Covers of Elliptic Curves and Slopes of Effective Divisors on the Moduli Space of Curves

A dissertation presented

by

Dawei Chen

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

Harvard University
Cambridge, Massachusetts

May 2008

Dissertation Advisor: Professor Joe Harris — Dawei Chen

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Abstract

Consider genus g curves that admit degree d covers to elliptic curves only branched at one point with a fixed ramification type. The locus of such covers forms a one parameter family Y that naturally maps into the moduli space of stable genus g curves $\overline{\mathcal{M}}_g$. We study the geometry of Y, and produce a combinatorial method by which to investigate its slope, irreducible components, genus and orbifold points. Moreover, a correspondence between our method and the viewpoint of square-tiled surfaces is established. We also use our results to study the lower bound for slopes of effective divisors on $\overline{\mathcal{M}}_g$.

Contents

Acknowledgments

I am grateful to my advisor Joe Harris who told me about this problem and gave me many valuable suggestions. I would also like to thank Noam Elkies, Thomas Lam and Curtis T. McMullen for useful conversations about this work.

My sincere thanks are due to Joe who led me into the beautiful world of algebraic geometry. His geometric insight, patience with students, along with his passion towards teaching have deeply influenced me. Without his guidance, I could not have got through this Ph.D. program so smoothly. There are many other people in Joe's family, who helped me like elder brothers, and whom I thank: Izzet Coskun, Jason Starr, Ravi Vakil, Brendan Hassett, Mike Roth and Dan Abramovich.

My time at the Harvard Mathematics Department has been quite enjoyable. Many people have contributed, sometimes in an unexpected way, to make the cubicle life more fun. My mathematical growth benefited a lot from conversations with a number of talented people, including Melissa Liu, Jesse Kass, Kwun Kwai, Ethan Cotterill, Ivan Petrakiev, Deepak Khosla, Sabin Cautis, Maksym Fedorchuk, David Smyth, Frederick van der Wyck and Sebastian Casalaina-Martin, among others. The team of department preceptors, especially Robin Gottlieb and Matthew Leingang, have done a great job to improve my teaching techniques. I really appreciate their efforts. I also want to express my gratitude to the department staff, especially to Irene Minder and Robbie Miller, who have always been there to help me, whenever I have non-mathematical trouble.

Finally, I would like to thank my parents and grandma for supporting me all the time. Their love is always the origin of the spiritual power in my life.

1 Introduction

Let $\overline{\mathcal{M}}_g$ denote the Deligne-Mumford moduli space of genus g stable curves. Our aim is to study the effective cone of $\text{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$. Recall that $\text{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$ is generated by the Hodge class λ and boundary classes $\delta_i, i = 0, 1, \ldots, [\frac{g}{2}]$. The study of effective divisors on $\overline{\mathcal{M}}_g$ has a long history dating back to [HMu], [H] and [EH]. The most important result in this circle of ideas is the following.

Theorem 1.1. *$\overline{\mathcal{M}}_g$ is of general type when $g \geq 24$.*

The idea of the proof is to write the canonical divisor in the form $K_{\overline{\mathcal{M}}_g} = A + E$, where A and E are ample and effective $\mathbb{Q}$-divisors respectively. Then K is big and $\overline{\mathcal{M}}_g$ is of general type. In practice, we know that $K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta - \delta_1$ and $\lambda - \alpha\delta$ is ample for $0 < \alpha \ll 1$, where δ is the total boundary. So as long as we obtain an effective divisor $D = a\lambda - \sum_{i=0}^{[\frac{g}{2}]} b_i\delta_i, a, b_i > 0$ with $\frac{a}{b_i} < \frac{13}{2}$ for all i and $\frac{a}{b_1} < \frac{13}{3}$, then we can write $K_{\overline{\mathcal{M}}_g}$ in the desired form $A + E$. Define the number $\frac{a}{min\ \{b_i\}}$ to be the slope $s(D)$ of the divisor D. Then the remaining task is to find effective divisors with small slope.

In [EH], the famous Brill-Noether divisor was shown to do the trick. Recall that when the Brill-Noether number $\rho(g, r, d) = g - (r+1)(g-d+r)$ is equal to -1, the Brill-Noether divisor BN_d^r consists of genus g curves which possess a g_d^r. Amazingly, $s(BN_d^r) = 6 + \frac{12}{g+1}$, the so-called Brill-Noether bound, is independent of d, r, and is less than $\frac{13}{2}$ if $g \geq 24$. For almost 20 years after the publication of [EH], no one succeeded in finding any effective divisor of slope lower than the Brill-Noether bound. This led to the formulation, in [HM], of the following slope conjecture.

Conjecture 1.2. *The Brill-Noether bound $6 + \frac{12}{g+1}$ provides a lower bound for slopes of effective divisors on $\overline{\mathcal{M}}_g$. The bound is sharp if and only if $g + 1$ is* *composite.*

Unfortunately, the slope conjecture is false. In [FP], Farkas and Popa study genus 10 curves contained in $K3$ surfaces, or equivalently curves embedded in $\mathbb{P}^4$ by g^4_{12}'s whose images are contained in quadric threefolds. These curves sweep out a divisor $\mathcal{K}$ in $\overline{\mathcal{M}}_{10}$. Surprisingly, $s(\mathcal{K}) = 7$, which is less than the Brill-Noether bound $7\frac{1}{11}$. So $\mathcal{K}$ serves as a first counterexample to the slope conjecture. Recently, in [Kh], [F1], [F2] and [F3], more counterexamples are constructed, and in these papers Farkas also announces that $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type.

The above results only tell us one side of the story, where slopes of effective divisors are concerned. On the other side, to the best of the author's knowledge, all effective divisors that have been studied thus far have slope greater than 6. So in some sense, the slope conjecture may not be far from the truth. A natural question is to ask:

Question 1.3. *Is there a lower bound for slopes of effective divisors on $\overline{\mathcal{M}}_g$?*

A uniform lower bound ε for all g would enable us to deal with the Schottky problem, see, e.g., [F4, p. 8].

The idea to obtain a lower bound is to construct moving curves in $\overline{\mathcal{M}}_g$. An irreducible curve $C \in \overline{\mathcal{M}}_g$ is called a moving curve if the family of its deformations fills in a Zariski open set of $\overline{\mathcal{M}}_g$. We can define the slope of C by $s(C) = \dfrac{C.\,\delta}{C.\,\lambda}$. If an effective divisor D does not contain C, then $D.\,C \geq 0$ and $s(D) \geq s(C)$. Hence, the smallest slope of curves in this family bounds the slope of effective divisors, since a divisor cannot contain all the deformations of a moving curve! Along these lines, in [HM], Harris and Morrison study genus g curves as degree d covers of $\mathbb{P}^1$ for large d. If we vary the branch points in $\mathbb{P}^1$, we just obtain moving curves in $\overline{\mathcal{M}}_g$. The slopes of such moving curves do provide good lower bounds when g is small. However, as $g \to \infty$, these bounds tend to 0. Therefore, the existence of a uniform positive lower bound for the slope for all g remains unknown.

The same circle of ideas is mentioned in [CHS, 2.3]. Instead of using covers of $\mathbb{P}^1$, Coskun, Harris and Starr choose canonical curves in $\mathbb{P}^{g-1}$ to study. In order to get moving curves in $\overline{\mathcal{M}}_g$, they impose on canonical curves some conditions like contacting fixed linear subspaces with suitable orders. This method works pretty well for genus up to 6. But the general case is hard to analyze. On the one hand, we do not have good knowledge about the geometry of canonical curves. On the other hand, to compute the intersection number with δ, we need the Gromov-Witten theory for high genus nodal curves in projective spaces, which is far from complete.

Here we push the idea in [HM] further to consider covers of elliptic curves. Take a cover $\pi : C \to E$ that only has one branch point at the marked point p of the elliptic curve E. If we vary the complex structure of E, C also varies and its locus in $\overline{\mathcal{M}}_g$ forms a 1-dimensional subscheme Y. The geometry of Y is our main interest. We need to do three things as follows.

Firstly, we want to calculate the slope of Y. Here we use a very classical approach originally due to Hurwitz. He used symmetric groups to enumerate covers of $\mathbb{P}^1$. Similarly, the information of a degree d branched cover of an elliptic curve can be encoded in a certain equivalence class of solution pairs in S_d. We fix the ramification type of the cover by a conjugacy class $\sigma = (l_1)\cdots(l_m)$ in S_d, i.e., $\pi^{-1}(p) = \sum_{i=1}^{m} l_i q_i$. Let $(1^{a_1}2^{a_2}\ldots d^{a_d})$ also denote a conjugacy class of S_d whose number of length i cycles equals a_i, $\sum_{i=1}^{d} i a_i = d$. For two pairs $(\alpha, \beta), (\alpha', \beta') \in S_d \times S_d$, we define an equivalence relation by $(\alpha, \beta) \sim (\alpha', \beta')$ if there exists an element $\tau \in S_d$ such that $\tau\alpha\tau^{-1} = \alpha'$ and $\tau\beta\tau^{-1} = \beta'$. Moreover, let $\langle \alpha, \beta \rangle$ denote the subgroup of S_d generated by α and β. Define a set

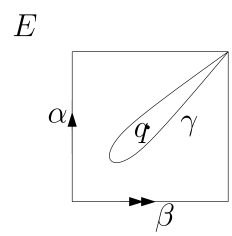
$$Cov^{g,d,\sigma} = \{(\alpha, \beta) \in S_d \times S_d \mid \alpha\beta\alpha^{-1}\beta^{-1} \in \sigma,$$

$$\langle\alpha, \beta\rangle \text{ is a transitive subgroup of } S_d\},$$

and its subset

$$Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\ldots d^{a_d}} = \{(\alpha, \beta) \in Cov^{g,d,\sigma} \mid \beta \in (1^{a_1}2^{a_2}\ldots d^{a_d})\}$$

based on the conjugacy type of β. Then the cover π corresponds to one equivalence class of solution pairs in $Cov^{g,d,\sigma}$. Readers should bear in mind that for this cover, α, β correspond to the monodromy images of a standard symplectic basis of $H_1(E)$. Pick a loop $\gamma = \alpha\beta\alpha^{-1}\beta^{-1}$ around the branched point q. The monodromy image of γ has to be an element in the ramification class σ. The transitivity condition imposed on $\langle\alpha, \beta\rangle$ is because we want the cover to be connected. See the following picture.



Let $N_{1^{a_1}2^{a_2}\ldots d^{a_d}}$ denote the order of the set $Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\ldots d^{a_d}}$ modulo the equivalence relation, i.e.,

$$N_{1^{a_1}2^{a_2}\ldots d^{a_d}} = |Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\ldots d^{a_d}} / \sim|.$$

When there is no ambiguity, we simply use Cov to denote $Cov^{g,d,\sigma}$. Let us further define a total sum

$$N = \sum_{1^{a_1}2^{a_2}\ldots d^{a_d}} N_{1^{a_1}2^{a_2}\ldots d^{a_d}}$$

and a weighted sum

$$M = \sum_{1^{a_1}2^{a_2}\ldots d^{a_d}} (\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_d}{d}) N_{1^{a_1}2^{a_2}\ldots d^{a_d}}.$$

It is clear that N and M only depend on g, d and σ. Now we can state the main result about the slope of Y.

Theorem 1.4. *In the above setting, we have the slope formula*

$$s(Y) = \frac{12M}{M + \left(d - \sum_{i=1}^{m} \frac{1}{l_i}\right) \frac{N}{12}}$$

Remark 1.5. Note that $s(Y)$ only depends on the ratio $\frac{N}{M}$.

Secondly, we have to study the irreducible components of Y. By using the symmetric group S_d again, we can see that two covers lie in the same component of Y if and only if the solution pair corresponding to one can be sent to the solution pair corresponding to the other via a monodromy action. More precisely, define two actions $a : (\alpha, \beta) \to (\alpha, \alpha\beta)$ and $b : (\alpha, \beta) \to (\alpha\beta, \beta)$ acting on the set Cov. It is not hard to check that both actions are well defined for $Cov/\sim$. Let G denote the group of actions generated by a, b and call G the monodromy group of Y. We give the following criterion.

Theorem 1.6. *Take two covers represented by two solution pairs (α, β) and (α', β') in $Cov/\sim$. Then the points in Y corresponding to these two covers are contained in the same component of Y if and only if there exists an element in the monodromy group G that sends (α, β) to (α', β').*

Finally, we have to verify whether or not Y is a moving curve in $\overline{\mathcal{M}}_g$. Readers may already be aware of the fact that Y is more likely rigid, since a cover corresponding to a point in Y only has one branched point, whose variation along the target elliptic curve does not change the moduli of this elliptic curve. Nevertheless, technically we do not need moving curves. Instead, when the degree d of the covers varies, as long as the union of all $Y_{g,d,\sigma}$ is Zariski dense in $\overline{\mathcal{M}}_g$, an effective divisor cannot contain all of them. So the smallest slope $s(Y)$ in this union still provides a lower bound for slopes of effective divisors. Luckily, for ramification class σ of suitable types, we have an affirmative answer to this density problem.

Theorem 1.7. *When g and $\sigma = (l_1)\cdots(l_m)$ are fixed but d varies, the image of the union of $Y_{g,d,\sigma}$ is dense in $\overline{\mathcal{M}}_g$ if and only if the number of ramification points is less than g, i.e., $|\{l_i : l_i \geq 2\}| < g$.*

We apply the above results to the case $g = 2$. About the slope, we have the following conclusion.

Theorem 1.8. *When $g = 2$, σ can be either of type $(3^1 1^{d-3})$ or of type $(2^2 1^{d-4})$. In both cases Y has constant slope $s(Y) = 10$ independent of d.*

Note that 10 is the sharp lower bound for slopes of effective divisors on $\overline{\mathcal{M}}_2$, cf. Remark 3.6. We will give a direct proof of Theorem 1.8 in section 4 but only for prime number d. The general case follows from an indirect argument in Remark 3.6.

We can further calculate the genus of Y. The following results are only for prime d.

Theorem 1.9. *When $g = 2$, d is prime and σ is of type $(3^1 1^{d-3})$,*

$$g(Y) = 1 + \frac{1}{8}(d-1)(d-2)(15d+23) - 6\Big(\sum_{\substack{a_1 l_1 + a_2 l_2 = d \\ l_1 > l_2}} (l_1, a_1)(l_2, a_2)\Big),$$

where (m, n) denotes the greatest common divisor of m and n. In particular, asymptotically $g(Y) \sim \frac{15}{8} d^3$.

Theorem 1.10. *When $g = 2$, d is prime and σ is of type $(2^2 1^{d-4})$,*

$$g(Y) = 1 + \frac{1}{12}(d-1)(d-3)(10d^2 - 13d - 14) - 6\Big(\sum_{\substack{a_1 l_1 + a_2 l_2 + a_3 l_3 = d \\ l_1 = l_2 + l_3 > l_2 > l_3}} \prod_{i=1}^{3} (l_i, a_i)$$

$$- \sum_{\substack{a_1 l_1 + a_2 l_2 = d \\ l_1 > l_2}} (l_1 - 2)(l_1, a_1)(l_2, a_2) - \sum_{2a_2 + a_1 = d} \frac{a_1 - 1}{(a_2, 2)} \Big).$$

In particular, asymptotically $g(Y) \sim \frac{5}{6} d^4$.

Theorems 1.9 and 1.10 will be proved in section 5 by considering local monodromy actions.

When g is 3, the combinatorics becomes harder. However, we check one case by computers: when σ is of type $(5^1 1^{d-5})$ and d is prime, it looks like that $s(Y)$ tends to 9, cf. Conjecture 3.8. Note that 9 is the sharp lower bound for slopes of effective divisors on $\overline{\mathcal{M}}_3$, cf. [HM, Cor. 3.30].

Unfortunately, although in principle by Theorems 1.4 and 1.6 we know almost everything about Y, in practice it seems that the combinatorial problems we encounter are quite complicated, especially when g is large. It would be very interesting to obtain more numerical evidences to see if this method does provide a lower bound for slopes of effective divisors.

This paper is organized in the following way. In section 2 we study the geometry of Y, namely we consider slope, monodromy and density. Then we apply the results to the genus 2 case and several examples of the genus 3 case in section 3. In section 4, a general method is introduced to express N and M in terms of some generating functions. Then we study the genus and orbifold points of Y in section 5. Finally in section 6, a correspondence between our viewpoint and the viewpoint of square-tiled surfaces is established. Throughout the paper, we work over the complex number field.

2 The Global Geometry of Y

In this section, we will focus on the slope, monodromy and density of Y and prove Theorems 1.4, 1.6 and 1.7 one by one. First, let us give a rigorous construction of Y.

Let $X \cong \mathbb{P}^1$ parameterize a general pencil of plane cubics. Blow up the 9 base points to obtain a smooth surface S that is an elliptic fibration over X. There are 12 rational nodal curves as special fibers over $b_1, \ldots, b_{12} \in X$. Fix a section Γ corresponding to the blow up of one of the 9 base points. With this section, X can be considered as a 12-sheeted cover of the moduli space of elliptic curves $\overline{\mathcal{M}}_{1,1}$. Take a general fiber (E, p) over $b \in X$. Consider all the possible degree d covers $\pi : C \to E$ from a genus g connected curve C to E branched only at p. Let $\pi^{-1}(p) = l_1 q_1 + \ldots + l_m q_m$, where $(l_1) \cdots (l_m)$ is a fixed conjugacy class σ of the symmetric group S_d. When b varies in $X^0 = X \backslash \{b_1, \ldots, b_{12}\}$, the locus of such covers also varies in a 1-dimensional Hurwitz scheme $Y^0_{g,d,\sigma}$. Now, if b approaches some point b_i, the cover degenerates to a cover of a rational nodal curve, in the sense of admissible covers. Hence, we can compactify $Y^0_{g,d,\sigma}$ by the space of admissible covers $Y_{g,d,\sigma}$. When there is no ambiguity, we will simply use Y^0 and Y instead.

The curve Y is an N-sheeted cover of X, possibly branched at $b_1, \ldots, b_{12}$. N is equal to the number of distinct genus g degree d covers of a general plane cubic only branched at one point of ramification type σ. Note that this number N equals the total sum

$$N = \sum_{1^{a_1} 2^{a_2} \ldots d^{a_d}} N_{1^{a_1} 2^{a_2} \ldots d^{a_d}}$$

defined in the introduction section.

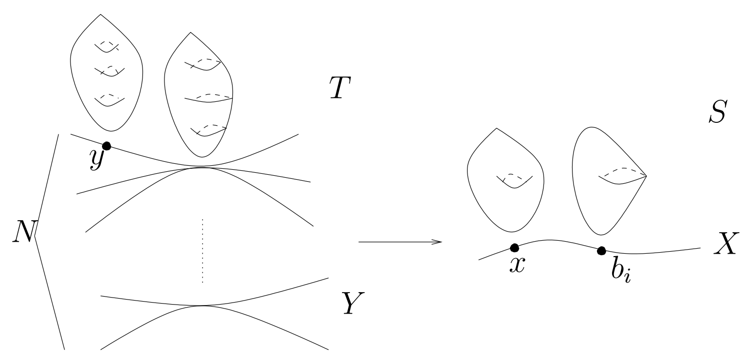
2.1 Slope

There is a natural map from the space of admissible covers Y to $\overline{\mathcal{M}}_g$. So it makes sense to talk about the intersections $Y.\delta$ and $Y.\lambda$ by pulling back the corresponding classes to Y. Now we can prove Theorem 1.4.

Proof. We have to figure out the two numbers $Y.\delta$ and $Y.\lambda$. The former can be worked out by the following argument. We want to establish a diagram of maps:

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\psi} & X \end{array}$$

such that over a point $y \in Y$ mapping to $x \in X$, the fiber over y is the corresponding cover of the fiber over x, i.e., T is the universal covering curve over Y. See the picture below.



When x is in X^0, locally we can always construct T. Now in a small neighborhood of b_i, we can identify all the smooth fibers to a fixed elliptic curve (E, p). Let α, β be a standard symplectic basis of $\pi_1(E, p)$ such that β is the vanishing cycle when the smooth elliptic curve degenerates to the rational nodal curve over b_i. We abuse notation and denote their monodromy images in S_d also by α, β, and assume $\beta \in (1^{a_1} 2^{a_2} \ldots d^{a_d})$. Now the key observation is that when the elliptic curve degenerates, the combinatorial type of the solution pair corresponding to the smooth cover determines the type of the degenerate cover. More precisely, the degenerate covering curve will have a_i nodes at which the map φ is given locally by $u \to x = u^i, v \to y = v^i$, where (u, v) and (x, y) are the local charts of T and S respectively. Each such node contributes $\frac{1}{i}$ to the intersection $Y.\, \delta$. Therefore, we have

$$Y.\, \delta = 12 \sum_{1^{a_1} 2^{a_2} \ldots d^{a_d}} (\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_d}{d}) N_{1^{a_1} 2^{a_2} \ldots d^{a_d}} = 12M.$$

The constant 12 comes from the number of singular fibers of S.

To calculate $Y.\, \lambda$, by Mumford's formula $\lambda = \dfrac{\delta + \kappa}{12}$, it suffices to work out the intersection $Y.\, \kappa = \omega_{T/Y}^2$. The ramification class σ is of type $(l_1) \ldots (l_m)$ so we have m sections $\Gamma_1, \ldots, \Gamma_m$ of T such that $\varphi^* \Gamma = \sum_{i=1}^{m} l_i \Gamma_i$. By Riemann-Hurwitz, the relative dualizing sheaves $\omega_{T/Y}$ and $\omega_{S/X}$ satisfy the relation

$$\omega_{T/Y} = \varphi^* \omega_{S/X} + \sum_{i=1}^{m} (l_i - 1) \Gamma_i.$$

Moreover,

$$\varphi_* \Gamma_i = N \cdot \Gamma, \ (\varphi^* \omega_{S/X})^2 = dN \cdot \omega_{S/X}^2 = 0,$$

and

$$\Gamma_i.\, \Gamma_j = 0, \ i \neq j.$$

So we get

$$l_i \Gamma_i^2 = \Gamma_i.\, (\varphi^* \Gamma) = (\varphi_* \Gamma_i).\, \Gamma = N(\Gamma^2) = -N,$$

and

$$\Gamma_i.\, (\varphi^* \omega_{S/X}) = (\varphi_* \Gamma_i).\, \omega_{S/X} = N(\Gamma.\, \omega_{S/X}) = N(-\Gamma^2) = N.$$

Now it is routine to check that

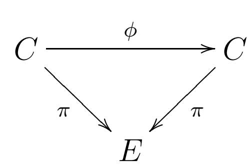
$$\omega_{T/Y}^2 = \left(\sum_{i=1}^{m} l_i - \sum_{i=1}^{m} \frac{1}{l_i} \right) N = \left(d - \sum_{i=1}^{m} \frac{1}{l_i} \right) N.$$

So we get the desired formula for $s(Y)$. $\square$

Remark 2.1. To complete the commutative diagram in the above proof, when the map is given locally by $u \to x = u^i, v \to y = v^i$ where (u, v) and (x, y) are the local charts of S and T respectively, a base change of degree divisible by i is necessary. So we can make a degree $d!$ base change once for all to realize such maps globally. After the base change, $Y.\,\delta$ and $Y.\,\lambda$ both have to be multiplied by $d!$, so the quotient $s(Y)$ remains the same.

A fancier interpretation is that by using a minimal base change, we pass from the space of admissible covers Y to the stack of admissible covers $\widetilde{Y}$ where a universal covering map lives. This viewpoint is crucial when we study the local geometry of Y in section 5.

Remark 2.2. Recall that in our original definition of a cover and its automorphism, we did not allow variation of the target elliptic curve. So an automorphism of a cover $\pi : C \to E$ is given by the following commutative diagram,



where ϕ is an automorphism of C.

However, the elliptic curve E always has an automorphism induced by its involution ι that sends (α, β) to $(\alpha^{-1}, \beta^{-1})$. If we further identify the two solution pairs $(\alpha, \beta) \sim (\alpha^{-1}, \beta^{-1})$, that is, we allow the following commutative diagram to be considered as an automorphism of the cover π,

$$\begin{array}{ccc} C & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\iota\circ\pi} & \\ E & \xrightarrow[\iota]{} & E \end{array}$$

we would get a new space Y'. Y can be viewed as a double cover of Y'. More precisely, for a component Z of Y, if a general cover in Z corresponding to a solution pair (α, β) does not have the automorphism induced by ι, i.e., there does not exist an element $\tau \in S_d$ such that $(\tau\alpha\tau^{-1}, \tau\beta\tau^{-1}) = (\alpha^{-1}, \beta^{-1})$, then Z is a double cover of the corresponding component Z' of Y'. On the other hand, if there is some $\tau \in S_d$ such that $(\tau\alpha\tau^{-1}, \tau\beta\tau^{-1}) = (\alpha^{-1}, \beta^{-1})$, then every cover in Z has the automorphism induced by ι. Hence, Z is a double curve and its reduced structure is the same as Z'. Note for the purpose of slope calculation that $s(Z)$ and $s(Z')$ are always the same.

2.2 Monodromy

The curve Y may be reducible. Actually if two pairs (α, β) and (α', β') generate non-conjugate subgroups of S_d, then the corresponding two covers must be contained in different components of Y.

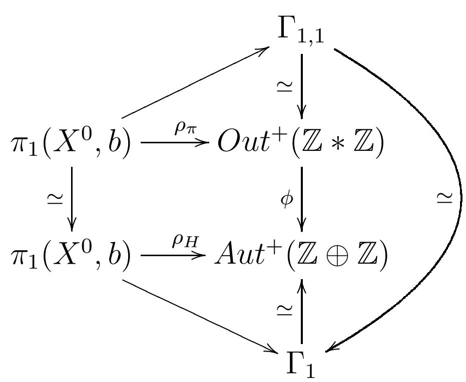
Now let us consider the monodromy of the map $Y \to X$. More precisely, we want to study the π_1-monodromy map $\rho_\pi : \pi_1(X^0, b) \to Out^+(\pi_1(X_b)) = Out^+(\mathbb{Z} * \mathbb{Z})$, where b is a fixed base point of X^0, X_b is the fiber over b with one marked point and $Out^+(\mathbb{Z} * \mathbb{Z}) = Aut(\mathbb{Z} * \mathbb{Z})/Inn(\mathbb{Z} * \mathbb{Z})$ is the orientable outer automorphism group.

Lemma 2.3. *The map ρ_π is surjective and its image $Out^+(\mathbb{Z} * \mathbb{Z}) \simeq SL_2(\mathbb{Z})$. Moreover, the group G of π_1-monodromy acting on $Cov/\sim$ can be generated by two operations: $a : (\alpha, \beta) \to (\alpha, \alpha\beta)$ and $b : (\alpha, \beta) \to (\alpha\beta, \beta)$.*

Proof. We give an indirect proof using the fact that the H_1-monodromy map ρ_H of a general pencil of plane cubics is surjective.

For H_1-monodromy, the marked point does not affect homology. So we consider the map $\rho_H : \pi_1(X^0, b) \to Out^+(H_1(X_b)) = Aut^+(\mathbb{Z} \oplus \mathbb{Z}) \simeq SL_2(\mathbb{Z})$. In order to show that ρ_H is surjective for a general pencil X, it suffices to exhibit a special pencil for which the claim holds. Actually some examples are explicitly worked out in [Sa] and the H_1-monodromy maps are surjective as we expect.

Now for π_1-monodromy, we can construct the following commutative diagram:



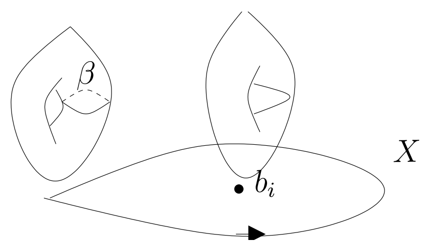
Here Γ_1 and $\Gamma_{1,1}$ are the mapping class groups for an ordinary torus and a torus with one marked point respectively. It is well known that Γ_1 and $Out^+(\mathbb{Z} * \mathbb{Z})$ are both isomorphic to $SL_2(\mathbb{Z})$. The map ϕ is induced by quotienting out commutators, i.e., $\mathbb{Z} * \mathbb{Z}/([\alpha, \beta]) \simeq \mathbb{Z} \oplus \mathbb{Z}$, where α and β are the generators of the free group $\mathbb{Z} * \mathbb{Z}$. Moreover, we have the well-known isomorphism $\Gamma_{1,1} \xrightarrow{\simeq} \Gamma_1$. So the conclusion that ρ_π is surjective follows from the fact that ρ_H is surjective.

Finally, $Out^+(\mathbb{Z} * \mathbb{Z})$ is generated by the two basic actions induced from Dehn twists along the two loops represented by α and β. So correspondingly the monodromy actions send (α, β) to $(\alpha, \alpha\beta)$ and $(\alpha\beta, \beta)$ respectively. □

Now Theorem 1.6 follows easily from the above lemma.

Remark 2.4. Since $(\alpha, \alpha\beta) \sim \alpha^{-1}(\alpha, \alpha\beta)\alpha = (\alpha, \beta\alpha)$, the monodromy actions are well defined up to the equivalence relation. The two actions a and b correspond to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ respectively, which generate $SL_2(\mathbb{Z})$.

Remark 2.5. Assume that β corresponds to the local vanishing cycle around a nodal fiber over $b_i \in X$. By the Picard-Lefschetz formula or Kodaira's classification of elliptic fibrations, going along a small loop around b_i once will send a solution pair (α, β) to $(\alpha\beta, \beta)$. See the following picture.



We call this action as the local monodromy action, in order to distinguish it from the global monodromy. It is useful when we study the genus of Y by applying the Riemann-Hurwitz formula to the map $Y \to X$, cf. section 5.

2.3 Density

In this section, whenever we mention density, we mean Zariski density. Notation is the same as before. Let σ be the conjugacy class $(l_1)\cdots(l_m)$ of S_d. We assume that $l_1, \ldots, l_k$ are greater than 1 and that the rest of the l_i's are all equal to 1. By Riemann-Hurwitz, $2g-2 = \sum_{i=1}^{k}(l_i - 1) = \left(\sum_{i=1}^{k} l_i\right) - k$. Now the pullback of a holomorphic 1-form from the target elliptic curve becomes a holomorphic 1-form ω on the covering curve C and $(\omega) = \sum_{i=1}^{k} \mu_i q_i$, where $\mu_i = l_i - 1$ and $\sum_{i=1}^{k} \mu_i = 2g-2$. So we can fix the ramification type by fixing $l_1, \ldots, l_k$ but adding more length-1 cycles to σ. Hence, the degree d of the map can vary from $2g-2+k$ to infinity.

Now we consider the union of all possible covers in the above sense. An equivalent statement of Theorem 1.7 is the following.

Theorem 2.6. *The image of* $\bigcup_{d=2g-2+k}^{\infty} Y^0_{g,d,\sigma}$ *is dense in* $\mathcal{M}_g$ *if and only if the inequality* $\sum_{i=1}^{k}(\mu_i - 1) \leq g-1$ *holds, i.e.,* $k \geq g-1$.

Proof. Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of $2g-2$. We consider the moduli space $\mathcal{H}(\mu)$ parameterizing $(C, \omega, q_1, \ldots, q_k)$ where $[C] \in \mathcal{M}_g$, $\omega \in H^0(K_C)$, and $(\omega) = \sum_{i=1}^{k} \mu_i q_i$. The dimension of $\mathcal{H}(\mu)$ is

$$3g - 3 + g - \left(\sum_{i=1}^{k}(\mu_i - 1)\right) = 2g - 1 + k = n.$$

The space $H(\mu)$ can locally be equipped with a coordinate system. Pick a basis $\gamma_1, \ldots, \gamma_n \in H_1(C, q_1, \ldots, q_k; \mathbb{Z})$ the relative homology of C with k marked points, such that $\gamma_1, \ldots, \gamma_{2g}$ are the standard symplectic basis of $H_1(C; \mathbb{Z})$ and γ_{2g+i} is a path connecting q_1 and q_{i+1}, $i = 1, \ldots, n-2g$. The period map $\Phi : (C, \omega) \to \mathbb{C}^n$ is given by

$$\Phi(C, \omega) = \Big(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_n} \omega\Big),$$

which provides a local coordinate system for $\mathcal{H}(\mu)$.

Now pick a torus T given by $\mathbb{C}/\Lambda$, where $\Lambda = \langle 1, \tau \rangle$ is a lattice. Consider the coordinate $\phi = \Phi(C, \omega) = (\phi_1, \ldots, \phi_n) \in \mathbb{C}^n$. We cite the following lemma from [EO, Lemma 3.1].

Lemma 2.7. $\phi_i \in \Lambda, i = 1, \ldots, 2g$ *if and only if the following conditions hold:*

(1) there exists a holomorphic map $f : C \to T$*;*

(2) $\omega = f^{-1}(dz)$*;*

(3) f *is ramified at* q_i *with ramification order* $\mu_i + 1 = l_i$, $i = 1, \ldots, k$*;*

(4) $f(q_{i+1}) - f(q_1) = \phi_{2g+i}$ *mod* Λ, $i = 1, \ldots, k-1$.

By virtue of this lemma, if we want to get a cover $C \to T$ branched at one point with ramification type $(l_1) \ldots (l_k)$, it is equivalent to $\phi_i = 0 \bmod \Lambda$, $i = 1, \ldots, n$.

Now we vary τ and such lattice points ϕ are dense in an open domain of $\mathbb{C}^n$. Let U be the union of such coverings $(C, \omega, q_1, \ldots, q_k)$, so U is dense in $\mathcal{H}(\mu)$. Now we have the following diagram:

$$\begin{array}{ccccc} U & \hookrightarrow & \mathcal{H}(\mu) & \hookrightarrow & \mathcal{H}_g \\ & \searrow & & & \downarrow \pi \\ & & & & \mathcal{M}_g \end{array}$$

where $\mathcal{H}_g$ is the Hodge bundle over $\mathcal{M}_g$. As long as $\mathcal{H}(\mu)$ dominates $\mathcal{M}_g$, i.e., for a general $[C] \in \mathcal{M}_g$ there exists $\omega \in H^0(K_C)$ such that $(\omega) = \sum_{i=1}^{k} \mu_i q_i$, then $\operatorname{Im} \bigcup_d^{\infty} Y^0_{g,d,\sigma} = \pi(U)$ must be dense in $\mathcal{M}_g$. Now we have to check that when $\sum_{i=1}^{k} (\mu_i - 1) \leq g - 1$, i.e., $k \geq g - 1$, $\mathcal{H}(\mu)$ does dominate $\mathcal{M}_g$.

The case $k > g - 1$ can be reduced to $k = g - 1$ since there is a natural stratification among all the moduli spaces $\mathcal{H}(\mu)$. When $k = g - 1$, we apply the De Jonquières' Formula from [ACGH, VIII §5]. Suppose that $a_1, \ldots, a_m$ are distinct integers and a_i appears n_i times in the partition μ of $2g - 2$, $\sum_{i=1}^{m} n_i = g - 1$ and $\sum_{i=1}^{m} n_i a_i = 2g - 2$. Further define $R(t) = 1 + \sum_{i=1}^{m} a_i^2 t_i$ and $P(t) = 1 + \sum_{i=1}^{m} a_i t_i$. Then on a genus g curve, the virtual number of canonical divisors having n_i points of multiplicity a_i is

$$\left[\frac{R(t)^g}{P(t)}\right]_{t_1^{n_1} \cdots t_m^{n_m}}.$$

As long as this number is nonzero, $\mathcal{H}(\mu)$ dominates $\mathcal{M}_g$.

Let $A = \sum_{i=1}^{m} a_i^2 t_i$ and $B = \sum_{i=1}^{m} a_i t_i$. Then

$$
\begin{aligned}
&\left[\frac{(1+A)^g}{1+B}\right]_{t_1^{n_1}\cdots t_m^{n_m}} \\
= &\left[\left(1+\binom{g}{1}A+\cdots+\binom{g}{g-1}A^{g-1}\right)\left(1-B+B^2-\cdots\right)\right]_{t_1^{n_1}\cdots t_m^{n_m}} \\
= &\left[\binom{g}{g-1}A^{g-1}-\binom{g}{g-2}A^{g-2}B+\cdots+(-B)^{g-1}\right]_{t_1^{n_1}\cdots t_m^{n_m}} \\
= &\left[\frac{A^g-(A-B)^g}{B}\right]_{t_1^{n_1}\cdots t_m^{n_m}} \\
= &\left[A^{g-1}+A^{g-2}(A-B)+\cdots+(A-B)^{g-1}\right]_{t_1^{n_1}\cdots t_m^{n_m}} > 0,
\end{aligned}
$$

since $A - B = \sum_{i=1}^{m}(a_i^2 - a_i)s_i$ has nonnegative coefficients and A has positive coefficients. □

Remark 2.8. The most general case is $k = 2g-2, \mu_i = 1, i = 1, \ldots, 2g-2$; then we want to analyze $\alpha\beta\alpha^{-1}\beta^{-1} \in \sigma = (2^{2g-2}1^{d-4g+4})$. Of course in this case the image of $\bigcup_{d=4g-4}^{\infty} Y^0_{g,d,\sigma}$ is dense in $\mathcal{M}_g$. The most special case is $k = 1, \mu_1 = 2g-2$; then correspondingly $\alpha\beta\alpha^{-1}\beta^{-1} \in \sigma = ((2g-1)^1 1^{d-2g+1})$. Obviously for dimension reasons, the image of $\bigcup_{d=2g-1}^{\infty} Y^0_{g,d,\sigma}$ has to be contained in a proper subvariety of $\mathcal{M}_g$.

Since the slope formula and the monodromy of Y are relatively simple to analyze when d is prime, from now on we will mainly focus on d prime for concrete examples. Therefore, the following result is also useful.

Claim 2.9. *The image of $\bigcup_{d \text{ prime}} Y^0_{g,d,\sigma}$ is dense in $\mathcal{M}_g$ if and only if the inequality $\sum_{i=1}^{k}(\mu_i - 1) \leq g-1$ holds, i.e., $k \geq g-1$.*

We illustrate the idea of the proof. Use the notation in the proof of Theorem 2.6 and write the coordinate $\phi_i = x_i + \sqrt{-1}y_i$. The degree d of the map from C to a standard torus T equals the area of C, which is given by

$$\frac{\sqrt{-1}}{2}\int_C \omega\wedge\overline{\omega} = \frac{\sqrt{-1}}{2}\sum_{i=1}^{g}(\phi_i\overline{\phi}_{g+i} - \overline{\phi}_i\phi_{g+i}) = \sum_{i=1}^{g}(x_i y_{g+i} - y_i x_{g+i}).$$

Now consider all the integer valued vectors $(x_1,\ldots,x_{2g},y_1,\ldots,y_{2g})$ such that $\sum_{i=1}^{g}(x_i y_{g+i} - y_i x_{g+i})$ is some prime number d. The above claim is equivalent to the density in $\mathbb{R}^{4g}$ of their union. Note for a fixed prime d that such integer points are always dense in the corresponding hypersurface. Since there are infinitely many such hypersurfaces, the result immediately follows.

3 Examples of Slope Calculation

In this section, we introduce one ad hoc method to calculate $N_{1^{a_1}2^{a_2}\dots d^{a_d}}$ for small g. For simplicity, we only deal with the case when d is prime. It will become clear that the method can also be used for general d but with more subtle analysis.

First, let us reduce the problem a little bit by getting rid of the equivalence relation. As introduced before, $Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}} = \{(\alpha,\beta)\in S_d\times S_d \mid \alpha\beta\alpha^{-1}\beta^{-1}\in\sigma,\ \langle\alpha,\beta\rangle \text{ is transitive},\ \beta\in(1^{a_1}2^{a_2}\dots d^{a_d})\}$. S_d acts on $Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}}$ by conjugation. Burnside's lemma tells us that

$$N_{1^{a_1}2^{a_2}\dots d^{a_d}} = \frac{1}{|S_d|}\sum_{\tau\in S_d}|Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}}(\tau)|,$$

where $Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}}(\tau) = \{(\alpha,\beta)\in Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}} \mid \tau\alpha=\alpha\tau, \tau\beta=\beta\tau\}$.

Lemma 3.1. *If τ commutes with all the elements in a transitive subgroup H of S_d, then τ must be of type (l^m) in S_d, $lm{=}d$.*

Proof. It suffices to show that for any $t\in\mathbb{Z}$, if τ^t fixes an element in $\{1,2,\dots,d\}$, then $\tau^t = id$.

Suppose $\tau^t(i)=i$. For any $j\neq i$, there exists $\xi\in H$, such that $\xi(i)=j$. But τ also commutes with ξ. By $\xi\tau^t\xi^{-1}=\tau^t$, we know that $\tau^t(j)=j$, so $\tau^t=id$. $\square$

When d is prime, the above τ must be either id or the long cycle class (d^1). Pick a long cycle $(12\cdots d)$, i.e., i is sent to $i+1$. By Burnside's lemma, we have

$$\begin{aligned} N_{1^{a_1}2^{a_2}\dots d^{a_d}} &= \frac{1}{d!}\Big(|Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}}| + (d-1)!|Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}}(12\cdots d)|\Big) \\ &= \frac{|Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}}|}{d!} + \frac{|Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}}(12\cdots d)|}{d}. \end{aligned}$$

Lemma 3.2. $Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\dots d^{a_d}}(12\cdots d)=\emptyset$.

Proof. We know that $(12\cdots d) = \alpha(12\cdots d)\alpha^{-1} = (\alpha(1)\alpha(2)\cdots\alpha(d))$, so $\alpha(i) = (i+s), i = 1, 2, \ldots, d$, where s is an integer independent of i. Similarly $\beta(j) = (j+t), j = 1, 2, \ldots, d$, and t is independent of j. Now it is easy to check that $\alpha\beta\alpha^{-1}\beta^{-1} = id$, so there is no solution pair in $Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\ldots d^{a_d}}(12\cdots d)$. □

Hence, finally we obtain the relation:

$$N_{1^{a_1}2^{a_2}\ldots d^{a_d}} = \frac{|Cov^{g,d,\sigma}_{1^{a_1}2^{a_2}\ldots d^{a_d}}|}{d!}.$$

3.1 $g = 2, \sigma = (3^1 1^{d-3})$

We want to find solution pairs $(\alpha, \beta) \in S_d \times S_d$ such that $\alpha\beta\alpha^{-1}\beta^{-1} \in (3^1 1^{d-3})$ and $\langle\alpha, \beta\rangle$ is transitive. Let $\gamma = (abc) \in (3^1 1^{d-3})$ satisfying the equality $\alpha\beta\alpha^{-1} = \gamma\beta$.

Now the key point is, $\alpha\beta\alpha^{-1}$ and β are in the same conjugacy class, so $\gamma\beta \sim \beta$. Note that a, b, c cannot be contained in three different cycles of β, since $(abc)(a\cdots)(b\cdots)(c\cdots) = (a\cdots b\cdots c\cdots)$ changes the conjugacy type of β. So only two cases are possible:

(1) $(abc)(a\cdots b\cdots c\cdots) = (a\cdots c\cdots b\cdots)$;

(2) $(abc)(a\cdots b\cdots)(c\cdots) = (a\cdots c\cdots)(b\cdots)$.

Using also the condition that $<\alpha, \beta>$ is transitive, in case (1) β must be of type (l^m), so β can only be the long cycle (d^1). Pick an element $\beta = (12\ldots d)$. There are $\binom{d}{3}$ choices for the cycle (abc); fix one. Now for α, we know that $(\alpha(1)\alpha(2)\cdots\alpha(d)) = \alpha(12\cdots d)\alpha^{-1} = (abc)(a\cdots b\cdots c\cdots) = (a\cdots c\cdots b\cdots)$, so there are d choices for α. Overall, we get

$$N_{d^1} = \frac{1}{d!}(d-1)!\binom{d}{3}d = \binom{d}{3}.$$

For case (2), we have $(abc)(\underbrace{a\cdots \overbrace{b\cdots}^{l_2}}_{l_1})(\underbrace{c\cdots}_{l_2}) = (a\cdots c\cdots)(b\cdots)$, so β is of type $(l_1^{a_1} l_2^{a_2})$, $l_1 > l_2, a_1 l_1 + a_2 l_2 = d$. There are $\dfrac{d!}{\prod_{i=1}^{2} (l_i^{a_i})(a_i!)}$ choices for β. Fix one choice; then there are $a_1 l_1 a_2 l_2$ choices for the cycle (abc). Fix one of these also; then because of the transitivity requirement, there are $\prod_{i=1}^{2} (l_i^{a_i})(a_i - 1)!$ choices for α. Multiplying all the numbers together and dividing by $d!$, we get

$$N_{l_1^{a_1} l_2^{a_2}} = l_1 l_2.$$

Therefore, we have

$$N = \binom{d}{3} + \sum_{\substack{a_1 l_1 + a_2 l_2 = d \\ l_1 > l_2}} l_1 l_2,$$

and

$$M = \frac{1}{d}\binom{d}{3} + \sum_{\substack{a_1 l_1 + a_2 l_2 = d \\ l_1 > l_2}} \left(\frac{a_1}{l_1} + \frac{a_2}{l_2}\right) l_1 l_2.$$

Theorem 3.3. *When d is prime and σ is of type $(3^1 1^{d-3})$, the slope $s(Y_{2,d,\sigma}) = 10$.*

Proof. By the slope formula, in this case

$$s(Y) = \frac{12M}{M + \frac{2}{9}N} = 10 \Longleftrightarrow \frac{N}{M} = \frac{9}{10}.$$

Actually when d is prime, we can verify that

$$N = \frac{3}{8}(d-2)(d-1)(d+1) \tag{1}$$

and

$$M = \frac{5}{12}(d-2)(d-1)(d+1). \tag{2}$$

□

To prove the above two equalities, we introduce the following functions

$$\sigma_i(n) = \sum_{k|n} k^i, i = 1, 2, \cdots .$$

These summations are quasi-modular forms with certain weights. Now define three series

$$P = 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k},$$

$$Q = 1 + 240\sum_{k=1}^{\infty} \frac{k^3q^k}{1-q^k},$$

and

$$R = 1 - 504\sum_{k=1}^{\infty} \frac{k^5q^k}{1-q^k}.$$

There are some fundamental relations among P, Q and R – the Ramanujan differential equations, cf. [BY]:

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12},\ q\frac{dQ}{dq} = \frac{PQ - R}{3},\ q\frac{dR}{dq} = \frac{PR - Q^2}{2}.$$

Lemma 3.4.

$$\sum_{k=1}^{d-1} \sigma_1(k)\sigma_1(d-k) = (\frac{1}{12} - \frac{d}{2})\sigma_1(d) + \frac{5}{12}\sigma_3(d).$$

Proof. Note that

$$\sum_{k=1}^{d-1} \sigma_1(k)\sigma_1(d-k) = \left[\Big(\sum_{k=1}^{\infty} \sigma_1(k)q^k\Big)^2\right]_d,$$

where $[\cdot]_d$ means the coefficient of the degree d term in the series expansion, and

$$\sum_{k=1}^{\infty} \sigma_1(k)q^k = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} kq^{kj} = \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}.$$

Similarly, we have

$$\sum_{k=1}^{\infty} \sigma_3(k) q^k = \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k}.$$

Hence, we get

$$\left[(\sum_{k=1}^{\infty} \sigma_1(k) q^k)^2 \right]_d = \left[(\frac{1-P}{24})^2 \right]_d = \left[\frac{1}{24^2} - \frac{1}{24 \cdot 12} + \frac{1}{24^2}(Q + 12q\frac{dP}{dq}) \right]_d$$

$$= (\frac{1}{12} - \frac{d}{2})\sigma_1(d) + \frac{5}{12}\sigma_3(d).$$

□

When d is prime, the right side of the last equality equals $\frac{1}{12}(d-1)(d+1)(5d-6)$. Moreover,

$$\sum_{\substack{a_1 l_1 + a_2 l_2 = d, \\ l_1 > l_2}} l_1 l_2 = \frac{1}{2} \left(\sum_{k=1}^{d-1} \sigma_1(k)\sigma_1(d-k) - (d-1) \right)$$

and

$$\begin{aligned} \sum_{\substack{a_1 l_1 + a_2 l_2 = d, \\ l_1 > l_2}} (\frac{a_1}{l_1} + \frac{a_2}{l_2}) l_1 l_2 &= \frac{1}{2} \left(\Big(\sum_{a_1 l_1 + a_2 l_2 = d} a_1 l_2 + a_2 l_1 \Big) - d(d-1) \right) \\ &= \Big(\sum_{a_1 l_1 + a_2 l_2 = d} l_1 l_2 \Big) - \frac{1}{2} d(d-1). \end{aligned}$$

Now the equalities (1), (2) for N and M follow immediately.

3.2 $g = 2, \sigma = (2^2 1^{d-4})$

We analyze this case similarly. The condition on α, β is $\alpha\beta\alpha^{-1} = (ab)(ce)\beta$. Now there are 4 possible cases:

(1) $(ab)(ce)(a \cdots c \cdots b \cdots e \cdots) = (a \cdots e \cdots b \cdots c \cdots)$;

(2) $(ab)(ce)(a \cdots b \cdots)(c \ldots)(e \cdots) = (c \cdots e \cdots)(a \cdots)(b \cdots)$;

(3) $(ab)(ce)(a \cdots c \cdots)(b \cdots e \cdots) = (c \cdots b \cdots)(e \cdots a \cdots)$;

(4) $(ab)(ce)(a\cdots b\cdots c\cdots)(e\cdots) = (b\cdots e\cdots c\cdots)(a\cdots)$.

For case (1), β is of type (l^m). Since d is prime, β must be the long cycle (d^1), so there are $(d-1)!$ choices for β. Pick one; then there are $\binom{d}{4}$ choices for the cycle $(ab)(ce)$. Fix one of these also; then there are d choices for α. Overall, we get

$$N_{d^1} = \binom{d}{4}.$$

For case (2), β is of type $(l_1^{a_1} l_2^{a_2} l_3^{a_3})$, where $l_1 = l_2 + l_3 > l_2 \geq l_3$.

(i) $l_2 > l_3$. There are $\dfrac{d!}{\prod_{i=1}^{3}(l_i^{a_i})(a_i!)}$ choices for β. Fix one of these; we then have $\prod_{i=1}^{3} l_i a_i$ choices for the cycle $(ab)(ce)$. Fix one of these too. To keep the transitivity of $\langle \alpha, \beta \rangle$, there are $\prod_{i=1}^{3}(l_i^{a_i})(a_i - 1)!$ choices for α. Hence, in this case we get

$$N_{l_1^{a_1} l_2^{a_2} l_3^{a_3}} = l_1 l_2 l_3.$$

(ii) $l_2 = l_3$. Since d is prime and $a_1(l_2 + l_3) + a_2 l_2 + a_3 l_3 = d$, we get $l_2 = l_3 = 1$ and $l_1 = 2$. So β is of type $(2^{a_2} 1^{a_1})$ and there are $\dfrac{d!}{2^{a_2} a_2! a_1!}$ choices. Fix β; $(ab)(ce)(ab)(c)(e) = (ce)(a)(b)$, so there are $a_2 \binom{a_1}{2}$ choices for the cycle $(ab)(ce)$. Fixing one of them, there are $2^{a_2+1}(a_2 - 1)!(a_1 - 1)!$ choices for α. Overall, we get

$$N_{2^{a_2} 1^{a_1}} = a_1 - 1.$$

For case (3),

$$(ab)(ce)(\underbrace{\overbrace{a\cdots}^{p_1}\overbrace{c\cdots}^{q_1}}_{l_1})(\underbrace{\overbrace{b\cdots}^{p_2}\overbrace{e\cdots}^{q_2}}_{l_2}) = (c\cdots b\cdots)(e\cdots a\cdots).$$

$l_1 > l_2 > 1, l_i = p_i + q_i$ and either $q_1 = q_2$ or $p_1 = p_2$. β is of type $(l_1^{a_1} l_2^{a_2})$: there are $\dfrac{d!}{\prod_{i=1}^{2}(l_i^{a_i})(a_i!)}$ choices. Pick one β and assume $p = p_1 = p_2$. Since d is prime, $q_1 \neq q_2$. The condition on p is that $1 \leq p \leq l_2 - 1$ so there are $l_2 - 1$ choices for p. Fix p; then there are $l_1 l_2 a_1 a_2$ choices for the cycle $(ab)(ce)$. Fixing one of these also, finally there are $\prod_{i=1}^{2}(l_i^{a_i})(a_i - 1)!$ choices for α . In total, we get $l_1 l_2 (l_2 - 1)(d!)$.

For case (4),

$$(ab)(ce)(\underbrace{\overbrace{a \cdots b}^{l_2} \cdots c \cdots}_{l_1})(\underbrace{e \cdots}_{l_2}) = (a \cdots)(b \cdots e \cdots c \cdots),$$

where $l_1 \geq l_2 + 2$. β has to be of type $(l_1^{a_1} l_2^{a_2})$ as in case (3). So there are also $\dfrac{d!}{\prod_{i=1}^{2}(l_i^{a_i})(a_i!)}$ choices for β. Fix one β; then there are $a_1 a_2 l_1 l_2 (l_1 - l_2 - 1)$ choices for the cycle $(ab)(ce)$. Pick one; then there are $\prod_{i=1}^{2}(l_i^{a_i})(a_i - 1)!$ choices for α. Overall, we get $l_1 l_2 (l_1 - l_2 - 1)(d!)$.

Now combining cases (3) and (4), finally we have

$$N_{l_1^{a_1} l_2^{a_2}} = l_1 l_2 (l_1 - 2).$$

Therefore, we have

$$N = \binom{d}{4} + \sum_{\substack{a_1 l_1 + a_2 l_2 + a_3 l_3 = d \\ l_1 = l_2 + l_3 > l_2 > l_3}} l_1 l_2 l_3$$

$$+ \sum_{\substack{a_1 l_1 + a_2 l_2 = d \\ l_1 > l_2}} l_1 l_2 (l_1 - 2) + \sum_{2a_2 + a_1 = d} (a_1 - 1),$$

and

$$M = \frac{1}{d}\binom{d}{4} + \sum_{\substack{a_1l_1+a_2l_2+a_3l_3=d \\ l_1=l_2+l_3>l_2>l_3}} (\frac{a_1}{l_1} + \frac{a_2}{l_2} + \frac{a_3}{l_3})l_1l_2l_3$$

$$+ \sum_{\substack{a_1l_1+a_2l_2=d \\ l_1>l_2}} (\frac{a_1}{l_1} + \frac{a_2}{l_2})l_1l_2(l_1-2) + \sum_{2a_2+a_1=d} (\frac{a_2}{2} + a_1)(a_1-1).$$

Theorem 3.5. *When d is prime and σ is of type $(2^2 1^{d-4})$, the slope $s(Y_{2,d,\sigma}) = 10$.*

Proof. By the slope formula, we know that

$$s(Y) = \frac{12M}{M + \frac{N}{4}} = 10 \Longleftrightarrow \frac{N}{M} = \frac{4}{5}.$$

Actually when d is prime, as in the proof of Theorem 3.3, one can similarly check that

$$N = \frac{1}{6}(d-3)(d-2)(d-1)(d+1) \tag{3}$$

and

$$M = \frac{5}{24}(d-3)(d-2)(d-1)(d+1). \tag{4}$$

Then we are done. □

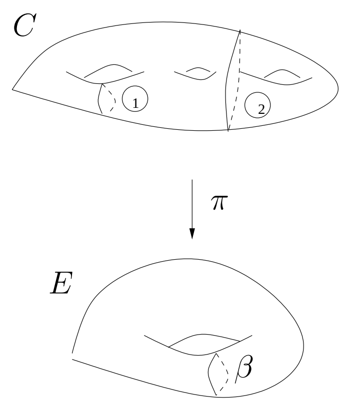
Combining Theorems 3.3 and 3.5, we have finished the proof of Theorem 1.8 for the case d prime. The general case follows from the remark below.

Remark 3.6. $\overline{\mathcal{M}}_2$ is special in that the following equality holds:

$$\lambda = \frac{\delta_0}{10} + \frac{\delta_1}{5}.$$

Hence, for a curve B in $\overline{\mathcal{M}}_2$ not entirely contained in the boundary, we always have that its slope $s(B) \leq 10$. Moreover, $s(B) = 10$ if and only if $B.\,\delta_1 = 0$.

There is an easy but indirect way to show that in general $Y_{g,d,\sigma}.\,\delta_i = 0, i > 0$, which was pointed out to the author by Curtis T. McMullen. The idea is that for a cover $\pi : C \to E$, when the vanishing cycle β shrinks to a node, any component γ of $\pi^{-1}(\beta)$ is not vanishing in $H_1(C)$ since β is not a zero cycle. Then when γ shrinks, it will form an internal node, i.e., the degenerate covering curve only lies in Δ_0, but not in $\Delta_i, i > 0$. As in the following picture, loop (1) may belong to $\pi^{-1}(\beta)$, but loop (2) cannot.



As $g = 2$, $Y_{2,d,\sigma}$ may be reducible, cf. section 5, [HL] and [CTM1]. However, by the above argument, the slope of each component of Y is always 10.

Remark 3.7. Another way to produce a one parameter family of degree d covers of elliptic curves is by fixing the j-invariant of the target elliptic curve and moving one branched point. For instance, consider degree d genus 2 covers of a fixed elliptic curve E simply branched at the marked point O and another point P. Let P vary, and then we have a 1-dimensional space W of admissible covers which maps to $\overline{\mathcal{M}}_2$. When P meets O, after blowing up we will get some nodal curves as admissible covers of E with a 2-marked stable rational tail. Using the method in [HM], it is easy to write down the intersection number $W.\,\delta_1$ by the monodromy data and verify that it is not vanishing for $d \geq 3$. By the same argument in Remark 3.6, the slope of W is strictly smaller than 10. Therefore, our original one parameter family Y provides a better lower bound for slopes than the family

W does, at least for the case $g = 2$.

3.3 $g = 3, \sigma = (5^1 1^{d-5})$

We still assume that d is prime. Since the analysis is almost the same as in the previous examples, we skip the discussion and state the result directly:

$$N_{d^1} = 8\binom{d}{5}, \; N_{2^{a_2}1^{a_1}} = 8(a_2 - 1),$$

$$N_{l_1^{a_1} l_2^{a_2}} = \frac{l_1 l_2}{2}(3l_1^2 + 3l_2^2 - 19l_1 - 11l_2 + 4d + 22), l_1 > l_2 > 1,$$

$$N_{l_1^{a_1} l_2^{a_2} l_3^{a_3}} = \begin{cases} 11 l_1 l_2 l_3, \; l_1 \neq l_2 + l_3 > l_2 > l_3; \\ 7 l_1 l_2 l_3, \; l_1 = l_2 + l_3 > l_2 > l_3. \end{cases}$$

Now the slope formula says that

$$s(Y) = \frac{12M}{M + \dfrac{2N}{5}}.$$

We calculated by computer for small prime numbers d and it seems that $s(Y)$ in this case decreases to 9. This evidence leads to the following conjecture.

Conjecture 3.8. *When σ is of type $(5^1 1^{d-5})$, we have*

$$\lim_{d\to\infty} s(Y_{3,d,\sigma}) = 9.$$

Remark 3.9. Note that for $\overline{\mathcal{M}}_3$, it is already known that the hyperelliptic divisor $\overline{H}$ has the smallest slope 9 among all effective divisors. So if the above conjecture is true, then these curves $Y_{3,d,\sigma}$ do provide the sharp lower bound for slopes of effective divisors on $\overline{\mathcal{M}}_3$.

Next, we study in detail the beginning case $d = 5$. The result is the following.

Claim 3.10. *For $g = 3, d = 5$ and σ of type (5^1), $Y_{3,5,(5^1)}$ has four irreducible components. Two of them have slope 9 and the other two have slope $9\frac{1}{3}$.*

Proof. The proof is nothing but to enumerate all the possible solution pairs (α, β) modulo equivalence relation, and further classify the orbits by the monodromy criterion, cf. Theorem 1.6.

In total there are 40 non-equivalent solutions in the following list:

(1) $\alpha = (12)(34), \beta = (12345)$; (2) $\alpha = (12)(35), \beta = (12345)$;

(3) $\alpha = (124), \beta = (12345)$; (4) $\alpha = (142), \beta = (12345)$;

(5) $\alpha = (12453), \beta = (12345)$; (6) $\alpha = (13254), \beta = (12345)$;

(7) $\alpha = (14)(25), \beta = (123)$; (8) $\alpha = (12435), \beta = (123)$;

(9) $\alpha = (13425), \beta = (123)$; (10) $\alpha = (15)(23), \beta = (12)(34)$;

(11) $\alpha = (135), \beta = (12)(34)$; (12) $\alpha = (12345), \beta = (12)(34)$;

(13) $\alpha = (12354), \beta = (12)(34)$; (14) $\alpha = (1243), \beta = (12345)$;

(15) $\alpha = (1342), \beta = (12345)$; (16) $\alpha = (15)(23), \beta = (1234)$;

(17) $\alpha = (15)(24), \beta = (1234)$; (18) $\alpha = (15)(34), \beta = (1234)$;

(19) $\alpha = (135), \beta = (1234)$; (20) $\alpha = (125)(34), \beta = (1234)$;

(21) $\alpha = (152)(34), \beta = (1234)$; (22) $\alpha = (1325), \beta = (1234)$;

(23) $\alpha = (1352), \beta = (1234)$; (24) $\alpha = (1523), \beta = (1234)$;

(25) $\alpha = (1253), \beta = (1234)$; (26) $\alpha = (12435), \beta = (1234)$;

(27) $\alpha = (14235), \beta = (1234)$; (28) $\alpha = (14)(23), \beta = (123)(45)$;

(29) $\alpha = (124), \beta = (123)(45)$; (30) $\alpha = (134), \beta = (123)(45)$;

(31) $\alpha = (145)(23), \beta = (123)(45)$; (32) $\alpha = (1245), \beta = (123)(45)$;

(33) $\alpha = (1345), \beta = (123)(45)$; (34) $\alpha = (1425), \beta = (123)$;

(35) $\alpha = (124)(35), \beta = (123)$; (36) $\alpha = (142)(35), \beta = (123)$;

(37) $\alpha = (143)(25), \beta = (12)(34)$; (38) $\alpha = (1345), \beta = (12)(34)$;

(39) $\alpha = (1354), \beta = (12)(34)$; (40) $\alpha = (1534), \beta = (12)(34)$.

Now by monodromy action, it is routine to check that:

(2),(10),(13) belong to the first component Z_1;

(1),(3),(4),(5),(6),(7),(8),(9),(11),(12) belong to the second component Z_2;

(14),(15),(16),(18),(22),(23),(24),(25),(26),(27),(38),(40) belong to the third component Z_3;

(17),(19),(20),(21),(28),(29),(30),(31),(32),(33),(34),(35),(36),(37),(39) belong to the last component Z_4.

Note that the slope formula can be readily used not only for the entire curve Y but also for its irreducible components. Plugging in the data listed above, we get

$$s(Z_1) = s(Z_3) = 9$$

and

$$s(Z_2) = s(Z_4) = 9\frac{1}{3}.$$

□

3.4 Hyperelliptic and hyperflex divisors on $\overline{\mathcal{M}}_3$

For a cover $C \to E$ in $Y_{3,d,(5^1 1^{d-5})}$, C necessarily has a holomorphic 1-form with a zero of order 4. Let

$$K = \{[C] \in \mathcal{M}_3 : K_C \text{ has a vanishing sequence} \geq (0,1,4) \text{ at some point } p \in C\}.$$

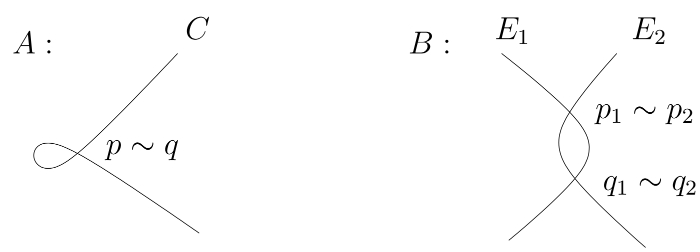
Then we know that $\text{Im}\, Y_{3,d,(5^1 1^{d-5})} \subset K$. We define another divisor $F \subset \mathcal{M}_3$ to be the locus of smooth plane quartics that have hyperflexes. The slope of $\overline{F}$ is $9\frac{5}{8}$, which was worked out by Cukierman, cf. [Cu]. We also know that $F \cup H = K$ and $F \cap H = \emptyset$. It would be interesting to get some information about the intersection of $\overline{H}$ and $\overline{F}$ at the boundary of $\overline{\mathcal{M}}_3$.

First, define codimension 2 loci

$$A = \{C/p \sim q : C \text{ is a genus 2 curve, } p \text{ and } q \text{ are conjugate points on } C\},$$

and

$$B = \{E_1 \sqcup E_2/p_1 \sim p_2, q_1 \sim q_2 : E_i \text{ is an elliptic curve, and } p_i, q_i \in E_i\}.$$



Curves in A and B are always double covers of rational curves in the sense of admissible covers. It is not hard to see that $\overline{H} \cap \Delta_0 = \overline{A} \cup \overline{B}$.

For the divisor $\overline{F}$, we have the following claim.

Claim 3.11. $\overline{F} \cap \Delta_0 \supset \overline{A} \cup \overline{B}$.

Proof. First, let us verify that $A \subset F$. Take a plane conic Q and a one parameter family C_s of general plane quartics, such that C_0 has a hyperflex line that is also tangent to Q at the hyperflex point. By stable reduction, as in [HM1, 3.C], we know that $\lim_{t \to 0}(tC_s + Q^2)$ is a general hyperelliptic curve for $s \neq 0$. Then taking the limit $\lim_{s \to 0} \lim_{t \to 0}(tC_s + Q^2)$ amounts to squeezing two Weierstrass points together. So we get a general element in A.

For B, take a banana curve $E_1 \sqcup E_2/p_1 \sim p_2, q_1 \sim q_2$. Consider a sub linear series of $|\mathcal{O}_{E_1}(2p_1 + 2q_1)|$ that contracts E_2 and maps E_1 to a tacnodal plane quartic. More precisely, map E_1 to $\mathbb{P}^3$ by the full linear system $|\mathcal{O}_{E_1}(2p_1 + 2q_1)|$. Take a point r collinear with p_1, q_1 and contained in a plane which meets E_1 at a quadruple point r_1, where $2r_1 \sim p_1 + q_1$. Project E_1 from r to $\mathbb{P}^2$. Then p_1, q_1 map to a tacnode of the image curve, and there is a hyperflex line passing through the image of r_1.

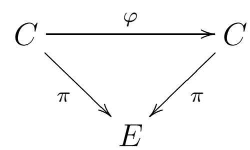
Now we want to show that the inclusion in the above claim is proper. Consider the case $g = 3, d = 5, \sigma = (5^1)$ in Claim 3.10. When the vanishing cycle β is of type (5^1), i.e., cases (1), (2), (3), (4), (5), (6), (14) and (15), the degenerate cover is a 5-sheeted admissible covering map from a 1-nodal geometric genus 2 curve to a rational nodal curve totally ramified at the node and another smooth point. Note that such a cover can be induced from a degree 5 covering map from a smooth genus 2 curve to $\mathbb{P}^1$ totally ramified at 3 points p, q and r. These points cannot be conjugate to each other simultaneously. Assume that p, q are not conjugate. Identify them and identify their images also. We get a covering curve $[C] \in \mathrm{Im}\, Y_{3,5,(5^1)} \cap \Delta_0 \subset \overline{K} \cap \Delta_0 = (\overline{F} \cup \overline{H}) \cap \Delta_0$. But by the construction, we know that $[C] \notin \overline{H} \cap \Delta_0 = \overline{A} \cup \overline{B}$. Therefore, we get the desired conclusion. $\square$

Remark 3.12. It would be interesting to do the stable reduction directly for the family $tC_0 + Q^2$.

4 Counting Weighted Connected Covers

In this section, we give a method to systematically calculate $N_{1^{a_1}2^{a_2}\ldots d^{a_d}}$. For simplicity, here we only consider the case when d is prime and σ is of type $(2^k1^{d-2k}) = \tau_{d,k}, k = 2g-2$. The general situation can be solved similarly without further difficulty.

For a cover $\pi : C \to E$, an automorphism φ of this cover is given by the following diagram:



If π corresponds to one solution pair $(\alpha, \beta) \in S_d \times S_d$, then the automorphism φ corresponds to an element $\tau \in S_d$ such that $(\tau\alpha\tau^{-1}, \tau\beta\tau^{-1}) = (\alpha, \beta)$. Hence, we have the correspondence $Aut(C, \pi) = Stab(\alpha, \beta)$, where $Stab(\alpha, \beta)$ is the set of stablizers of the S_d conjugate action.

In many cases people are interested in the weighted Hurwitz numbers, i.e., counting a cover (C, π) with weight $\dfrac{1}{|Aut(C, \pi)|}$. Hence, we define a weighted number

$$\widetilde{N}_{\wp}^{d,k} = \sum_{(\alpha,\beta)\in Cov_{\wp}^{g,d,\tau_{d,k}}/\sim} \frac{1}{|Stab(\alpha, \beta)|} = \frac{1}{d!}|Cov_{\wp}^{g,d,\tau_{d,k}}|,$$

where $\wp$ denotes the conjugacy class $(1^{a_1}2^{a_2}\ldots d^{a_d})$. In particular when d is prime, the cover has no non-trivial automorphism since $\langle\alpha, \beta\rangle$ is transitive, cf. Lemma 3.1 and 3.2. So we can pretend to count weighted connected covers instead.

Now we want to get rid of the transitivity condition imposed on $\langle\alpha, \beta\rangle$. So we further define a set

$$\widehat{Cov}_{\wp}^{d,k} := \{(\alpha, \beta) \in S_d \times S_d \mid \beta \in \wp,\ \alpha\beta\alpha^{-1}\beta^{-1} \in \tau_{d,k}\},$$

and let

$$\widehat{N}_{\wp}^{d,k} = |\widehat{Cov}_{\wp}^{d,k}|.$$

We fix an element $\tau \in \tau_{d,k}$ and look for pairs $(\gamma, \beta) \in \wp \times \wp$ such that $\gamma\beta^{-1} = \tau$. For such a pair, there are $\dfrac{|S_d|}{|\wp|}$ choices for α that satisfy the equality $\alpha\beta\alpha^{-1} = \gamma$. Hence,

$$\widehat{N}_{\wp}^{d,k} = |\tau_{d,k}| \cdot \frac{|S_d|}{|\wp|} \cdot |\{(\gamma, \beta) \in \wp \times \wp : \gamma\beta^{-1} = \tau\}|.$$

By [St, 7.68 a], we know that

$$|\{(\gamma, \beta) \in \wp \times \wp : \gamma\beta^{-1} = \tau\}| = \frac{|\wp|^2}{|S_d|} \cdot \left(\sum_{\chi} \frac{1}{\deg(\chi)} |\chi(\wp)|^2 \chi(\tau) \right),$$

where χ runs over all irreducible characters of S_d. Therefore, we get the following expression

$$\widehat{N}_{\wp}^{d,k} = |\wp| \cdot |\tau_{d,k}| \cdot \left(\sum_{\chi} \frac{1}{\deg(\chi)} |\chi(\wp)|^2 \chi(\tau_{d,k}) \right).$$

Now our task is to derive $\widetilde{N}_{\wp}^{d,k}$ from $\widehat{N}_{\wp}^{d,k}$. Take a solution pair $(\alpha, \beta) \in \widehat{Cov}_{\wp}^{d,k}$. The subgroup $\langle \alpha, \beta \rangle$ of S_d may not be transitive. Consider the orbits and the action of α, β on them, which correspond to the following data:

$$\{(\alpha_i, \beta_i), i = 1, \ldots, m \mid \alpha\beta\alpha^{-1}\beta^{-1} \in (2^{k_i} 1^{d_i - 2k_i}) = \tau_{d_i, k_i},$$

$$\beta_i \in \wp_i \text{ a conjugacy class of } S_{d_i},\ \bigcup_{i=1}^{m} \wp_i = \wp,\ \sum_{i=1}^{m} k_i = k,$$

$$\sum_{i=1}^{m} d_i = d,\ \langle \alpha_i, \beta_i \rangle \text{ is a transitive subgroup of } S_{d_i}\}.$$

Two data $(\wp_i, k_i, d_i)$ and $(\wp_j, k_j, d_j)$ are of the same type if $\wp_i \sim \wp_j, k_i = k_j$

and $d_i = d_j$. Hence, we get the following equality

$$\widehat{N}_{\wp}^{d,k} = \sum \binom{d}{\underbrace{d_1,\ldots,d_1}_{p_1},\ldots,\underbrace{d_m,\ldots,d_m}_{p_m}} \frac{(d_1!)^{p_1}\cdots(d_m!)^{p_m}}{(p_1!)\cdots(p_m!)} (\widetilde{N}_{\wp_1}^{d_1,k_1})^{p_1}\cdots(\widetilde{N}_{\wp_m}^{d_m,k_m})^{p_m},$$

where the condition on the summation is $\sum_{i=1}^{m} p_i d_i = d, \sum_{i=1}^{m} p_i k_i = k$ and $\bigcup_{i=1}^{m} \wp_i^{p_i} = \wp$. Simplifying the above expression, we obtain

$$\widehat{N}_{\wp}^{d,k} = \sum (d!) \prod_{i=1}^{m} \frac{(\widetilde{N}_{\wp_i}^{d_i,k_i})^{p_i}}{p_i!}.$$

If $\wp$ is of type $(1^{a_1}\cdots d^{a_d})$, a datum $(\wp_i, k_i, d_i)$ corresponds to a vector $(a_{i1},\ldots,a_{id},k_i), 0 \le a_{ij} \le a_j, 0 \le k_i \le k, \sum_{j=1}^{d} j a_{ij} = d_i$ with integer entries. Put all the vectors into a matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1d} & k_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{md} & k_m \end{pmatrix}$$

Then the summation runs over all possible $(p_1,\ldots,p_m)$ satisfying

$$(p_1,\ldots,p_m)\cdot A = (a_1,\ldots,a_d,k).$$

Now slightly change the notation. Define an index set

$$I = \{(a_1,a_2,\cdots) \mid a_i \ge 0, \text{ and there are only finite many non-zero entries}\}.$$

For $\wp$ of type $(1^{a_1}2^{a_2}\cdots d^{a_d})$, write it as $(1^{a_1}2^{a_2}\cdots)$. So it is determined by an element $(a_1,a_2,\ldots) \in I$, and $d = a_1 + 2a_2 + \cdots$.

We define two generating functions as follows:

$$\widehat{Z}(y; x_1, x_2, \ldots) = \sum_{I,k} \frac{\widehat{N}_{\wp}^{d,k}}{(a_1 + 2a_2 + \cdots)!} \cdot y^k x_1^{a_1} x_2^{a_2} \cdots ,$$

and

$$\widetilde{Z}(y; x_1, x_2, \ldots) = \sum_{I,k} \widetilde{N}_{\wp}^{d,k} \cdot y^k x_1^{a_1} x_2^{a_2} \cdots ,$$

where $d = a_1 + 2a_2 + \cdots$ and $\wp$ is of type $(1^{a_1} 2^{a_2} \cdots)$ for each term.

Therefore, finally we obtain the relation between the generating functions for the number of connected and possibly disconnected covers:

$$\widehat{Z} = exp(\widetilde{Z}) - 1.$$

Remark 4.1. If k is odd or $2k > d = a_1 + 2a_2 + \cdots$, then obviously $\widehat{N}_{\wp}^{d,k} = \widetilde{N}_{\wp}^{d,k} = 0$. The evaluation of a character χ on a conjugacy class $\wp$ can be worked out by standard formulae from the representation theory of S_d. However, when d is large, it seems hard to evaluate the quotient $\frac{N}{M}$ even by computer. So the estimate of $\lim_{d\to\infty} s(Y_{g,d,\sigma})$ remains mysterious to us.

5 The Local Geometry of Y

In this section, we study the local monodromy action of the map $Y \to X$ mentioned in Remark 2.5, and via it we obtain information like the genus, orbifold points and orbifold Euler characteristic of Y.

For each component of Y, we take its reduced scheme structure. An orbifold point of Y possibly occurs when a smooth cover degenerates to a singular cover, and its orbifold order depends on the information about the degree of a local minimal base change and the order of its extra automorphisms (not induced from nearby covers), cf. Remark 2.1. The following two examples would illustrate the idea.

5.1 Basic Examples

Example 5.1. $g = 2, d = 3, \sigma = (3^1)$

There are three non-equivalent solution pairs (α, β):

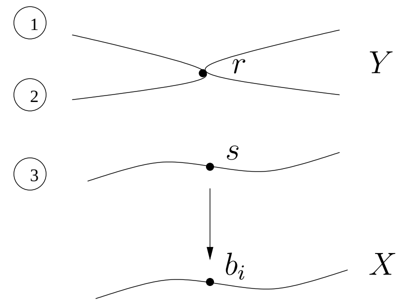
(1) $\alpha = (13), \beta = (12)$;

(2) $\alpha = (123), \beta = (12)$;

(3) $\alpha = (12), \beta = (123)$.

When β corresponds to the vanishing cycle, the local monodromy action $(\alpha, \beta) \to (\alpha\beta, \beta)$ can switch sheets (1) and (2) but keep (3) unchanged, and the global monodromy can send all three sheets to each other. So Y is a degree 3 connected cover of X simply branched at $b_1, \ldots, b_{12}$. By Riemann-Hurwitz, $2g(Y) - 2 = 3(2g(X) - 2) + 12$, so $g(Y) = 4$.

Furthermore, over b_i, sheets (1) and (2) meet at the ramification point r, and the other pre-image point s lies in sheet (3).

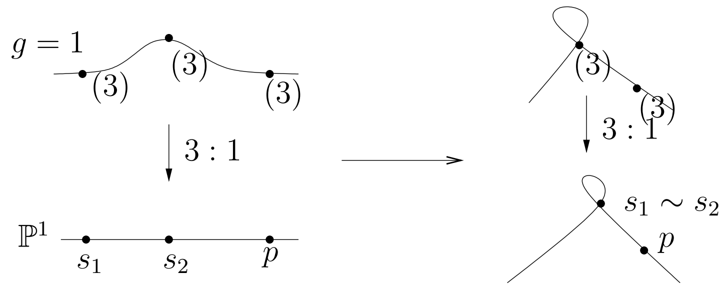


Since in case (3), β is a length 3 cycle, as in Remark 2.1 a degree 3 base change is necessary to complete the universal covering map locally over sheet (3). So s is an orbifold point with structure group $\mathbb{Z}/3$. By contrast, locally around r, we need a degree 2 base change since β contains length 2 cycles. But sheets (1) and (2) meet at r, so the cover corresponding to r does not have an extra order 2 automorphism compared with nearby smooth covers. Hence, r is not an orbifold point of Y. Finally, by the orbifold Euler characteristic formula, we have

$$\chi(Y) = 2 - 2g(Y) - 12(1 - \frac{1}{3}) = -14.$$

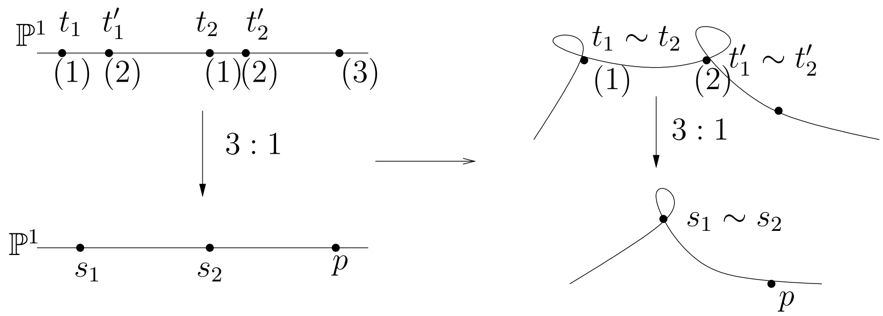
Remark 5.2. By using the results about $g(Y)$ in section 5.2 and 5.3, we can work out the orbifold Euler characteristic of $Y_{g,d,\sigma}$ in the same way when $g = 2, \sigma = (3^1 1^{d-3})$ or $(2^2 1^{d-4})$ and d is prime. Note that the result for general d is obtained in [Ba], although only the case $\sigma = (3^1 1^{d-3})$ is discussed there.

It is also natural to study the degenerate covers directly and recover the information obtained from the local monodromy action. The cover corresponding to s can be induced from a degree 3 covering map from an elliptic curve to $\mathbb{P}^1$ totally branched over 3 points s_1, s_2, p by identifying s_1, s_2 and their pre-images. The following picture reveals the idea. A ramification point with order k is marked by (k) in the picture.



Hence, we look at triples $(\tau_{s_1}, \tau_{s_2}, \tau_p)$ in S_3 such that $\tau_{s_1}\tau_{s_2}\tau_p = id$ and τ_i is of type (3^1). The only possible solution is $\tau_{s_1} = \tau_{s_2} = \tau_p$ and therefore this cover has an order 3 automorphism, which coincides with our above discussion.

For r, we look at a degree 3 covering map π from $\mathbb{P}^1$ to $\mathbb{P}^1$ simply branched at s_1, s_2 and totally branched at p. Assume that $\pi^{-1}(s_i) = t_i + 2t_i', i = 1, 2$. Identify $s_1 \sim s_2, t_1 \sim t_2, t_1' \sim t_2'$. Then we can recover the map corresponding to r.

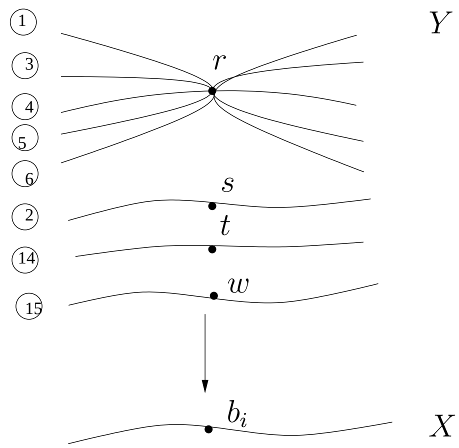


Hence, consider triples $(\tau_{s_1}, \tau_{s_2}, \tau_p)$ in S_3 such that $\tau_{s_1}\tau_{s_2}\tau_p = id$, τ_{s_1} and τ_{s_2} are simple transpositions but τ_p is a (3^1) cycle. Modulo the S_3 conjugation action, there is a unique solution corresponding to $\tau_{s_1} = (12), \tau_{s_2} = (13)$. If we switch τ_{s_1} and τ_{s_2}, we get back the same cover, so it has the automorphism induced from the involution ι. But it does not have other automorphisms. This result also coincides with our previous analysis.

Example 5.3. $g = 3, d = 5, \sigma = (5^1)$

We only focus on the case when the vanishing cycle β is of type (5^1), i.e., cases (1), (2), (3), (4), (5), (6), (14) and (15) in the proof of Claim 3.10. The local monodromy acts transitively on (1),(3),(4),(5),(6), but keeps (2),(14),(15) fixed.

Let r be the point over b_i where the sheets (1),(3),(4),(5),(6) meet, and s, t, w be the other 3 pre-images of b_i contained in (2),(14),(15) respectively. Since locally around b_i we need a degree 5 base change, s, t, w are orbifold points with structure group $\mathbb{Z}/5$, and r is not an orbifold point.



Also note that if we consider Y' instead of Y, then (1)-(6) are still distinct sheets in Y' and they all have automorphisms induced from the elliptic involution ι. However, (14) has to be identified with (15) under the new equivalence relation $(\alpha, \beta) \sim (\alpha^{-1}, \beta^{-1})$ for Y'.

Again, we can study the degenerate covers directly as the previous example. In this case, the covers corresponding to r, s, t, w can be induced by degree 5 covers from genus 2 curves to $\mathbb{P}^1$ totally branched over 3 points s_1, s_2, p. We then identify s_1, s_2 and their pre-image points to obtain the desired singular covers. Now we need to analyze the solution triples $(\tau_{s_1}, \tau_{s_2}, \tau_p)$ in S_5 such that $\tau_{s_1}\tau_{s_2}\tau_p = id$ and they are all (5^1) cycles. Take $\tau_{s_1} = (12345)$, then τ_{s_2} can only be $\tau_{s_1}^k, k = 1, 2, 3$ or (12453), modulo the S_5 conjugation action. For the case $\tau_{s_2} = \tau_{s_1}^2$, after switching τ_{s_1} and τ_{s_2}, we get a cover equivalent to the case $\tau_{s_2} = \tau_{s_1}^3$. So these two covers can be exchanged by the involution of the target rational nodal curve and they correspond to t and w in sheets (14) and (15). For $\tau_{s_2} = (12453)$, switching τ_{s_1} and τ_{s_2}, we get the same cover. Moreover, this cover does not have automorphisms except the one induced by the involution ι. So it corresponds to the non-orbifold point r. Finally, if $\tau_{s_2} = \tau_{s_1}$, this cover has the automorphism induced by ι and another order 5 automorphism, so it corresponds to s in sheet (2).

Now we study in general the orbits of local monodromy for the case $g = 2$ and d prime. Starting from one solution pair (α, β) where β is the vanishing cycle, the local monodromy action can send (α, β) to $(\alpha\beta^k, \beta)$, so these two sheets have to meet at the same degenerate cover over b_i. We also assume that $\alpha\beta\alpha^{-1}\beta^{-1} = \gamma \in \sigma$, where σ is of type $(3^1 1^{d-3})$ or $(2^2 1^{d-4})$.

5.2 $g = 2, \sigma = (3^1 1^{d-3})$

We give a proof of Theorem 1.9. All the numbers N and $N_{1^{a_1}2^{a_2}\dots d^{a_d}}$ in the proof below are from section 3.1.

Proof. Let $\alpha\beta\alpha^{-1}\beta^{-1} = \gamma = (abc)$ be a fixed cycle in S_d. We look for k such that there exists an element $\tau \in S_d$, $\tau(\alpha\beta^k, \beta)\tau^{-1} = (\alpha, \beta)$. Note that such τ must satisfy $\tau\gamma\tau^{-1} = \gamma$, since

$$\tau\gamma\tau^{-1} = \tau(\alpha\beta^k)\beta(\alpha\beta^k)^{-1}\beta^{-1}\tau^{-1} = \alpha\beta\alpha^{-1}\beta^{-1} = \gamma.$$

If β is of type (d^1), from $\tau\beta\tau^{-1} = \beta$, we get $\tau = \beta^m$ for some integer m. So if $\tau(abc)\tau^{-1} = (a+m\ b+m\ c+m) = (abc)$, then $d|3m$. As long as $d \geq 5$ is prime, τ must be id and $d|k$. Since $N_{d^1} = \binom{d}{3}$, we get $\frac{1}{d}\binom{d}{3} = \frac{1}{6}(d-1)(d-2)$ orbits each of which has cardinality d. From the viewpoint of the covering map $Y \to X$, the d sheets in one orbit meet at a degenerate cover which is not an orbifold point of Y, since locally around such a point we need a degree d base change to realize the universal covering map in the proof of Theorem 1.4.

If β is of type $(l_1^{a_1} l_2^{a_2})$, $l_1 > l_2$, we know $(l_1, l_2) = 1$. Without loss of generality, assume that

$$\beta = (t_{11}t_{12}\cdots t_{1l_1})\cdots(t_{a_11}\cdots t_{a_1l_1})$$

$$\cdot(s_{11}s_{12}\cdots s_{1l_2})\cdots(s_{a_21}\cdots s_{a_2l_2}),$$

and that $a = t_{11}, b = t_{1\ l_1-l_2+1}, c = s_{11}$. From the condition $\tau\beta\tau^{-1} = \beta, \tau\gamma\tau^{-1} = \gamma$ and d prime, we can verify that τ fixes all the elements in the cycles $(t_{11}t_{12}\cdots t_{1l_1})$ and $(s_{11}s_{12}\cdots s_{1l_2})$. Then we have

$$\alpha\beta\alpha^{-1} = \gamma\beta = (t_{11}\cdots t_{1\ l_1-l_2}s_{11}\cdots s_{1l_2})(t_{21}\cdots t_{2l_1})\cdots(t_{a_11}\cdots t_{a_1l_1})$$

$$\cdot(t_{1\ l_1-l_2+1}\cdots t_{1l_1})(s_{21}\cdots s_{2l_2})\cdots(s_{a_21}\cdots s_{a_2l_2}).$$

So we can assume that α sends the cycle $(t_{a_11}\cdots t_{a_1l_1})$ to $(t_{11}\cdots t_{1\ l_1-l_2}s_{11}\cdots s_{1l_2})$ and the cycle $(s_{a_21}\cdots s_{a_2l_2})$ to $(t_{1\ l_1-l_2+1}\cdots t_{1l_1})$. Furthermore, assume that $\alpha(t_{ij}) = t_{i+1\ j}, 1 \leq i < a_1 - 1, 1 \leq j \leq l_1, \alpha(s_{ij}) = s_{i+1\ j}, 1 \leq i < a_2 - 1, 1 \leq j \leq l_2$, but $\alpha(t_{a_1-1\ j}) = t_{a_1\ j+w_1}$ and $\alpha(s_{a_2-1\ j}) = s_{a_2\ j+w_2}$, i.e., these actions are twisted by twist parameters w_1 and w_2, whose geometric meaning can be more clearly seen in [HL] and the next section. Now, α contains a cycle $(t_{11}t_{21}\cdots t_{a_1-1\ 1}t_{a_1\ 1+w_1}\alpha(t_{a_1\ 1+w_1})\cdots)$ and the corresponding cycle in $\alpha\beta^k$ is $(t_{11}t_{2\ 1+k}\cdots t_{a_1\ 1+(a_1-1)k+w_1}\alpha(t_{a_1\ 1+a_1k+w_1})\cdots)$. Since $\tau\alpha\beta^k\tau^{-1} = \alpha$ and $t_{11}, \alpha(t_{a_1\ 1+w_1}), \alpha(t_{a_1\ 1+a_1k+w_1})$ are all fixed by τ, we get $l_1|a_1k$.

Similarly we have $l_2|a_2k$. One can check that these two conditions on k are also sufficient for the existence of τ. Since $N_{l_1^{a_1}l_2^{a_2}} = l_1l_2$, for β of such type we just get $(l_1, a_1)(l_2, a_2)$ orbits and each orbit has cardinality $\dfrac{l_1l_2}{(l_1, a_1)(l_2, a_2)}$.

Consider the map $Y \to X$. By the Riemann-Hurwitz formula, we have

$$2g(Y) - 2 = -2N + 12\bigg(\frac{(d-1)(d-2)}{6}(d-1)$$

$$+\sum_{\substack{a_1l_1+a_2l_2=d\\ l_1>l_2}}(l_1,a_1)(l_2,a_2)\Big(\frac{l_1l_2}{(l_1,a_1)(l_2,a_2)}-1\Big)\Big).$$

After simplifying, we get the desired expression in the theorem.

Furthermore, we already know that $N\sim\frac{3}{8}d^3$ and $\sum_{\substack{a_1l_1+a_2l_2=d\\ l_1>l_2}} l_1l_2\sim\frac{5}{24}d^3$. The fact that $\sum_{\substack{a_1l_1+a_2l_2=d\\ l_1>l_2}}(l_1,a_1)(l_2,a_2)$ has lower order than d^3 follows from [HL, 7.4]. So we also obtain the asymptotic result for $g(Y)$. □

Remark 5.4. In the next section, we will see that the above Y has two irreducible components Z_1 and Z_2 that do not intersect, cf. Theorem 6.8. So actually we have $g(Y)=g(Z_1)+g(Z_2)-1$.

Remark 5.5. Note that a similar genus formula for the case $g=2,\sigma=(3^11^{d-3})$ and d prime is also obtained in [HL] using the technique of square-tiled surfaces. However, our space of admissible covers Y differs from the Teichmüller discs defined in [HL] in that Y is closed and it is over a pencil of plane cubics rather than $\overline{\mathcal{M}}_{1,1}$.

5.3 $g=2,\sigma=(2^21^{d-4})$

Now we study the other case: $\sigma=(2^21^{d-4})$, and prove Theorem 1.10. All the numbers N and $N_{1^{a_1}2^{a_2}\dots d^{a_d}}$ in the following proof are from section 3.2.

Proof. Let $\alpha\beta\alpha^{-1}\beta^{-1}=\gamma=(ab)(ce)$ be a fixed cycle. We still look for k such that there exists $\tau\in S_d$, $\tau(\alpha\beta^k,\beta)\tau^{-1}=(\alpha,\beta)$. As before, such τ must satisfy $\tau\gamma\tau^{-1}=\gamma$. The discussion is very similar to the one above. We analyze the solution pairs (α,β) case by case based on the type of the vanishing cycle β.

If β is of type (d^1), assume that $\beta=(12\cdots d)$. From $\tau\beta\tau^{-1}=\beta$ we get that τ must be β^m for some integer m. But $\tau\gamma\tau^{-1}=(a{+}m\ b{+}m)(c{+}m\ e{+}m)\neq(ab)(ce)$, since $4m \not\equiv 0 \pmod{d}$. $N_{d^1} = \binom{d}{4}$, so in this case we get

$$\frac{1}{d}\binom{d}{4} = \frac{1}{24}(d-1)(d-2)(d-3)$$

orbits, each of which contains d sheets meeting at one degenerate cover over b_i.

If β is of type $(l_1^{a_1} l_2^{a_2} l_3^{a_3})$, $l_1 = l_2 + l_3 > l_2 > l_3$, we can write β as

$$(t_{11} \cdots t_{1l_1}) \cdots (t_{a_1 1} \cdots t_{a_1 l_1})$$

$$\cdot (r_{11} \cdots r_{1l_2}) \cdots (r_{a_2 1} \cdots r_{a_2 l_2})$$

$$\cdot (s_{11} \cdots s_{1l_3}) \cdots (s_{a_3 1} \cdots s_{a_3 l_3}),$$

and $\gamma = (t_{11} t_{1\ l_2+1})(r_{11} s_{11})$. By $\tau\gamma\tau^{-1} = \gamma, \tau\beta\tau^{-1} = \beta$ and l_1, l_2 co-prime, we know that τ fixes all the elements in $(t_{11} \cdots t_{1l_1}), (r_{11} \cdots r_{1l_2})$ and $(s_{11} \cdots s_{1l_3})$. Then we have

$$\alpha\beta\alpha^{-1} = \gamma\beta = (r_{11} \cdots r_{1l_2} s_{11} \cdots s_{1l_3})(t_{21} \cdots t_{2l_1}) \cdots (t_{a_1 1} \cdots t_{a_1 l_1})$$

$$\cdot (t_{11} \cdots t_{1l_2})(r_{21} \cdots r_{2l_2}) \cdots (r_{a_2 1} \cdots r_{a_2 l_2})$$

$$\cdot (t_{1\ l_2+1} \cdots t_{1l_1})(s_{21} \cdots s_{2l_3}) \cdots (s_{a_3 1} \cdots s_{a_3 l_3}).$$

So we can assume that α sends the cycle $(t_{a_1 1} \cdots t_{a_1 l_1})$ to $(r_{11} \cdots r_{1l_2} s_{11} \cdots s_{1l_3})$, $(r_{a_2 1} \cdots r_{a_2 l_2})$ to $(t_{11} \cdots t_{1l_2})$, and $(s_{a_3 1} \cdots s_{a_3 l_3})$ to $(t_{1\ l_2+1} \cdots t_{1l_1})$. $\alpha(t_{ij}) = t_{i+1\ j}, 1 \leq i < a_1 - 1, 1 \leq j \leq l_1, \alpha(r_{ij}) = r_{i+1\ j}, 1 \leq i < a_2 - 1, 1 \leq j \leq l_2, \alpha(s_{ij}) = s_{i+1\ j}, 1 \leq i < a_3 - 1, 1 \leq j \leq l_3$, but $\alpha(t_{a_1-1\ j}) = t_{a_1\ j+w_1}, \alpha(r_{a_2-1\ j}) = r_{a_2\ j+w_2}, \alpha(s_{a_3-1\ j}) = s_{a_3\ j+w_3}$, i.e., these actions are twisted by twist parameters w_1, w_2 and w_3. α contains a cycle $(t_{11} t_{21} \cdots t_{a_1-1\ 1} t_{a_1\ 1+w_1} \alpha(t_{a_1\ 1+w_1}) \cdots)$ and the corresponding cycle in $\alpha\beta^k$ is $(t_{11} t_{2\ 1+k} \cdots t_{a_1\ 1+(a_1-1)k+w_1} \alpha(t_{a_1\ 1+a_1 k+w_1}) \cdots)$. Since $\tau\alpha\beta^k\tau^{-1} = \alpha$ and

$t_{11}, \alpha(t_{a_1 1+w_1}), \alpha(t_{a_1\ 1+a_1k+w_1})$ are all fixed by τ, we get $l_1|a_1k$.

Similarly, we have $l_i|a_ik, i = 2, 3$. These conditions are also sufficient. Since $N_{l_1^{a_1}l_2^{a_2}l_3^{a_3}} = l_1l_2l_3$, we get $\prod_{i=1}^{3}(l_i, a_i)$ orbits in this case, and each orbit contains $\dfrac{l_1l_2l_3}{(l_1, a_1)(l_2, a_2)(l_3, a_3)}$ elements.

If β is of type $(l_1^{a_1}l_2^{a_2})$, $l_1 > l_2 > 1$, we can write

$$\beta = (t_{11} \cdots t_{1l_1}) \cdots (t_{a_1 1} \cdots t_{a_1 l_1})$$

$$\cdot(s_{11} \cdots s_{1l_2}) \cdots (s_{a_2 1} \cdots s_{a_2 l_2}).$$

If $\gamma = (t_{11}s_{11})(t_{1\ p+1}s_{1\ p+1}), 0 < p < l_2$, then

$$\alpha\beta\alpha^{-1} = \gamma\beta = (t_{1\ p+1} \cdots t_{1l_1}s_{11} \cdots s_{1p})(t_{21} \cdots t_{2l_1}) \cdots (t_{a_1 1} \cdots t_{a_1 l_1})$$

$$\cdot(t_{11} \cdots t_{1p}s_{1\ p+1} \cdots s_{1l_2})(s_{21} \cdots s_{2l_2}) \cdots (s_{a_2 1} \cdots s_{a_2 l_2}).$$

We can always assume that α sends the cycle $(t_{a_1 1} \cdots t_{a_1 l_1})$ to $(s_{11} \cdots s_{1p}t_{1\ p+1} \cdots t_{1l_1})$ and $(s_{a_2 1} \cdots s_{a_2 l_2})$ to $(t_{11} \cdots t_{1p}s_{1\ p+1} \cdots s_{1l_2})$. Moreover, $\alpha(t_{ij}) = t_{i+1\ j}, 1 \leq i < a_1 - 1$ and $\alpha(s_{ij}) = s_{i+1\ j}, 1 \leq i < a_2 - 1$, but $\alpha(t_{a_1-1\ j}) = t_{a_1\ j+w_1}$ and $\alpha(s_{a_2-1\ j}) = s_{a_2\ j+w_2}$, where w_1 and w_2 are the twist parameters.

Now by $\tau\gamma\tau^{-1} = \gamma$, if $\tau(t_{11}) = t_{1\ p+1}, \tau(s_{11}) = s_{1\ p+1}, \tau(t_{1\ p+1}) = t_{11}$ and $\tau(s_{1\ p+1}) = s_{11}$, we get $l_i|2p, i = 1, 2$, which is impossible. Hence, τ acts trivially on the elements in the cycles $(t_{11} \cdots t_{1l_1})$ and $(s_{11} \cdots s_{1l_2})$.

The cycle of α starting from t_{11} can be written as

$$(t_{11}t_{21} \cdots t_{a_1-1\ 1}t_{a_1\ 1+w_1}\alpha(t_{a_1\ 1+w_1}) \cdots),$$

and the corresponding cycle of $\alpha\beta^k$ is

$$(t_{11}t_{2\ 1+k}\cdots t_{a_1\ 1+(a_1-1)k+w_1}\alpha(t_{a_1\ 1+a_1k+w_1})\cdots).$$

Since $\tau\alpha\beta^k\tau^{-1}=\alpha$ and $t_{11},\alpha(t_{a_1\ 1+w_1}),\alpha(t_{a_1\ 1+a_1k+w_1})$ are all fixed by τ, we get $l_1|a_1k$.

Similarly, we have $l_2|a_2k$. These two conditions on k are also sufficient to find a desired τ. Hence in this case, there are $(l_2-1)(l_1,a_1)(l_2,a_2)$ orbits and each orbit contains $\dfrac{l_1l_2}{(l_1,a_1)(l_2,a_2)}$ elements.

There is one more case when β is of type $(l_1^{a_1}l_2^{a_2})$, $l_1>l_2>1$; namely, $\gamma=(t_{11}t_{1\ l_2+1})(t_{1\ l_2+1+m}s_{11})$ and

$$\alpha\beta\alpha^{-1}=\gamma\beta=(t_{1\ l_2+1}\cdots t_{1\ l_2+m}s_{11}\cdots s_{1l_2}t_{1\ l_2+m+1}\cdots t_{1l_1})\cdots(t_{a_11}\cdots t_{a_1l_1})$$

$$\cdot(t_{11}\cdots t_{1l_2})\cdots(s_{a_21}\cdots s_{a_2l_2}).$$

We can also check that τ acts trivially on $(t_{11}\cdots t_{1l_1})$ and $(s_{11}\cdots s_{1l_2})$.

Assume that, similarly to the last case, α acts with twist parameters w_1 and w_2 at the end. Then α contains one cycle

$$(t_{11}t_{21}\cdots t_{a_1-1\ 1}t_{a_1\ 1+w_1}\alpha(t_{a_1\ 1+w_1})\cdots)$$

and the corresponding cycle in $\alpha\beta^k$ is

$$(t_{11}t_{2\ 1+k}\cdots t_{a_1-1\ 1+(a_1-2)k}t_{a_1\ 1+w_1+(a_1-1)k}\alpha(t_{a_1\ 1+w_1+a_1k})\cdots).$$

Since $\tau\alpha\beta^k\tau^{-1}=\alpha$ and $t_{11},\alpha(t_{a_1\ 1+w_1}),\alpha(t_{a_1\ 1+w_1+a_1k})$ are all fixed by τ, we get $l_1|a_1k$. Similarly we have $l_2|a_2k$. So in this case there are $(l_1-l_2-1)(l_1,a_1)(l_2,a_2)$ orbits and each orbit contains $\dfrac{l_1l_2}{(l_1,a_1)(l_2,a_2)}$ elements.

The last case is when β is of type $(2^{a_2}1^{a_1})$. Assume $\beta = (s_1t_1)\cdots(s_{a_2}t_{a_2})(r_1)\cdots(r_{a_1})$ and $\gamma = (s_1t_1)(r_1r_2)$. Then we have

$$\alpha\beta\alpha^{-1} = \gamma\beta = (r_1r_2)(s_2t_2)\cdots(s_{a_2}t_{a_2})(s_1)(t_1)(r_3)\cdots(r_{a_1}).$$

We can further assume $\alpha(s_i) = s_{i+1}, \alpha(t_i) = t_{i+1}$ for $1 \le i \le a_2 - 1$ and $\alpha(s_{a_2}) = r_1, \alpha(t_{a_2}) = r_2$. Since $\beta^2 = id$, we only need to check when there exists τ such that $\tau\alpha\beta\tau^{-1} = \alpha$ and $\tau\alpha\tau^{-1} = \beta$.

It is not hard to see that α can be of type

$$(s_1\cdots s_{a_2}r_1r_3\cdots r_k)(t_1\cdots t_{a_2}r_2r_{k+1}\cdots r_{a_1})$$

or

$$(s_1\cdots s_{a_2}r_1r_3\cdots r_kt_1\cdots t_{a_2}r_2r_{k+1}\cdots r_{a_1}).$$

If a_2 is odd, then $\alpha\beta$ can be

$$(s_1t_2\cdots s_{a_2}r_2r_{k+1}\cdots r_{a_1}t_1s_2\cdots t_{a_2}r_1r_3\cdots r_k)$$

or

$$(s_1t_2\cdots s_{a_2}r_2\cdots r_{a_1})(t_1s_2\cdots t_{a_2}r_1\cdots r_k)$$

respectively. Note that here $\alpha\beta$ is not of the same type as α, so τ does not exist.

If a_2 is even, then $\alpha\beta$ can be

$$(s_1t_2\cdots s_{a_2-1}t_{a_2}r_1r_3\cdots r_k)(t_1s_2\cdots t_{a_2-1}s_{a_2}r_2r_{k+1}\cdots r_{a_1})$$

or

$$(s_1t_2\cdots s_{a_2-1}t_{a_2}r_1r_3\cdots r_kt_1s_2\cdots t_{a_2-1}s_{a_2}r_2\cdots r_{a_1}).$$

Here it is easy to check that the desired τ always exists.

Therefore, since $N_{2^{a_2}1^{a_1}} = a_1 - 1$, we get $\dfrac{a_1 - 1}{(a_2, 2)}$ orbits and each orbit contains $(a_2, 2)$ elements.

Putting all the results together, we obtain, by the Riemann-Hurwitz formula, for the map $Y \to X$

$$2g(Y) - 2 = -2N + 12\bigg(\frac{1}{24}(d-1)^2(d-2)(d-3) + \sum_{2a_2+a_1=d}\Big(a_1 - 1 - \frac{a_1-1}{(a_2,2)}\Big)$$

$$+ \sum_{\substack{a_1l_1+a_2l_2+a_3l_3=d \\ l_1=l_2+l_3>l_2>l_3}}\Big(l_1l_2l_3 - \prod_{i=1}^{3}(l_i, a_i)\Big) + \sum_{\substack{a_1l_1+a_2l_2=d \\ l_1>l_2}}(l_1-2)\Big(l_1l_2 - (l_1, a_1)(l_2, a_2)\Big)\bigg).$$

After simplifying, we get exactly the expression in the theorem.

The asymptotic behavior of $g(Y)$ follows from the result about the expression of N. Again, the terms involving (l_i, a_i) do not affect the asymptotic order. □

Remark 5.6. In the next section we will see that the above Y has at least two components, based on the different types of subgroups generated by α, β.

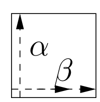
6 Square-Tiled Surfaces

In this section, we establish a correspondence between our method and the work in [HL] from the viewpoint of square-tiled surfaces.

The idea is quite simple. Take a standard torus E. If C is a cover of E, then C can be realized as a possibly degenerate lattice polygon with some edges and vertices identified. It covers d unit squares if the degree of the map is d. We will explain the details by some examples for the case $g = 2$. As before, we only consider the situation when d is prime.

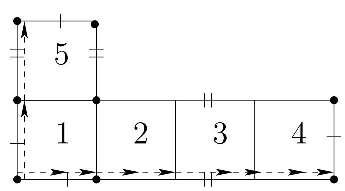
6.1 $g = 2, \sigma = (3^1 1^{d-3})$

In this case, if there is a degree d covering map $C \to E$ only ramified at one point $q \in C$, then C can be realized as an octagon of area d. All of its vertices are identified to be the unique ramification point q marked with • in the following picture. Take also two loops α and β of the torus E as in the picture.



Mark the unit squares covered by the octagon by $1, 2, \ldots, d$, and consider the monodromy images in S_d induced from α and β. Let us look at two examples.

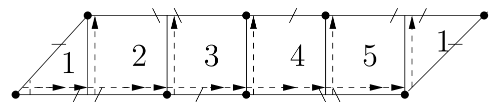
Example 6.1. Consider the following octagon.



It should correspond to a degree 5 cover of E. We still abuse notation and use α, β to denote also their monodromy images. It is easy to see $\alpha = (15)$ and

$\beta = (1234)$. Then we can check that $\alpha\beta\alpha^{-1}\beta^{-1} = (152) \in (3^1 1^2)$ has the desired ramification type.

Example 6.2. Consider another octagon as follows.

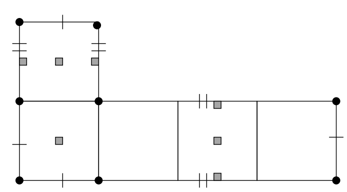


The area is still 5. This time we get $\alpha = (12435)$ and $\beta = (12345)$. So $\alpha\beta\alpha^{-1}\beta^{-1} = (134) \in (3^1 1^2)$ has the required ramification type.

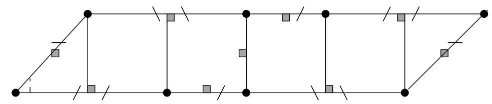
Remark 6.3. In [HL], the square-tiled octagons can be of two types: one-cylinder type and two-cylinder type. The one-cylinder type corresponds to $\beta \in (d^1)$ as in example 1, and the two-cylinder type corresponds to $\beta \in (l_1^{a_1} l_2^{a_2})$ as in example 2.

Moreover, also in [HL, Prop. 4.3], it is shown that we can mark the 6 Weierstrass points of C. 1 or 3 out of the 6 points are integer points, which provides two different parities invariant under the monodromy action. It follows immediately that $Y_{2,d,(3^1 1^{d-3})}$ has at least two components. On the other hand, by our method using S_d, one can check that 1 and 3 integer Weierstrass points correspond to $\langle \alpha, \beta \rangle = S_d$ and A_d respectively.

For instance, in the first example, $\langle \alpha, \beta \rangle = S_5$. Of course the ramification point is one Weierstrass point, and we mark the others with $\square$ in the following picture. Note that only the ramification point is an integer Weierstrass point.



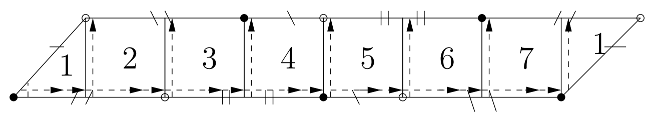
In the second example, $\langle \alpha, \beta \rangle = A_5$. From the picture below, we can see that there are exactly 3 integer Weierstrass points.



6.2 $g = 2, \sigma = (2^2 1^{d-4})$

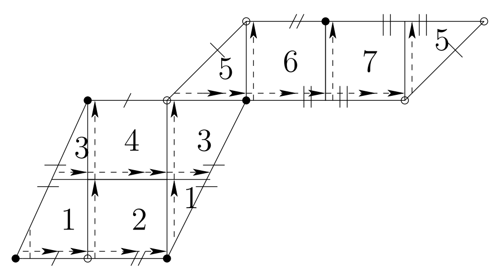
In this case we will have 2 ramification points on C. We mark them with $\bullet$ and $\circ$ respectively. As in [CTM2], C can be realized as a decagon. Let us look at some examples.

Example 6.4. Consider a decagon in the following picture and mark the unit squares.



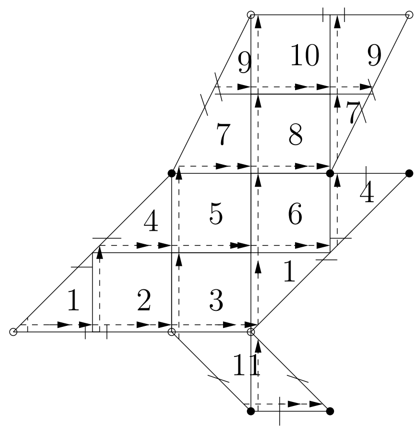
This case $d = 7$. We get $\alpha = (1264537)$ and $\beta = (1234567)$. So $\alpha\beta\alpha^{-1}\beta^{-1} = (16)(25) \in (2^2 1^3)$. Note that β is of type (d^1). In general, this is the one-cylinder type corresponding to $\beta \in (d^1)$ in our previous discussion.

Example 6.5. Consider another decagon with area 7.



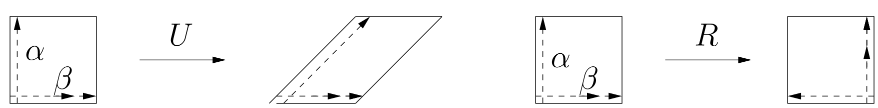
$\alpha = (1357624)$ and $\beta = (12)(34)(567)$ so $\alpha\beta\alpha^{-1}\beta^{-1} = (16)(25) \in (2^2 1^3)$. In general, this is the two-cylinder type corresponding to $\beta \in (l_1^{a_1} l_2^{a_2})$.

Example 6.6. For the case $\beta \in (l_1^{a_1} l_2^{a_2} l_3^{a_3})$, we can consider an example such as the following.

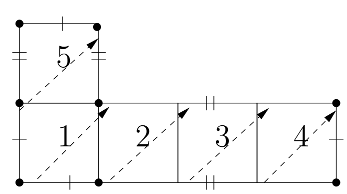


$\alpha = (168\ 10)(24\ 11\ 3579)$ and $\beta = (123)(456)(78)(9\ 10)$. So $\alpha\beta\alpha^{-1}\beta^{-1} = (13)(7\ 11) \in (2^2 1^7)$. In general, this corresponds to the three-cylinder type.

Remark 6.7. In [HL, 3.4], there are two actions U and R defined by the following pictures.



Actually they correspond to our monodromy actions $(\alpha, \beta) \to (\alpha\beta, \beta)$ and $(\alpha, \beta) \to (\beta^{-1}, \alpha)$ respectively. For instance, in the first example, applying the action U, the direction of β does not change but α changes to a direction α' parallel to the diagonal.



After the action U, we get $\alpha' = (12345) = (15) \cdot (1234) = \alpha\beta$. Hence, the method in [HL] to work out the number of components of Y by the monodromy actions can be similarly carried out here.

The reader may also be aware of the correspondence between our monodromy actions and the butterfly moves defined in [CTM1]. Actually, the result in [CTM1, Thm. 1.1] is more general, not only for d prime. We simply cite the result as the following theorem.

Theorem 6.8. *The Teichmüller curve* $Y_{2,d,(3^1 1^{d-3})}$ *is irreducible for* d *even or* $d = 3$*, and has exactly two components for* $d > 3$ *odd.*

However, to the best of the author's knowledge, for the case $\sigma = (2^2 1^{d-4})$ the question below is still unknown.

Question 6.9. *How many irreducible components does the curve* $Y_{2,d,(2^2 1^{d-4})}$ *have?*

It is relatively easy to get a lower bound for the number of components. For instance, when $d > 5$ is odd, pick a solution pair $\alpha = (1352467 \cdots d), \beta = (12)(34)$. Then $\alpha\beta\alpha^{-1}\beta^{-1} = (12)(56) \in \sigma = (2^2 1^{d-4})$, and $\langle \alpha, \beta \rangle$ is a subgroup of A_d. We can take another solution pair $\alpha = (13245 \cdots d), \beta = (12)$. Then $\alpha\beta\alpha^{-1}\beta^{-1} = (12)(34) \in \sigma = (2^2 1^{d-4})$, and $\langle \alpha, \beta \rangle = S_d$. Hence, in this case $Y_{2,d,(2^2 1^{d-4})}$ has at least two components.

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