde{BN}^1_3$. Therefore, $f^*\widetilde{BN}^1_3$ is extremal and rigid on $\overline{\mathcal{M}}_{1,n}$.

Recall that the divisor $D_{\mathbf{a}} = D_{-\mathbf{a}}$ on $\overline{\mathcal{M}}_{1,n}$ parameterizes $(E; p_1, \ldots, p_n)$ where $\sum_{i=1}^n a_i p_i \sim 0$ in E, $\sum_{i=1}^n a_i = 0$ and $\gcd(a_1, \ldots, a_n) = 1$. Since $D_{\mathbf{a}}$ and $f^{-1}\widetilde{BN}^1_3$ are rigid, in order to prove that they span different extremal rays, it suffices to show that they have different supports. If $f^{-1}\widetilde{BN}^1_3$ and $D_{\mathbf{a}}$ are set-theoretically the same, then $a_i = 0$ for $i > 8$ since there is no constraint imposed to p_i for $i > 8$ in the definition of $\widetilde{BN}^1_3$. Moreover, by the symmetry between the four pairs of nodes $\{p_{2k-1}, p_{2k}\}$, we conclude that $\{a_{2k-1}, a_{2k}\} = \{c, c\}$ or $\{-c, -c\}$ for $1 \leq k \leq 4$. It follows that $c = 1$ and without loss of generality, say, $a_1 = a_2 = a_3 = a_4 = 1$, $a_5 = a_6 = a_7 = a_8 = -1$ (up to reordering the four nodes). Nevertheless, the resulting relation $p_1 + p_2 + p_3 + p_4 \sim p_5 + p_6 + p_7 + p_8$ is not invariant under the symmetry between $\{p_3, p_4\}$ and $\{p_5, p_6\}$, leading to a contradiction. □

Remark 3.4. We claim that $\overline{BN}^1_3$ is nonempty, i.e. $\pi^{-1}BN^1_3$ consists of the main component $\widetilde{BN}^1_3$ as well as some boundary divisors. For example, take a pencil of plane cubics and mark a base point as p_8. Attach it to $\mathbb{P}^1$ at another base point and mark seven general points in $\mathbb{P}^1$ as $p_1, \ldots, p_7$. We obtain a curve C moving in $\delta_{0;\{1,\ldots,7\}}$ with the following intersection numbers:

$$C \cdot \lambda = 1,$$

$$C \cdot \delta_{0;\{1,\ldots,7\}} = -1,$$
$$C \cdot \delta_{0;S} = 0, \quad S \neq \{1,\ldots,7\}.$$

It follows that $C \cdot \pi^* BN^1_3 < 0$, hence $\pi^* BN^1_3$ contains $\Delta_{0;S}$ for any $|S| = 7$. It would be interesting to calculate directly the class of the main component $\widetilde{BN}^1_3$.

4. Pulling back the d-gonal divisor in $\overline{\mathcal{M}}_{2d-1}$

Consider $\pi : \overline{\mathcal{M}}_{1,4d-4} \to \overline{\mathcal{M}}_g$, where $g = 2d-1$. Let $BN^1_d \subset \overline{\mathcal{M}}_g$ be the Brill-Noether divisor parameterizing d-gonal curves. By [HMu], it has class

$$\begin{aligned} BN^1_d &= c\left((2d+2)\lambda - \frac{d}{3}\delta_{\mathrm{irr}} - \sum_{i=1}^{d-1} i(2d-1-i)\delta_i\right) \\ &= c\left((2d+2)\lambda - \frac{d}{3}\delta - \sum_{i=1}^{d-1}\left(i(2d-1-i) - \frac{d}{3}\right)\delta_i\right), \end{aligned} \tag{4}$$

where $c = \frac{3(2d-4)!}{d!(d-2)!}$. In this section we will show that the main component of $\pi^* BN^1_d$ is extremal.

Consider the surface $S = E \times \mathbb{P}^1$, where E is a smooth genus one curve. Let π_0 and π_1 be the projections to $\mathbb{P}^1$ and E, respectively. We have

$$\mathrm{Pic}(S) \cong \mathbb{Z}[e] \oplus \pi_1^* \mathrm{Pic}(E),$$

where $e = \pi_0^* \mathcal{O}_{\mathbb{P}^1}(1)$ represents the genus one fiber class. Let D be a divisor of degree d on E. Projecting a curve in the linear system $|e + \pi_1^* D|$ via π_0, we obtain a degree d cover of $\mathbb{P}^1$. Take $2d-2$ general genus one fibers $E_1, \ldots, E_{2d-2}$ and fix a general point $p_{2k-1} \in E_k$ for $k = 1, \ldots, 2d-2$. Since $\dim |e + \pi_1^* D| = 2d-1$, we obtain a pencil of curves C_b in S that pass through all the fixed p_{2k-1}. Denote by p_{2k} an intersection point of C_b with E_i other than p_{2k-1}. Then $(C_b; p_1, \ldots, p_{4d-4})$ is a genus one curve with $4d-4$ marked points. Note that there are $d-1$ choices for each p_{2k}. After a base change of degree $(d-1)^{2d-2}$, we obtain a one-dimensional family B of curves in $\overline{\mathcal{M}}_{1,4d-4}$. Since p_{2k-1} and p_{2k} are both contained in E_k, projecting to $\mathbb{P}^1$ realizes $\pi(E, p_1, \ldots, p_{4d-4})$ as a d-gonal curve.

Note that C_b has the same j-invariant as that of E, because it admits a one-to-one map to E via π_1. It implies that

$$B \cdot \lambda = B \cdot \delta_{\mathrm{irr}} = 0.$$

Since $(e + \pi_1^* D)^2 = 2d$, besides $p_1, p_3, \ldots, p_{4d-5}$ there are two other base points q_1 and q_2 in the pencil. Hence it contains $2d$ singular curves before the base change, each of which consists of a genus zero fiber passing through $s = p_{2k-1}$ or $s = q_i$ union a (unique) genus one curve in $|e + \pi_1^*(D-s)|$ passing through the remaining base points.

Example 4.1. Consider $d = 3$ and $\pi : \overline{\mathcal{M}}_{1,8} \to \overline{\mathcal{M}}_5$. Make a base change of degree 2^4, so that we can distinguish the two remaining intersection points of E_k and C_b besides p_{2k-1} for $k = 1,2,3,4$. See Figure 2 for the configuration.

If p_{2k} coincides with p_{2k-1}, it contributes $2 \cdot 2^3 = 2^4$ to $B \cdot \delta_{0;\{2k-1,2k\}}$ (due to the base change). For a singular curve in B, if $s = q_i$ for $i = 1,2$, it contributes 1 to $\delta_{0;S}$ where $S \subset \{2,4,6,8\}$. If $s = p_{2k-1}$, then it contributes 2 to $\delta_{0;S\cup\{2k-1\}}$ for $S \subset \{2,4,6,8\}\backslash\{2k\}$, where the number 2 comes from the choice of p_{2k}.

We thus obtain that the only nonzero intersections of B with boundary divisors are as follows:

$$B \cdot \delta_{0;\{2k-1,2k\}} = 16,$$
$$B \cdot \delta_{0;S} = 2, \quad S \subset \{2,4,6,8\},$$
$$B \cdot \delta_{0;S\cup\{2k-1\}} = 2, \quad S \subset \{2,4,6,8\}\backslash\{2k\}.$$

It follows that

$$B \cdot \pi^* BN^1_3 = -128 + 2(1 \cdot 6 + 2 \cdot 4 + 3 \cdot 1) + 2 \cdot 4 \cdot (1 \cdot 3 + 2 \cdot 3 + 3 \cdot 1) = 2 > 0.$$

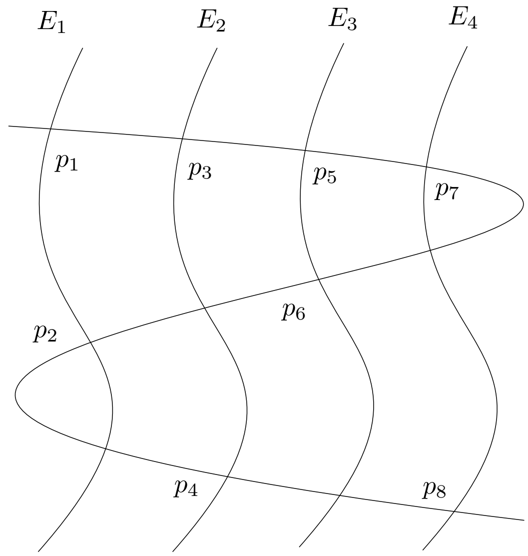

FIGURE 2. A genus one curve marked at $p_1, \ldots, p_8$

Nevertheless, for $d \geq 4$ the above intersection number turns out to be negative.

Lemma 4.2. *In the above setting, $B \cdot \pi^* BN^1_d < 0$ for $d \geq 4$.*

Proof. The calculation is similar to Example 4.1. Make a degree $(d-1)^{2d-2}$ base change, so that we can distinguish the marked point p_{2k} out of the remaining $d-1$ intersection points of E_k and C_b for $k = 1, \ldots, 2d-2$. If p_{2k} coincides with p_{2k-1}, it contributes $(d-1)(d-1)^{2d-3} = (d-1)^{2d-2}$ to $B \cdot \delta_{0;\{2k-1,2k\}}$. For a singular curve in B, if it passes through $s = q_i$ for $i = 1$ or 2, both of them contribute $(d-2)^{2d-2-|S|}$ to $\delta_{0;S}$ for $S \subset \{2, 4, \ldots, 4d-4\}$, due to the choice of the marked points in $\{2, 4, \ldots, 4d-4\}\backslash S$. If $s = p_{2k-1}$, it contributes $(d-1)(d-2)^{2d-3-|S|}$ to $\delta_{0;S\cup\{2k-1\}}$ for $S \subset \{2, 4, \ldots, 4d-4\}\backslash\{2k\}$, due to the choice of p_{2k} and the marked points in $\{2, 4, \ldots, 4d-4\}\backslash(\{2k\}\cup S)$.

We thus obtain that

$$B \cdot \delta_{0;\{2k-1,2k\}} = (d-1)^{2d-2},$$

$$B \cdot \delta_{0;S} = 2(d-2)^{2d-2-|S|}, \quad S \subset \{2, 4, \ldots, 4d-4\},$$

$$B \cdot \delta_{0;S\cup\{2k-1\}} = (d-1)(d-2)^{2d-3-|S|}, \quad S \subset \{2, 4, \ldots, 4d-4\}\backslash\{2k\}.$$

In $\pi^* BN^1_d / c$, the coefficients of $\delta_{0;\{2k-1,2k\}}$, of $\delta_{0;S}$ for $S \subset \{2, 4, \ldots, 4d-4\}$ and of $\delta_{0;S\cup\{2k-1\}}$ for $S \subset \{2, 4, \ldots, 4d-4\}\backslash\{2k\}$ are

$$2 - \frac{4}{3}d, \quad \frac{d}{3}(|S|-1), \quad \frac{d}{3}|S|,$$

respectively, by Proposition 2.1 and (4). It follows that

$$\begin{aligned}
\frac{1}{c}B\cdot\pi^*BN_d^1 &= \Big(2-\frac{4}{3}d\Big)2(d-1)^{2d-1}+\frac{d}{3}\sum_{s=1}^{2d-2}2(d-2)^{2d-2-s}(s-1)\binom{2d-2}{s}\\
&\quad +\frac{d}{3}(2d-2)\sum_{s=1}^{2d-3}(d-1)(d-2)^{2d-3-s}s\binom{2d-3}{s}\\
&= \Big(2-\frac{4}{3}d\Big)2(d-1)^{2d-1}+\frac{2d}{3}\Big((d-1)^{2d-2}+(d-2)^{2d-2}\Big)\\
&\quad +\frac{2d}{3}(2d-3)(d-1)^{2d-2}\\
&= \frac{2}{3}\Big(d(d-2)^{2d-2}-2(d-3)(d-1)^{2d-1}\Big).
\end{aligned}$$

It is easy to check that $B\cdot\pi^*BN_d^1<0$ for $d\geq 4$. □

Taking away possible boundary components, denote by $\widetilde{BN}_d^1$ the main component of $\pi^{-1}BN_d^1$.

Lemma 4.3. *The main component $\widetilde{BN}_d^1$ is irreducible.*

Proof. Let $U\subset\widetilde{BN}_d^1$ be the open dense subset parameterizing simply branched degree d, genus one covers $f: E\to\mathbb{P}^1$ with a general choice of m pairs of points $(p_1,p_2),\ldots,(p_{2m-1},p_{2m})$ in E, where $m=2d-2$ and $f(p_{2j-1})=f(p_{2j})=q_j\in\mathbb{P}^1$. It suffices to show that U is irreducible.

Let $b_1,\ldots,b_{2d}\in\mathbb{P}^1$ be the set of branch points of f. The branch data $\phi_1,\ldots,\phi_{2d}$ associated to the branch points can be arranged as

$$(1,2),(1,2),(1,2),(1,2),(2,3),(2,3),(3,4),(3,4),\ldots,(d-1,d),(d-1,d),$$

see e.g. [EEHS, page 100]. In other words, the monodromy induced by a closed, suitably oriented loop centering around b_i is the permutation $\phi_i\in S_d$.

The choice of p_{2j-1} and p_{2j} amounts to choosing two distinct numbers a_{2j-1} and a_{2j} out of $\{1,\ldots,d\}$, i.e. specifying two of the d sheets of f over q_j. Without loss of generality, assume that $a_{2j-1}<a_{2j}$. Vary q_j along the loops centering around the b_i with branch data $(a_{2j-1}-1,a_{2j-1}),(a_{2j-1}-2,a_{2j-1}-1),\ldots,(1,2)$ successively. When q_j comes back to the original position, a_{2j-1} is transformed to 1. Next, vary q_j along the loops centering around the b_i with branch data $(a_{2j}-1,a_{2j}),(a_{2j}-2,a_{2j}-1),\ldots,(2,3)$ successively. As a result, a_{2j} is transformed to 2. Finally, if we vary q_j along the loop around b_1 with branch datum $(1,2)$, the ordered pair $(1,2)$ is transformed to $(2,1)$. Meanwhile, the other pairs (a_{2k-1},a_{2k}) are unchanged since we did not vary q_k for $k\neq j$. Carrying out this process for $j=1,\ldots,m$ one by one, eventually all the pairs (a_{2j-1},a_{2j}) can be transformed to $(1,2)$.

Let W be the open dense subset of the Hurwitz space of degree d, genus one, simply branched covers of $\mathbb{P}^1$. The above process implies that the monodromy of $U\to W$ is transitive, where the map is finite of degree $(d(d-1))^m$ forgetting the p_i. Since W is irreducible, see [Hu, EEHS], it follows that U is irreducible. □

Theorem 4.4. *For $d\geq 4$, $\widetilde{BN}_d^1$ spans an extremal ray of $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,4d-4})$ and it is different from the ray of any $D_{\mathbf{a}}$.*

Proof. By Lemma 4.2, $B\cdot\pi^*BN_d^1<0$ and it is not entirely contained in the boundary of $\overline{\mathcal{M}}_{1,4d-4}$, hence $B\cdot\widetilde{BN}_d^1<0$. Moreover, by the construction of B, it is a moving curve in $\widetilde{BN}_d^1$. It follows that $\widetilde{BN}_d^1$ spans an extremal ray of $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,4d-4})$. The same argument as in the proof of Theorem 3.3 shows that $\widetilde{BN}_d^1$ and $D_{\mathbf{a}}$ span different extremal rays of $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,4d-4})$. □

5. Pulling back the Gieseker-Petri divisor in $\overline{\mathcal{M}}_4$

Consider the gluing map $\pi : \overline{\mathcal{M}}_{1,6} \to \overline{\mathcal{M}}_4$. Let GP denote the Gieseker-Petri divisor in $\overline{\mathcal{M}}_4$. The divisor GP has several geometric interpretations. Here we view it as the closure of the locus in $\overline{\mathcal{M}}_4$ parameterizing genus four curves whose canonical images are contained in a quadric cone in $\mathbb{P}^3$. By [EH, Theorem 2], it has class

$$\begin{aligned} GP &= 34\lambda - 4\delta_{\text{irr}} - 14\delta_1 - 18\delta_2 \\ &= 34\lambda - 4\delta - 10\delta_1 - 14\delta_2. \end{aligned}$$

By Proposition 2.1, we obtain:

$$\pi^* GP = 10\lambda + 4\sum_{S \notin \cup \Lambda_i} (|S|-1)\delta_{0;S} + 10\delta_{0;\{1,2,3,4,5,6\}} \tag{5}$$
$$-6\big(\delta_{0;\{1,2\}} + \delta_{0;\{3,4\}} + \delta_{0;\{5,6\}}\big) - 2\big(\delta_{0;\{1,2,3,4\}} + \delta_{0;\{1,2,5,6\}} + \delta_{0;\{3,4,5,6\}}\big).$$

In principle, $\pi^{-1}GP$ may contain some boundary components of $\overline{\mathcal{M}}_{1,6}$, as we saw in Remark 3.4. Denote by $\widetilde{GP}$ the main component of $\pi^{-1}GP$, i.e. $\widetilde{GP}$ is the closure of $(\pi^{-1}GP) \cap \mathcal{M}_{1,6}$.

The main result of this section is that $\widetilde{GP}$ spans an extremal ray of $\overline{\text{Eff}}(\overline{\mathcal{M}}_{1,6})$. We prove it by constructing a moving curve $B \subset \widetilde{GP}$ such that $B \cdot \widetilde{GP} < 0$.

We begin by observing that the general element $(E; p_1, ..., p_6) \in \widetilde{GP}$ is obtained as the (marked) normalization of a 3-nodal elliptic curve $C \subset \mathbb{F}_2$ where $\mathbb{F}_2$ is the second Hirzebruch surface. Let σ be the directrix class on $\mathbb{F}_2$ and let τ be the section class satisfying $\tau \cdot \sigma = 0$. The class of C is 3τ. We will essentially vary the 3-nodal curve C. However, it will also turn out to be necessary to "vary" the ambient surface $\mathbb{F}_2$. Let us first construct the variation of the ambient surface.

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$, with projections π_1 and π_2. Let f_1 and f_2 denote the classes of the fibers of π_1 and π_2, respectively. Let $\mathcal{O}_Q(m, n)$ denote the line bundle associated to the divisor class $mf_1 + nf_2$. Let $V = \mathcal{O}_Q \oplus \mathcal{O}_Q(1, 2)$ be a rank two vector bundle on Q. Consider the $\mathbb{P}^1$-bundle

$$Y = \text{Proj}(V),$$

with its projection $p : Y \to Q$. Let $\phi : Y \to \mathbb{P}^1$ be the composite $\pi_1 \circ p$. Then ϕ is a non-trivial family of $\mathbb{F}_2$'s. The threefold Y will essentially serve as the ambient space of the varying family of 3-nodal elliptic curves.

Let us understand the geometry of Y. The Picard group of Y is $\mathbb{Z}[\zeta, f_1, f_2]$, where here, and in everything that follows, we suppress pullbacks p^*'s from notation. The projection p has a distinguished section $\Sigma \subset Y$, which can be described as the union of the (-2)-curves in the $\mathbb{F}_2$'s. The class of Σ is

$$\Sigma = \zeta - f_1 - 2f_2.$$

We also select a disjoint section $\Pi \subset Y$ having divisor class

$$\Pi = \zeta.$$

In terms of the family $\phi : Y \to \mathbb{P}^1$, the section Π provides a family of sections complementary to the directrices of the varying $\mathbb{F}_2$ fibers. The projection p obviously restricts to an isomorphism on the section Π, so it makes sense to refer to the rulings of Π by f_1 and f_2 as well.

We will now blow up Y along the union of three disjoint curves. Restricted to each fiber $\mathbb{F}_2$, this amounts to blowing up three points corresponding to the nodes of the desired 3-nodal curve C.

Let Z_1, Z_2 and Z_3 be three disjoint curves of the ruling class f_2 on Π. Let $Z = Z_1 \cup Z_2 \cup Z_3$ and let $X = \text{Bl}_Z Y$ with the blow down map $\beta : X \to Y$. Denote by $\varphi : X \to \mathbb{P}^1$ the composite $\phi \circ \beta$. Then φ is a family of surfaces S_t, each being the blow up of $\mathbb{F}_2$ at three points $z_1(t), z_2(t)$ and $z_3(t)$ for $t \in \mathbb{P}^1$. The Picard group of X is $\mathbb{Z}[\zeta, f_1, f_2, e_1, e_2, e_3]$ where the e_i are the classes of the respective exceptional divisors E_i. Let $e = e_1 + e_2 + e_3$ and $E = E_1 \cup E_2 \cup E_3$. Since $N_{Z_i/Y}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$, each exceptional divisor E_i is isomorphic to the first Hirzebruch surface $\mathbb{F}_1$.

Now we consider the divisor class $3\zeta+(a-2)f_1-2e$ on X, where $a\gg 0$. Let l be the class of a line and let r be the class of a ruling on $E_i\cong\mathbb{F}_1$. Note that $\zeta|_{E_i}=f_1|_{E_i}=r$ and $e_i|_{E_i}=r-l$. Therefore,

$$\big(3\zeta+(a-2)f_1-2e\big)|_{E_i}=2l+(a-1)r.$$

Lemma 5.1. *Let D be a general divisor with divisor class $3\zeta+(a-2)f_1-2e$ on X, where $a\gg 0$.*

- (i) *D is a smooth surface in X. Under the map $\varphi: D\to\mathbb{P}^1$, the fibers have genus one and the singular fibers are at worst nodal.*
- (ii) *The curves $C_i=D\cap E_i$ provide smooth 2-sections to the family $\varphi: D\to\mathbb{P}^1$ for $i=1,2,3$. Moreover, the induced double cover $\varphi: C_i\to\mathbb{P}^1$ has $2a$ ramification points.*
- (iii) *There are exactly $(a+1)$ singular fibers in the family $\varphi: D\to\mathbb{P}^1$ having rational tails. Every rational tail meets C_i once for each i.*

Proof. Let S be a fiber of $\varphi: X\to\mathbb{P}^1$. So $S=\mathrm{Bl}_3\,\mathbb{F}_2$. We first show that the restriction map

$$\rho: H^0(X,\mathcal{O}_X(D))\to H^0(S,\mathcal{O}_S(D))$$

is surjective. By the long exact sequence, it is enough to show that

$$h^1(X,\mathcal{O}_X(D-S)=0.$$

By the Leray spectral sequence for φ, $h^1(X,\mathcal{O}_X(D-S))=0$ if

$$h^0(R^1\varphi_*(\mathcal{O}_X(D-S))=h^1(\varphi_*\mathcal{O}_X(D-S))=0.$$

The sheaf $R^1\varphi_*\mathcal{O}_X(D-S)$ vanishes, because the line bundle class $3\tau-2e$ on $S=\mathrm{Bl}_3\,\mathbb{F}_2$ has no higher cohomology. Furthermore, by push-pull, we have

$$\varphi_*\mathcal{O}_X(D-S)\cong\varphi_*(\mathcal{O}_X(3\zeta-2e))\otimes\mathcal{O}_{\mathbb{P}^1}(a-3),$$

which has vanishing higher cohomology for $a\gg 0$. Therefore, ρ is surjective. The divisor class $3\tau-2e$ on $\mathrm{Bl}_3\,\mathbb{F}_2$ is base point free. By adjunction, curves parameterized by the linear system $|3\tau-2e|$ have genus one. Furthermore, the locus of curves with worse than nodal singularities is of codimension 2 in $|3\tau-2e|$. Therefore, (i) follows from Bertini's theorem.

In exactly the same way, one can show that the restriction map $H^0(\mathcal{O}_X(D))\to H^0(\mathcal{O}_{E_i}(D))$ is surjective when $a\gg 0$. The adjunction formula applied to the curve class $2l+(a-1)r$ implies that C_i has genus $a-1$, thus proving (ii).

The proper transform Π' of Π in X is isomorphic to Π under the blow down map $\beta: X\to Y$. The intersection $D\cdot\Pi'$ has divisor class $(a+1)\widetilde{f_1}$ in Π', where $\widetilde{f_1}$ is the ruling class corresponding to f_1 under the isomorphism $\Pi'\cong\Pi$. It gives rise to $(a+1)$ disjoint curves that are precisely the rational tails in the family $\varphi: D\to\mathbb{P}^1$. Hence (iii) follows right away. □

Our next goal is to compute the degree of δ_{irr} restricted to $\varphi: D\to\mathbb{P}^1$, i.e. the number of nodal fibers in this family which do not have a rational tail. It is equal to 12λ by (1). Before doing so, we gather some relevant intersection products in the Chow ring of X.

Lemma 5.2. *The following intersection numbers hold in the Chow ring of X:*

$$f_i^2=(\zeta-f_1-2f_2)\cdot\zeta=0,$$
$$f_2\cdot e_i=(\zeta-f_1)\cdot e_i=0,$$
$$\zeta^3=4,\quad e_i^3=-1,$$
$$e_i^2\cdot f_1=e_i^2\cdot\zeta=-1,$$
$$\zeta^2\cdot f_1=2,\quad \zeta^2\cdot f_2=\zeta\cdot f_1\cdot f_2=1.$$

Furthermore, let l_i and r_i be the line and ruling classes of $E_i\cong\mathbb{F}_1$. Then $e_i^2=r_i-l_i$ and $\zeta\cdot e_i=f_1\cdot e_i=r_i$.

Proof. We will prove only those which are not immediately clear. First, we see that

$$\zeta^2 = (f_1 + 2f_2) \cdot \zeta$$

which follows from the general fact that

$$\zeta^r = \sum_{i=1}^{r} (-1)^{i+1} c_i(V) \cdot \zeta^{r-i}$$

for any projective bundle $\operatorname{Proj} V$, where V is a rank r vector bundle and ζ is the universal line bundle class of $\operatorname{Proj} V$. Here in our case ζ is the class of Π. From this, one easily derives the formula $\zeta^3 = 4$ as well as all other formulas involving only ζ, f_1 and f_2.

We now deal with the intersections involving the exceptional divisor classes e_i. If $W \subset G$ is a smooth subscheme of a variety G and if $\widetilde{G} = \operatorname{Bl}_W G$, then the exceptional divisor $E \subset \widetilde{G}$ is isomorphic to $\operatorname{Proj}(I_W/I_W^2)$, and $\mathcal{O}_E(E) = \mathcal{O}_E(-1)$ has class $r - l$. In our setting, the conormal bundle of Z_i in Y is $\mathcal{O}_{Z_i} \oplus \mathcal{O}_{Z_i}(-1)$. Therefore, E_i is abstractly isomorphic to $\mathbb{F}_1$, and $\mathcal{O}_{E_i}(1)$ is the line bundle associated to the directrix $\sigma \subset \mathbb{F}_1$. The class of the directrix is also given by $l_i - r_i$, and therefore $e_i^2 = r_i - l_i$. Furthermore, since $\zeta \cdot Z_i = f_1 \cdot Z_i = 1$ in Y, we conclude that $\zeta \cdot e_i = f_1 \cdot e_i = r_i$.

The remaining intersection products follow from those explained above. □

Lemma 5.3. *For the family $\varphi : D \to \mathbb{P}^1$, $\delta_{\mathrm{irr}} = 12\lambda = 12(a-1)$.*

Proof. First observe that $\lambda = \chi(\mathcal{O}_D)$. Secondly, by Noether's formula we know that

$$12\chi(\mathcal{O}_D) = K_D^2 + c_2(T_D).$$

Since the fibers of φ have genus one, the intersection number K_D^2 is $-(a+1)$, where each rational tail contributes -1.

So we need only to compute $c_2(T_D)$. Using the exact sequence

$$0 \to T_D \to T_X|_D \to N_{D/X} \to 0,$$

we see that

$$\begin{aligned} c_2(T_D) &= c_2(T_X|_D) - c_1(T_D) \cdot c_1(N_{D/X}) \\ &= c_2(T_X|_D) + K_D \cdot c_1(N_{D/X}) \\ &= c_2(T_X) \cdot D + (D + K_X) \cdot D^2. \end{aligned} \tag{6}$$

Equality (6) indicates that we need only to compute $c_2(T_X)$, and then we can proceed by using the intersection formulas provided in Lemma 5.2. Consider the sequence

$$0 \to T_X \to \beta^* T_Y \to T_{E/Z}(E) \to 0, \tag{7}$$

from which we conclude

$$c_2(T_X) = i_*[-\omega_{E/Z}] + E \cdot K_X + \beta^* c_2(T_Y) \tag{8}$$

where $i : E \to X$ is the inclusion map. For (8), we can use the Grothendieck-Riemann-Roch formula for the inclusion i to deduce

$$c_1(T_{E/Z}(E)) = E,$$
$$c_2(T_{E/Z}(E)) = i_*[c_1(\omega_{E/Z})] = r - 2l.$$

Using the exact sequence

$$0 \to \mathcal{O}_Y(\Sigma + \Pi) \to T_Y \to p^*(T_Q) \to 0,$$

we obtain

$$c_2(T_Y) = -K_Y \cdot (\Sigma + \Pi) + 4f_1 \cdot f_2. \tag{9}$$

Finally, we can compute the quantity $c_2(T_D)$ using the equality (6) and the intersection numbers given in Lemma 5.2. The end result is

$$c_2(T_D) = 13a - 11. \tag{10}$$

Therefore, $12\lambda = -(a+1) + 13a - 11 = 12a - 12$ as claimed. □

There are three types of singular fibers in the genus one family $\varphi : D \to \mathbb{P}^1$:

(i) Irreducible nodal fibers.
(ii) Fibers having a rational tail. The rational tail meets each 2-section C_i once.
(iii) Two $\mathbb{P}^1$'s joined at two nodes. One component will be the directrix $\sigma \subset \mathrm{Bl}_3 \mathbb{F}_2$. The other component will intersect each C_i twice.

In order to obtain a family of genus one curves with six ordered marked points, we need to perform a base change $T \to \mathbb{P}^1$ of degree $2^3 = 8$ to distinguish p_{2k-1} and p_{2k} that are glued as a node for $k = 1, 2, 3$. We name the sections so that the marked points $\{p_{2k-1}, p_{2k}\}$ correspond to the 2-section C_k. Then the curve T carries over it a family of genus one curves with six ordered marked points.

Using Lemmas 5.1, 5.3 and the above analysis of singular fibers of φ, we obtain the following intersection numbers:

(i) $T \cdot \lambda = 8(a-1)$.
(ii) $T \cdot \delta_{0;\{i,j,k\}} = a + 1$ for each triple $(i, j, k) \in \{1, 2\} \times \{3, 4\} \times \{5, 6\}$.
(iii) $T \cdot \delta_{0;\{2k-1,2k\}} = 4 \cdot (2a) = 8a$ for $k = 1, 2, 3$.

All intersections of T with other boundary divisors of $\overline{\mathcal{M}}_{1,6}$ are trivial. Plugging these numbers into the expression of $\pi^*(GP)$ in (5) yields:

$$T \cdot (\pi^* GP) = -16.$$

Theorem 5.4. *The divisor $\widetilde{GP}$ is extremal in $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,6})$. For $n \geq 6$, the pullback of $\widetilde{GP}$ to $\overline{\mathcal{M}}_{1,n}$ spans an extremal ray of $\overline{\mathrm{Eff}}(\overline{\mathcal{M}}_{1,n})$, which is different from the rays spanned by $D_{\mathbf{a}}$'s.*

Proof. Since T parameterizes curves in the main component $\widetilde{GP}$, we conclude that

$$T \cdot \widetilde{GP} \leq T \cdot (\pi^* GP) < 0.$$

Furthermore, the family T is evidently moving in $\widetilde{GP}$, hence $\widetilde{GP}$ is an extremal divisor. Using the symmetry between the three pairs of marked points $\{p_{2k-1}, p_{2k}\}$ for $k = 1, 2, 3$, the same argument as in the proof of Theorem 3.3 shows the second part of the claim. □

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