HILBERT SCHEME OF A PAIR OF CODIMENSION TWO LINEAR SUBSPACES

DAWEI CHEN, IZZET COSKUN, AND SCOTT NOLLET

Abstract. We study the component H_n of the Hilbert scheme whose general point parameterizes a pair of codimension two linear subspaces in $\mathbb{P}^n$ for $n \geq 3$. We show that H_n is smooth and isomorphic to the blow-up of the symmetric square of $\mathbb{G}(n-2, n)$ along the diagonal. Further H_n intersects only one other component in the full Hilbert scheme, transversely. We determine the stable base locus decomposition of its effective cone and give modular interpretations of the corresponding models, hence conclude that H_n is a Mori dream space.

1. Introduction

The Hilbert scheme $\mathrm{Hilb}^{p(m)}(\mathbb{P}^n)$ parameterizes closed subschemes in $\mathbb{P}^n$ with fixed Hilbert polynomial $p(m)$. Grothendieck [G] proved that $\mathrm{Hilb}^{p(m)}(\mathbb{P}^n)$ exists as a projective scheme and Hartshorne [Ha] showed that it is connected. In general, $\mathrm{Hilb}^{p(m)}(\mathbb{P}^n)$ can be very complicated, possibly having many components of various dimensions or generically non-reduced components. Investigating the geometry of particular components has been one of the main themes in the study of the Hilbert schemes. For example, Piene and Schlessinger showed that $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$ has two smooth components which meet transversely and gave an explicit description of the component whose general member is a twisted cubic curve [PS].

In this paper, we study the component of the Hilbert scheme whose general point parameterizes a pair of codimension two linear subspaces in $\mathbb{P}^n$. Let X be a pair of general codimension two linear subspaces Λ_{n-2} and Λ'_{n-2} in $\mathbb{P}^n$ that intersect along a codimension four linear subspace Λ_{n-4}. The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\Lambda_{n-2}} \oplus \mathcal{O}_{\Lambda'_{n-2}} \to \mathcal{O}_{\Lambda_{n-4}} \to 0$$

implies that X has the Hilbert polynomial

$$P_n(m) = 2\binom{n-2+m}{m} - \binom{n-4+m}{m}.$$

Since a degree two irreducible, reduced, codimension two subscheme of $\mathbb{P}^n$ is contained in a hyperplane but X is not, there exists an irreducible component H_n of the Hilbert scheme $\mathrm{Hilb}^{P_n}(\mathbb{P}^n)$ whose general point parameterizes X.

For $n = 2$, it is well-known that H_2 is the full Hilbert scheme $\mathrm{Hilb}^2(\mathbb{P}^2)$ parameterizing length-2 zero dimensional subschemes of $\mathbb{P}^2$ and is isomorphic to the blow-up of $\mathrm{Sym}^2\mathbb{P}^2$ along the diagonal. For $n = 3$, the Hilbert polynomial of a pair of skew lines in $\mathbb{P}^3$ is $2m+2$. The structure of $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ was sketched in [H, 1.b] and elaborated in [L, 3.5, 4.2]. It consists of two irreducible components H_3 and H'_3, of respective dimensions 8 and 11. The general point of H_3 parameterizes a pair of skew lines while the general point of H'_3 parameterizes a plane conic union an isolated point. In Theorem 1.1 below, we prove that both H_3 and H'_3 are smooth, as suggested in [H, 1.b] and [L, Conjecture 3.5.10].

2000 *Mathematics Subject Classification.* Primary: 14E05, 14E30, 14M15, 14D22.

During the preparation of this article the first author was partially supported by the 2009 Spring Algebraic Geometry Program at MSRI and the second author was partially supported by the NSF grant DMS-0737581 and an Alfred P. Sloan Foundation Fellowship.

Moreover, in spite of the rapid growth in the number of irreducible components of $\mathrm{Hilb}^{P_n(m)}(\mathbb{P}^n)$ for $n > 3$ (Remark 2.7), we provide a good understanding of the component H_n for all n, showing it to be smooth, isomorphic to the blow-up of $\mathrm{Sym}^2\mathbb{G}(n-2,n)$ along the diagonal, and completely working out its Mori theory.

Theorem 1.1. *Let $n \geq 3$ be an integer.*

(1) A subscheme parameterized by H_n is projectively equivalent to one of the following four types:

(I) A pair of codimension two linear subspaces intersecting along a codimension four linear subspace.

(II) A pure double structure supported on a codimension two linear subspace.

(III) A pair of codimension two linear subspaces intersecting along a codimension three linear subspace with an embedded component determined by the square of the ideal of the intersection.

(IV) A double structure contained in a hyperplane and supported on a codimension two linear subspace with an embedded component determined by the square of the ideal of a codimension three linear subspace.

The loci (I), (II), (III) and (IV) have dimensions $4n-4$, $4n-5$, $3n-2$ and $3n-3$, respectively. The closure of (I) is H_n. The closures of (II) and (III) intersect along (IV).

(2) In the full Hilbert scheme $Hilb^{P_n}(\mathbb{P}^n)$, H_n intersects only one other component H'_n of dimension $7n-10$ whose general point parameterizes a quadric $(n-2)$-fold Q union a codimension three linear subspace Λ_{n-3}, where $Q \cap \Lambda_{n-3}$ is a codimension four linear subspace. Moreover, H_n and H'_n intersect transversely along the loci (III) $\cup$ (IV).

(3) The component H_n is smooth and isomorphic to the blow-up of $Sym^2\mathbb{G}(n-2,n)$ along the diagonal. For $n = 3$, the other component H'_3 is smooth and isomorphic to the blow-up of $\mathbb{P}^3 \times Hilb^{2m+1}(\mathbb{P}^3)$ along the incidence correspondence $\{p \in C\}$, where p denotes a point in $\mathbb{P}^3$ and C denotes a conic parameterized by $Hilb^{2m+1}(\mathbb{P}^3)$.

To study the Mori theory of H_n, we introduce the following divisor classes on H_n.

Definition 1.2. Let $n \geq 3$ be an integer.

Let M be the divisor class of the locus of subschemes that intersect a fixed line.

Let N be the divisor class of the locus of generically non-reduced subschemes. By Theorem 1.1, N consists of the loci (II) and (IV).

Consider the locus of subschemes whose intersection with a fixed plane consists of two points, which are collinear with a fixed point on that plane. Let F be the divisor class parameterizing the closure of this locus in H_n.

Let E be the divisor class of the locus of subschemes such that the intersection of the two subspaces in the pair intersects a fixed $\mathbb{P}^3$. For $n = 3$, E parameterizes the locus of two incident lines with a spatial embedded point at their intersection.

Since H_n is smooth, the Weil divisors defined above are Cartier. For a divisor D, let $\mathbf{B}(\mathrm{D})$ be its stable base locus. Denote by $[D_1, D_2]$, (D_1, D_2) and $[D_1, D_2)$ the convex cones consisting of divisors of type $aD_1 + bD_2$, where $a, b \geq 0$, $a, b > 0$ and $a > 0, b \geq 0$, respectively. Our next result describes the stable base locus decomposition for the effective cone of H_n.

Theorem 1.3. *Let $n \geq 3$ be an integer.*

(1) The Picard group of H_n is generated by M and F. The divisor class N is linearly equivalent to $2M - 2F$. The divisor class E is linearly equivalent to $2F - M$. Moreover, two divisors on H_n are linearly equivalent iff they are numerically equivalent.

(2) The ample cone of H_n is (F, M). The effective cone of H_n is $[N, E]$. For a divisor D in the chamber $[F, M]$, D is base-point-free. For D in the chamber $(M, N]$, $\mathbf{B}(D)$ consists of N, which is equal to the loci (II) and (IV). For D in the chamber $[E, F)$, $\mathbf{B}(D)$ consists of the loci (III) and (IV). The moving cone of H_3 is $[F, M]$ and the moving cone of H_n is $[E, M]$ for $n \geq 4$.

Definition 1.4. For an effective divisor D on a variety X, let $P(D)$ denote its Proj model

$$\mathrm{Proj}\big(\bigoplus_{m\geq 0} H^0(X, mD)\big)$$

assuming the section ring of D is finitely generated. Let $\psi_D\colon X \dashrightarrow P(D)$ (morphism or rational map) denote the map induced by D.

In order to describe all possible models of H_n, we define two spaces Ψ_n and Θ_n as follows.

Definition 1.5. Let Ψ_n denote the $\mathbb{G}(3,5)$ bundle over $\mathbb{G}(3,n)$ whose fiber over a base point $[\Lambda_3]$ parameterizes codimension two linear sections of the Plücker embedding of the Grassmannian of lines in Λ_3. In particular, Ψ_3 is isomorphic to $\mathbb{G}(3,5)$.

Consider the Plücker embedding $\mathbb{G}(n-2,n) \cong \mathbb{G}(1,n) \hookrightarrow \mathbb{P}^N$. There is a subset of $\mathbb{G}(N-2,N)$ parameterizing codimension two linear sections of $\mathbb{G}(n-2,n) \cong \mathbb{G}(1,n)$ that are the intersections of two Schubert varieties $\Sigma_1 \cap \Sigma_1'$. Let Θ_n denote (the normalization of) the closure of this subset in $\mathbb{G}(N-2,N)$.

The reader can refer to Remark 3.10 for another geometric interpretation of Ψ_n and Θ_n.

Our third result describes the model $P(D)$ for any effective divisor D on H_n. Since the locus (III) is divisorial iff $n=3$, the results for $n=3$ and $n\geq 4$ are slightly different.

Theorem 1.6. *Let $n \geq 3$ be an integer. Let D denote an effective divisor on H_n.*

(1) For D in the chamber (F, M), the model $P(D)$ is isomorphic to H_n.

(2) For D in the chamber $[M, N)$, the morphism ψ_D contracts the loci (II) and (IV). The resulting model $P(D)$ is isomorphic to $Sym^2\mathbb{G}(n-2,n)$.

(3) The morphism ψ_F contracts the loci (III) and (IV). The model $P(F)$ is isomorphic to Θ_n.

(4) For $n=3$ and D in the chamber $(E, F]$, the morphism ψ_D contracts the divisor E and $P(D)$ is isomorphic to $\Psi_3 \cong \mathbb{G}(3,5)$.

(5) For $n \geq 4$ and D in the chamber (E, F), the birational map ψ_D is a flip over Θ_n and the flipping space $P(D)$ is isomorphic to Ψ_n. Further the birational transform of E on Ψ_n induces a morphism that contracts the $\mathbb{G}(3,5)$ bundle structure to the base $\mathbb{G}(3,n)$.

A variety X is called a Mori dream space if Mori's program can be carried out for every effective divisor on X [HK]. By Theorems 1.3 and 1.6, this holds for H_n.

Corollary 1.7. *The Hilbert component H_n is a Mori dream space.*

This corollary also follows from [BCHM, Corollary 1.3.1] in view of Propositions 3.3 and 3.11.

This paper is organized as follows. In section 2, we describe the stratification of H_n and prove Theorem 1.1. In section 3, we study the divisor theory of H_n and prove Theorems 1.3, 1.6. Throughout the paper, we work over an algebraically closed field k of characteristic zero. We often denote an m-dimensional linear subspace of $\mathbb{P}^n$ by Λ_m. The ideal of a subscheme of $\mathbb{P}^n$ is always saturated. All divisors considered here are Cartier.

Acknowledgements. We would like to thank Lawrence Ein, Tommaso de Fernex, Joe Harris, Anatoly Libgober, Mihnea Popa and Christian Schnell for useful conversations related to this paper. Part of this work was done when the first two authors were visiting MSRI for the Spring 2009 Algebraic Geometry Program. They would like to thank MSRI for providing support and a wonderful, stimulating work environment.

2. Description of H_n

In this section, let $n \geq 3$ denote an integer. Let $S = k[x_0, \ldots, x_n]$ denote the coordinate ring of $\mathbb{P}^n$. We begin by determining the double structures of pure dimension supported on a codimension two linear subspace of $\mathbb{P}^n$. The following lemma generalizes the classification of double lines given in $\mathbb{P}^3$ [M, N].

Lemma 2.1. *Let X be a pure codimension two subscheme of $\mathbb{P}^n$ which is a double structure supported on a codimension two linear subspace Λ_{n-2}: $x_0 = x_1 = 0$. Then the ideal of X can be written as $(x_0^2, x_0x_1, x_1^2, x_0G - x_1F)$, where F and G are degree $k \geq 0$, relatively prime homogeneous polynomials in $x_2, \ldots, x_n$. Moreover, the Hilbert polynomial of X equals P_n iff F and G are linear.*

Proof. The ideal I_X of X satisfies $I^2_{\Lambda_{n-2}} \subset I_X \subset I_{\Lambda_{n-2}}$. Therefore, I_X contains the ideal (x_0^2, x_0x_1, x_1^2) and a polynomial of type $x_0G_0 - x_1F_0$, where F_0 and G_0 are homogenous polynomials of the same degree in $x_2, \ldots, x_n$. Suppose $G_0 = GH$ and $F_0 = FH$, where F, G are relatively prime of degree $k \geq 0$. Then the ideal $((x_0, x_1)^2, x_0G - x_1F)$ is the total ideal for a double structure Y on Λ_{n-2} without embedded components. Away from the proper closed subset $H = 0$, we have the containment $X \subset Y$, and therefore $X = Y$ since X has no embedded components. Moreover, $I_X = I_Y = ((x_0, x_1)^2, x_0G - x_1F)$.

Let us compute the Hilbert polynomial of I_X. For $m \gg 0$, an element $H \in (S/I_X)_m$ can be written as $A + x_0B_0 - x_1B_1$, where A and B_i are homogeneous polynomials of degree m and $m-1$, respectively, in $x_2, \ldots, x_n$. Moreover, $x_0B_0 - x_1B_1$ is divisible by $x_0G - x_1F$ iff $B_0 = CG$ and $B_1 = CF$ for a degree $m-1-k$ homogeneous polynomial C in $x_2, \ldots, x_n$. Hence, we know dim $(S/I_X)_m = \binom{n-2+m}{m} + 2\binom{n-2+m-1}{m-1} - \binom{n-2+m-1-k}{m-1-k}$, which equals P_n iff $k = 1$. □

Now we classify all subschemes parameterized by H_n up to projective equivalence.

Proof of Theorem 1.1 (1). We want to show any subscheme X parameterized by H_n belongs to one of the four loci. Let I_X denote the ideal of X. Note that X_{red} has degree two or one. In the former case, X_{red} consists of a pair of codimension two linear subspaces. If their intersection has dimension $n-4$, then the Hilbert polynomial of X_{red} equals P_n, hence $X = X_{red}$. Without loss of generality, assume I_X is $(x_0x_2, x_0x_3, x_1x_2, x_1x_3)$, namely, X consists of two linear subspaces $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$. This corresponds to the locus (I).

If the two components of X_{red} intersect along a codimension three linear subspace, without loss of generality, assume $I_{X_{red}}$ is (x_0, x_1x_2), namely, it consists of two linear subspaces $x_0 = x_1 = 0$ and $x_0 = x_2 = 0$. Note that I_X is contained in $I_{X_{red}}$. Since a one dimensional flat family in (I) specializes to X, there is a spatial embedded component of X whose support is contained in the linear intersection of the two components of X_{red}. Then I_X is contained in $(x_0, x_1x_2) \cap (x_0, x_1, x_2)^2 = (x_0^2, x_0x_1, x_0x_2, x_1x_2)$, whose Hilbert polynomial equals P_n. Hence, I_X equals $(x_0^2, x_0x_1, x_0x_2, x_1x_2)$ and X has an embedded component supported on the codimension three linear intersection of the two components. This corresponds to the locus (III). The embedded structure is uniquely determined by the square of the ideal of the codimension three linear intersection. One can take a family with general member $(x_0, x_1) \cap (x_0 + tx_3, x_2)$ whose flat limit is X. Hence, the locus of (III) is in the closure of the locus (I).

If X_{red} has degree one, it is a codimension two linear subspace. Hence, X is a generically double structure supported on X_{red}. Suppose X_{red} is defined by $x_0 = x_1 = 0$. Let X' be the non-reduced subscheme of X of pure dimension $n-2$ supported on X_{red}. By Lemma 2.1, $I_{X'}$ equals $(x_0^2, x_0x_1, x_1^2, x_0G - x_1F)$, where F and G are degree k homogeneous polynomials in $x_2, \ldots, x_n$ without common factors. Since $h^0(I_X(2))$ has dimension ≥ 4 by semi-continuity and I_X is contained in $I_{X'}$, we know F, G must be linear and I_X equals $(x_0^2, x_0x_1, x_1^2, x_0G - x_1F)$, whose Hilbert polynomial is P_n by Lemma 2.1. This corresponds to the locus (II). One can take a flat family with general member $(x_0, x_1) \cap (x_0 + tF, x_1 + tG)$ that specializes to X, where F and G are defined as above. Hence, the locus of (II) is in the closure of the locus (I).

If F and G are linearly dependent, one can assume $F = G = x_2$. Since I_X contains $(x_0^2, x_0x_1, x_1^2, x_0x_2 - x_1x_2)$, whose Hilbert polynomial equals P_n, I_X must equal $(x_0^2, x_0x_1, x_1^2, x_0x_2 - x_1x_2) = (x_0 - x_1, x_0^2) \cap$

$(x_0, x_1, x_2)^2$. From this expression, we see that X consists of a double structure contained in the hyperplane $x_0 - x_1 = 0$ along with an embedded component supported on the codimension three linear subspace $x_0 = x_1 = x_2 = 0$. This corresponds to the locus (IV).

The dimension counts for the above loci are standard. The locus (I) is open and dense in H_n. We have seen that subschemes of type (II) can degenerate to (IV). For an ideal $(x_0^2, x_0x_1, x_0x_2, x_1x_2)$ of type (III), one can replace x_2 by $x_1 + tx_2$. The flat limit lies in (IV). So subschemes of type (III) can also degenerate to (IV). □

Let Δ denote the diagonal of $\mathrm{Sym}^2\mathbb{G}(n-2, n)$. One can regard $\mathrm{Sym}^2\mathbb{G}(n-2, n)$ as the Chow variety parameterizing a pair of codimension two and degree one cycles in $\mathbb{P}^n$. Let $\mathrm{Bl}_\Delta\mathrm{Sym}^2\mathbb{G}(n-2, n)$ denote the blow-up of $\mathrm{Sym}^2\mathbb{G}(n-2, n)$ along the diagonal.

Lemma 2.2. *$Bl_\Delta Sym^2\mathbb{G}(n-2, n)$ is a smooth variety.*

Proof. Let X be a nonsingular variety. Denote by Y the blow-up of $X \times X$ along the diagonal. There is a natural involution acting on Y and the quotient space is $\mathrm{Bl}_\Delta\mathrm{Sym}^2X$. Since the smooth exceptional divisor is the fixed locus, we conclude that $\mathrm{Bl}_\Delta\mathrm{Sym}^2X$ is smooth. In particular, $\mathrm{Bl}_\Delta\mathrm{Sym}^2\mathbb{G}(n-2, n)$ is smooth. □

Proposition 2.3. *There is a bijective morphism δ: $Bl_\Delta Sym^2\mathbb{G}(n-2, n) \to H_n$.*

Proof. A generically reduced subscheme parameterized by H_n is uniquely determined by a pair of codimension two linear subspaces, cf. Theorem 1.1 (1). Hence, there is a natural bijection between $\mathrm{Sym}^2\mathbb{G}(n-2, n)\backslash\Delta$ and $H_n\backslash(\mathrm{II}) \cup (\mathrm{IV})$. Fix a codimension two linear subspace Λ_{n-2} given by $x_0 = x_1 = 0$ and consider another linear subspace $x_0 + tF = x_1 + tG = 0$ approaching Λ_{n-2} as $t \to 0$, where F and G are linear functions in $x_2, \ldots, x_n$. Note that (F, G) can be regarded as an element $\phi \in \mathrm{Hom}\ (\mathbb{A}^{n-1}, \mathbb{A}^{n+1}/\mathbb{A}^{n-1})$ of the tangent space of $\mathbb{G}(n-2, n)$ at $[\Lambda]$. We have seen in the proof of Theorem 1.1 (1) that if F and G are linearly independent, the limit scheme is uniquely determined in (II). If F and G are dependent, then the limit scheme is determined in (IV). Using the universal property of the Hilbert scheme, we thus obtain the desired bijective morphism. □

Next, we will prove Theorem 1.1 (2) regarding the smoothness of H_n and how it intersects other components in the full Hilbert scheme. Note that if a point parameterizing a subscheme X is in the singular locus of H_n, then all the points parameterizing subschemes projectively equivalent to X lie in the singular locus of H_n. Since the locus (IV) is contained in the closure of the locus (I), (II) or (III) and each locus is homogeneous, if H_n is smooth along (IV), then it must be smooth everywhere. So it suffices to analyze the deformation space of a subscheme of type (IV). We invoke the following result, which transforms the study of the deformation of a subscheme to that of its ideal.

Theorem 2.4 (Comparison Theorem, [PS])**.** *If the ideal I defining a subscheme $X \subset \mathbb{P}^n$ is generated by homogeneous polynomials $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$, for which*

$$(k[x_0, \ldots, x_n]/I)_d \cong H^0(\mathcal{O}_X(d))$$

for $d = d_1, \ldots, d_r$, then there is an isomorphism between the universal deformation space of I and that of X.

From now on, fix a subscheme X of type (IV) with ideal $I = (x_0^2, x_0x_1, x_1^2, x_0x_2)$.

Lemma 2.5. *The hypothesis of the Comparison Theorem holds for X.*

Proof. Let $J = (x_0, x_1^2)$ be the double structure contained in $x_0 = 0$ and supported on $x_0 = x_1 = 0$. Let $K = J/I$, which is isomorphic to $S/(x_0, x_1, x_2)$ twisted by -1 as an S-module. Using the exact sequence

$$0 \to H^0_{\mathbf{m}}(M) \to M \to \bigoplus_d H^0(\widetilde{M}(d)) \to H^1_{\mathbf{m}}(M) \to 0$$

where M is a graded S-module and $\widetilde{M}$ is the corresponding quasicoherent sheaf on $\mathbb{P}^n$, we get the positive graded pieces of the local cohomology $H^i_{\mathbf{m}}(K)$ and $H^i_{\mathbf{m}}(S/J)$ are vanishing for $i = 0, 1$. Then the local cohomology sequence associated to the exact sequence

$$0 \to K \to S/I \to S/J \to 0$$

shows that $(S/I)_d \to H^0(\mathcal{O}_X(d))$ is an isomorphism for all $d > 0$. Therefore, the completion of $\mathrm{Hilb}^{P_n}(\mathbb{P}^n)$ at $[X]$ can be identified as the universal deformation space of the ideal I of X. □

We follow the method in [PS] to write down a universal deformation space of the ideal I.

Proposition 2.6. *The tangent space of $\mathrm{Hilb}^{P_n}(\mathbb{P}^n)$ at $[X]$ has dimension $8n-12$. The ideal I of X has a universal deformation space of type $\mathbb{A}^{4n-4} \cup \mathbb{A}^{7n-10}$, where the two components intersect transversely along $\mathbb{A}^{3n-2}$.*

Proof. By Lemma 2.5, the tangent space of $\mathrm{Hilb}^{P_n}(\mathbb{P}^n)$ at $[X]$ can be identified as $\mathrm{Hom}_S(I/I^2, S/I)_0$. Consider the following presentation of S/I over $S = k[x_0, \ldots, x_n]$:

$$0 \longrightarrow S(-4) \xrightarrow{\ \nu\ } S(-3)^4 \xrightarrow{\ \mu\ } S(-2)^4 \xrightarrow{\ \lambda\ } S \longrightarrow S/I \longrightarrow 0,$$

where the maps are given by

$$\lambda = (x_0x_1, x_0x_2, x_0^2, x_1^2),\ \ \mu = \begin{pmatrix} x_1 & x_2 & x_0 & 0 \\ 0 & -x_1 & 0 & x_0 \\ 0 & 0 & -x_1 & -x_2 \\ -x_0 & 0 & 0 & 0 \end{pmatrix},\ \ \nu = \begin{pmatrix} 0 \\ x_0 \\ -x_2 \\ x_1 \end{pmatrix}$$

An element $\phi \in \mathrm{Hom}_S(I/I^2, S/I)_0$ satisfies

$$x_1\phi(x_0x_1) = x_0\phi(x_1^2),\ \ x_2\phi(x_0x_1) = x_1\phi(x_0x_2),$$
$$x_0\phi(x_0x_1) = x_1\phi(x_0^2),\ \ x_0\phi(x_0x_2) = x_2\phi(x_0^2)$$

modulo I. Then one can check that $\mathrm{Hom}_S(I/I^2, S/I)_0$ is generated by the following elements:

$$\phi(x_0x_1) = x_0\sum_{i\geq 3} a_ix_i + x_1\sum_{i\geq 2} b_ix_i,$$

$$\phi(x_0x_2) = x_0\sum_{i\geq 3} c_ix_i + x_1\sum_{i\geq 2} d_ix_i + x_2\sum_{i\geq 2} b_ix_i,$$

$$\phi(x_0^2) = x_0\sum_{i\geq 3} e_ix_i,$$

$$\phi(x_1^2) = x_0\sum_{i\geq 3} f_ix_i + x_1\sum_{i\geq 2} g_ix_i + x_2\sum_{i\geq 2} h_ix_i,$$

where $a_i, b_i, \ldots, h_i$ are independent parameters. Hence, $\mathrm{Hom}_S(I/I^2, S/I)_0$ has dimension $8n-12$.

Let us write down a group of generators for $\mathrm{Hom}_S(I/I^2, S/I)_0$. For $3 \leq i \leq n$, let

$$\frac{\partial}{\partial t_{0i}} = x_i\frac{\partial}{\partial x_0} = \begin{pmatrix} x_1x_i \\ x_2x_i \\ 2x_0x_i \\ 0 \end{pmatrix},\ \ \frac{\partial}{\partial t_{1i}} = x_i\frac{\partial}{\partial x_1} = \begin{pmatrix} x_0x_i \\ 0 \\ 0 \\ 2x_1x_i \end{pmatrix},\ \ \frac{\partial}{\partial t_{2i}} = x_i\frac{\partial}{\partial x_2} = \begin{pmatrix} 0 \\ x_0x_i \\ 0 \\ 0 \end{pmatrix}$$

Also let

$$\frac{\partial}{\partial t_{01}} = x_1\frac{\partial}{\partial x_0} = \begin{pmatrix} 0 \\ x_1x_2 \\ 0 \\ 0 \end{pmatrix},\ \ \frac{\partial}{\partial t_{02}} = x_2\frac{\partial}{\partial x_0} = \begin{pmatrix} x_1x_2 \\ x_2^2 \\ 0 \\ 0 \end{pmatrix},\ \ \frac{\partial}{\partial t_{12}} = x_2\frac{\partial}{\partial x_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2x_1x_2 \end{pmatrix}$$

Note that X uniquely determines a $(\Lambda_{n-3} \subset \Lambda_{n-2} \subset \Lambda_{n-1})$ flag and vice versa, where Λ_k is a k-dimensional linear subspace of $\mathbb{P}^n$. The above $3n-3$ elements $\frac{\partial}{\partial t_{ij}}, 0 \le i \le 2, i < j \le n$ provide the trivial deformations for the ideal I of X, which correspond to moving the flag determined by X.

Moreover, for $i \geq 3$, let

$$\frac{\partial}{\partial u_{1i}} = \begin{pmatrix} 0 \\ x_1x_i \\ 0 \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial u_{2i}} = \begin{pmatrix} 0 \\ 0 \\ x_0x_i \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial u_{3i}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1x_i \end{pmatrix},$$

$$\frac{\partial}{\partial u_{4i}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2x_i \end{pmatrix}, \quad \frac{\partial}{\partial u_{5i}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_0x_i \end{pmatrix}, \quad \frac{\partial}{\partial u_6} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2^2 \end{pmatrix}$$

These $\frac{\partial}{\partial u_{ij}}, 1 \le i \le 5, 3 \le j \le n$ and $\frac{\partial}{\partial u_6}$ provide a versal deformation space for I. Along with those $\frac{\partial}{\partial t_{ij}}$, they form a basis for $\mathrm{Hom}_S(I/I^2, S/I)_0$. For $1 \le i \le 5$, define

$$v_i = \sum_{j \geq 3} u_{ij} x_j.$$

Consider the following homogeneous perturbations of λ, μ and ν:

$$\lambda' = (x_0x_1 - u_6x_2v_1,\ x_0x_2 + x_1v_1,\ x_0^2 + x_0v_2 + u_6v_1^2,$$
$$x_1^2 + x_1v_3 + x_2v_4 + x_0v_5 + u_6x_2^2 + v_2v_5),$$

$$\mu' = \begin{pmatrix} x_1 + v_3 & x_2 & x_0 + v_2 & -v_1 \\ v_4 + u_6x_2 & -x_1 & u_6v_1 & x_0 + v_2 \\ v_5 & 0 & -x_1 & -x_2 \\ -x_0 & v_1 & 0 & 0 \end{pmatrix},$$

$$\nu' = \begin{pmatrix} v_1 \\ x_0 + v_2 \\ -x_2 \\ x_1 \end{pmatrix}$$

Differentiating the above in terms of u_{ij}, we get the deformation corresponding to $\frac{\partial}{\partial u_{ij}}$. Note that $\lambda' \cdot \mu' \equiv \mu' \cdot \nu' \equiv 0 \bmod (v_1v_2, v_1v_3, v_1v_4, v_1v_5)$ i.e. mod $(u_{1i}u_{2j}, \ldots, u_{1i}u_{5j})$ for $3 \le i, j \le n$ and no higher order terms arise in these relations. Hence, the versal deformation space of I is isomorphic to

$$\mathrm{Spec}\big(k[u_{1i}, \ldots, u_{5i}, u_6]/(u_{1i}u_{2j}, \ldots, u_{1i}u_{5j})\big).$$

To add the trivial deformations corresponding to $\frac{\partial}{\partial t_{ij}}$, we can take

$$x_0 = x_0 + \sum_{i \geq 1} t_{0i}x_i,\ x_1 = x_1 + \sum_{i \geq 2} t_{1i}x_i,\ x_2 = x_2 + \sum_{i \geq 3} t_{2i}x_i.$$

Hence, the universal deformation of I is given by

$$\mathrm{Spec}\big(k[u_{1i}, \ldots, u_{5i}, u_6, t_{ij}]/(u_{1i}u_{2j}, \ldots, u_{1i}u_{5j})\big).$$

It is isomorphic to $\mathbb{A}^{4n-4} \cup \mathbb{A}^{7n-10}$, where $\mathbb{A}^{4n-4}$ has coordinates $u_{1i}, 3 \le i \le n, u_6, t_{ij}, 0 \le i \le 2, i < j \le n$ and $\mathbb{A}^{7n-10}$ has coordinates $u_{ij}, 2 \le i \le 5, 3 \le j \le n, u_6, t_{ij}, 0 \le i \le 2, i < j \le n$. They intersect transversely along $\mathbb{A}^{3n-2}$, whose coordinates are given by $u_6, t_{ij}, 0 \le i \le 2, i < j \le n$. □

Now we are ready to prove Theorem 1.1 (2).

Proof of Theorem 1.1 (2). By Proposition 2.6, the universal deformation space of the ideal I of X has two components. The first $\mathbb{A}^{4n-4}$ corresponds to the deformation of X along the $(4n-4)$-dimensional Hilbert component H_n. The second $\mathbb{A}^{7n-10}$ implies there exists another Hilbert component of dimension at most $7n-10$, which also contains the locus (IV). Below we will describe that component.

Consider the incidence correspondence $\Sigma = \{(Q, \Lambda_{n-3}, \Lambda_{n-4})\}$, where Q is a quadric $(n-2)$-fold in $\mathbb{P}^n$, Λ^k denotes a k-dimensional linear subspace and $\Lambda_{n-4} = Q \cap \Lambda_{n-3}$. Using the projection $\Sigma \to \mathbb{G}(n-4, n)$, Σ is irreducible and has dimension $7n-10$. For an element parameterized by Σ, let X' be the subscheme $Q \cup \Lambda_{n-3}$ in $\mathbb{P}^n$. By the exact sequence

$$0 \to \mathcal{O}_{X'} \to \mathcal{O}_Q \oplus \mathcal{O}_{\Lambda_{n-3}} \to \mathcal{O}_{\Lambda_{n-4}} \to 0,$$

the Hilbert polynomial of X' equals P_n. Hence, there is a component H'_n of $\text{Hilb}^{P_n}(\mathbb{P}^n)$ that parameterizes X'. We will show that X' can specialize to the subschemes of type (III). Without loss of generality, assume the hyperplane spanned by Q is $x_0 = 0$ and Λ_{n-3} is given by $x_1 = x_2 = x_3 = 0$. Then Λ_{n-4} is specified by $x_0 = x_1 = x_2 = x_3 = 0$. Let Q degenerate to a pair of codimension two linear subspaces in $x_0 = 0$, say, it has ideal (x_0, x_1x_2). Then let Λ_{n-3} approach the intersection $x_0 = x_1 = x_2 = 0$ of the two subspaces by writing the ideal of Λ_3 as $(x_1, x_2, tx_3 + (1-t)x_0)$. For general t, the union of Q and Λ_{n-3} has ideal $(x_0, x_1x_2) \cdot (x_1, x_2, tx_3 + (1-t)x_0)$. As $t \to 0$, we see the limit ideal must contain $(x_0^2, x_0x_1, x_0x_2, x_1x_2) = (x_0, x_1x_2) \cap (x_0, x_1, x_2)^2$, which defines a subscheme with Hilbert polynomial P_n parameterized by the locus (III). Hence, the limit ideal equals $(x_0^2, x_0x_1, x_0x_2, x_1x_2)$ and H'_n contains the locus (III). Geometrically, the approaching direction of Λ_{n-3} provides the embedded structure supported on the codimension three linear subspace $x_0 = x_1 = x_2 = 0$. Since subschemes of type (III) can specialize to type (IV), we know H'_n also contains the locus (IV).

Since a subscheme of type (II) does not possess a $(n-3)$-dimensional component, H'_n does not intersect the locus (II). Hence, $H_n \cap H'_n$ consists of (III) and (IV). Because the two components in the universal deformation space of a subscheme of type (IV) intersect transversely and (IV) is the specialization of (III), we know H_n and H'_n intersect transversely along (III) $\cup$ (IV). □

Remark 2.7. The number of irreducible components of $\text{Hilb}^{P_n}(\mathbb{P}^n)$ may increase rapidly with n.

For $n = 3$, there are only two components H_3 and H'_3.

For $n = 4$, there is one more component H''_4, whose general points parameterize a quadric surface Q and a line L_0 intersecting at two points along with an isolated point q. Since H_4 does not intersect H''_4, by Hartshorne's connectedness theorem [Ha], H'_4 necessarily intersects H''_4. We can see how they intersect as follows. On the one hand, for a quadric surface Q and a line L intersecting at a point in $\mathbb{P}^4$ parameterized by H'_4, let L degenerate to L_0, which intersects Q at two points. Then an embedded point will arise at an intersection point p to make the Hilbert polynomial correct. On the other hand, let the isolated point q approach p, whose limit will also yield the embedded structure at p. This shows how H'_4 and H''_4 intersect.

For $n = 5$, in addition to H_5 and H'_5, there are at least four other components of the full Hilbert scheme. The general points of the first one parameterize a quadric three-fold and a plane in a four dimensional linear subspace $\Lambda_4 \subset \mathbb{P}^5$, along with a line intersecting the quadric at a point. The second parameterizes a quadric three-fold and a plane in Λ_4, along with a line intersecting the plane at a point. The third parameterizes a quadric three-fold, a plane and a line in $\mathbb{P}^4$, along with an isolated point. The last one parameterizes a quadric three-fold, a plane and a line in $\mathbb{P}^4$ such that the line intersects the plane, along with two isolated points.

For $n = 6$, in the same way one can list more than twenty components.

Now we prove Theorem 1.1 (3).

Proof of Theorem 1.1 (3). By Proposition 2.6, the universal deformation space of a subscheme of type (IV) is isomorphic to $\mathbb{A}^{4n-4} \cup \mathbb{A}^{7n-10}$, where the two components correspond to deformations along H_n and H'_n, respectively. This shows the $(4n-4)$-dimensional component H_n and the $(7n-10)$-dimensional component H'_n are both smooth along the locus (IV). Since the closures of (I), (II) or (III) all contain (IV), H_n is smooth everywhere. By Zariski's Main Theorem, the bijective morphism δ in Proposition 2.3 is an isomorphism.

For $n = 3$, by [H, 1.b] and [L, 3.5, 4.2], $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ consists of two components H_3 and H'_3. The second component H'_3 parameterizes a conic union an isolated point. In [L, Theorem 3.5.1], a similar bijective morphism as Proposition 2.3 was established, σ: $\mathrm{Bl}_\Sigma(\mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3)) \to H'_3$, where Σ denotes the incidence correspondence $\{p \in C\}$ for a point p and a conic C. In order to show σ is an isomorphism, it suffices to prove the smoothness of H'_3. A subscheme C parameterized by H'_3 can specialize to a planar double line with an embedded point. If the embedded point is spatial, this is of type (IV). We have seen in the previous paragraph that H'_3 is smooth along (IV).

Let $k[x, y, z, w]$ denote the coordinate ring of $\mathbb{P}^3$. If the embedded point is also in that plane, the ideal of C is equivalent to $I = (z, xy^2, y^3)$. Let $\mathcal{N}$ and $\mathcal{N}'$ denote the normal sheaves of C in $\mathbb{P}^3$ and in the plane $z = 0$, respectively. Let $S' = k[x, y, w]$ and $I' = (xy^2, y^3)$ be the ideal of C in S'. We want to show $h^0(\mathcal{N}) = 11$. By the exact sequence

$$0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{O}_C(1) \to 0$$

and $h^0(\mathcal{O}_C(1)) = 4$, it suffices to show $h^0(\mathcal{N}') = 7$. One checks that the condition of Theorem 2.4 holds for C regarded as a subscheme of the plane $z = 0$. Then we only need to verify $\dim \mathrm{Hom}_{S'}(I', S'/I')_0 = 7$. An element ϕ in $\mathrm{Hom}_{S'}(I', S'/I')_0$ satisfies $y\phi(xy^2) = x\phi(y^3)$ modulo I'. One checks that

$$\phi(xy^2) = a_1x^2y + a_2x^3 + a_3w^2x + a_4wx^2 + a_5wxy + a_6wy^2,$$

$$\phi(y^3) = a_2x^2y + a_3w^2y + a_4wxy + a_7wy^2$$

with 7 parameters $a_1, \dots, a_7$ generate $\mathrm{Hom}_{S'}(I', S'/I')_0$. Hence, $\dim \mathrm{Hom}_{S'}(I', S'/I')_0 = 7$ and $h^0(\mathcal{N}) = 11$, which implies that the tangent space of H'_3 at $[C]$ has dimension 11. Combining with the previous paragraph, we know that H'_3 is smooth everywhere, so it is isomorphic to $\mathrm{Bl}_\Sigma(\mathbb{P}^3 \times \mathrm{Hilb}^{2m+1}(\mathbb{P}^3))$. □

Our complete analysis of $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ extends $\mathrm{Hilb}^{2m+2}(\mathbb{P}^n)$ with $n \geq 4$.

Corollary 2.8. *For $n \geq 4$, $\mathrm{Hilb}^{2m+2}(\mathbb{P}^n)$ consists of two components W_n and W'_n. The component W_n has dimension $4n-4$ and its general point parameterizes a pair of skew lines. The component W'_n has dimension $4n-1$ and its general point parameterizes a conic union an isolated point. Both W_n and W'_n are smooth. They intersect transversely along a $(4n-5)$-dimensional locus E_n whose general points parameterize a pair of coplanar lines with a spatial embedded point at their intersection. In particular, W_n is an H_3 bundle over $\mathbb{G}(3, n)$.*

Proof. Using the arguments in [PS, Lemma 1] and [L, Lemma 3.5.3], a subscheme C in $\mathbb{P}^n$ with Hilbert polynomial $2m+2$ is contained in a linear subspace $\mathbb{P}^3 \subset \mathbb{P}^n$. Hence, $\mathrm{Hilb}^{2m+2}(\mathbb{P}^n)$ has two components W_n and W'_n, whose general points parameterize a pair of skew lines and a conic union an isolated point, respectively. We have $\dim W_n = \dim \mathbb{G}(3, n) + \dim H_3 = 4n-4$ and $\dim W'_n = \dim \mathbb{G}(3, n) + \dim H'_3 = 4n-1$. Moreover, W_n and W'_n intersect along the $(4n-5)$-dimensional locus E_n whose general point parameterizes two incident lines with a spatial embedded point at their intersection.

For $C \subset \mathbb{P}^3 \subset \mathbb{P}^n$, let $\mathcal{N}$ and $\mathcal{N}'$ denote the normal sheaves of C in $\mathbb{P}^n$ and $\mathbb{P}^3$, respectively. By the exact sequence

$$0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}_{\mathbb{P}^3/\mathbb{P}^n}|_C \to 0,$$

we get $h^0(\mathcal{N}) = h^0(\mathcal{N}') + (n-3)h^0(\mathcal{O}_C(1)) = h^0(\mathcal{N}') + 4(n-3)$. Since $h^0(\mathcal{N}')$ equals the dimension of the tangent space of $\mathrm{Hilb}^{2m+2}(\mathbb{P}^3)$ at $[C]$, we know that $h^0(\mathcal{N})$ equals the dimension of W_n or W'_n for $[C] \in W_n \backslash E_n$ or $W'_n \backslash E_n$, respectively. Hence, $W_n \backslash E_n$ and $W'_n \backslash E_n$ are smooth.

For $[C] \in E_n$, note that C spans $\mathbb{P}^3$. The deformation of $C \subset \mathbb{P}^3$ in Proposition 2.6 along with the deformation corresponding to perturbing $\mathbb{P}^3$ in $\mathbb{P}^n$ provide a $4n$ dimensional universal deformation space for $C \subset \mathbb{P}^n$. This space is isomorphic to $\mathbb{A}^{4n-4} \cup \mathbb{A}^{4n-1}$, where $\mathbb{A}^{4n-4} \cap \mathbb{A}^{4n-1} = \mathbb{A}^{4n-5}$. This shows that W_n and W'_n are smooth along E_n and they intersect transversely.

Finally, a subscheme C parameterized by W_n uniquely determines a $\mathbb{P}^3$ spanned by C. So W_n admits a fibration over $\mathbb{G}(3,n)$ with fiber isomorphic to H_3. In contrast, W'_n does not admit a natural fibration over $\mathbb{G}(3,n)$, since a plane conic with a point on that plane only span $\mathbb{P}^2$ rather than $\mathbb{P}^3$. □

3. Mori theory of H_n

In this section, we will prove Theorems 1.3 and 1.6. To study the geometry of the divisors defined in Definition 1.2, we calculate their intersection numbers with the following test curves.

Definition 3.1. We introduce effective curves in H_n as follows.

Let B_1 denote a pencil of codimension two linear subspaces contained in a hyperplane union a fixed codimension two linear subspace in $\mathbb{P}^n$.

Let B_2 denote a pencil of codimension two linear subspaces contained in a hyperplane union a fixed codimension two subspace in this pencil. Put an embedded structure at the base $\mathbb{P}^{n-3}$ of the pencil given by the square of its ideal.

Take a pencil of lines from a ruling class of a quadric surface in $\mathbb{P}^3$. Each line along with a fixed line in that ruling class and a fixed codimension four subspace can span a pair of codimension two linear subspaces in $\mathbb{P}^n$. Let B_3 denote this family in H_n.

Let B_4 denote a pencil of subschemes defined by the ideal $(x_0^2, x_0x_1, x_1^2, tx_0x_3 - sx_1x_2)$, where $[s,t]$ denote the coordinates of $\mathbb{P}^1$.

Lemma 3.2. *We have the following intersection numbers:*

$$B_1.\,M = 1,\ B_1.\,N = 0,\ B_1.\,F = 1,\ B_1.\,E = 1,$$
$$B_2.\,M = 1,\ B_2.\,N = 2,\ B_2.\,F = 0,$$
$$B_3.\,M = 2,\ B_3.\,N = 2,\ B_3.\,E = 0,$$
$$B_4.\,M = 0,\ B_4.\,F = 1,\ B_4.\,E = 2.$$

Proof. Let us verify the intersection numbers involving B_1. The others can be checked similarly.

Suppose B_1 is given by a pencil of codimension two linear subspaces $(x_0, sx_1 + tx_2)$ union a fixed codimension two linear subspace Λ_{n-2} defined by $(x_0 - x_3, x_1 + x_3)$. The pencil has a base codimension three linear subspace Λ_{n-3}: $x_0 = x_1 = x_2 = 0$.

Take a line L that defines M whose ideal is $(x_2, \ldots, x_n)$. There is a unique subscheme with $[s,t] = [0,1]$ in B_1 intersecting with L. To check that B_1 intersects M transversely, around $(x_0, x_2) \cap (x_0 - x_3, x_1 + x_3)$, subschemes in H_n have ideal $(x_0 + a_1x_1 + \sum_{i=3}^n a_ix_i, x_2 + b_1x_1 + \sum_{i=3}^n b_ix_i) \cap (x_0 - x_3 + \sum_{i=2}^n c_ix_i, x_1 + x_3 + \sum_{i=2}^n d_ix_i)$, where a_i, b_i, c_i, d_i yield a local chart for H_n. The divisor M corresponds to the locus $b_1 = 0$. The pencil B_1 corresponds to the locus where all a_i, b_i, c_i, d_i are zero except b_1. Hence, B_1 intersects M transversely at their unique meeting point, so $B_1.\,M = 1$.

Since Λ_{n-2} is not in the pencil, there is no generically non-reduced subscheme parameterized by B_1. So B_1 does not intersect N.

For E, take its defining Λ_3 with ideal $(x_4, \ldots, x_n)$. There is a unique subscheme in B_1 with $[s,t] = [1,0]$ intersecting Λ_3. To check that B_1 intersects E transversely, around $(x_0, x_1) \cap (x_0 - x_3, x_1 + x_3)$, subschemes in H_n have ideal $(x_0 + \sum_{i=2}^n a_ix_i, x_1 + \sum_{i=2}^n b_ix_i) \cap (x_0 - x_3 + \sum_{i=2}^n c_ix_i, x_1 + x_3 +$

$\sum_{i=2}^{n} d_i x_i)$, where a_i, b_i, c_i, d_i yield a local chart for H_n. The pencil B_1 corresponds to the locus where all a_i, b_i, c_i, d_i are zero except b_2. The divisor E corresponds to the locus $(a_3+1)(b_2-d_2) = (a_2-c_2)(b_3-1)$. Hence, B_1 and E intersect transversely, so $B_1.\, E = 1$.

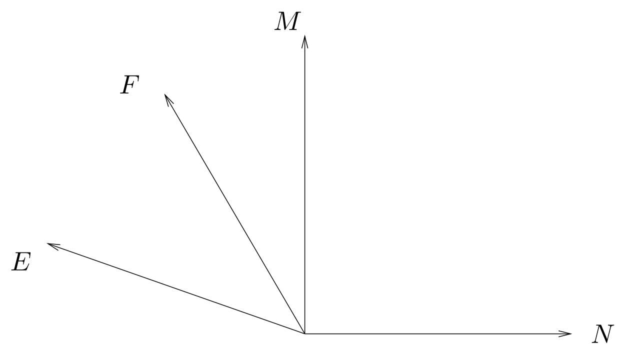
Take a general point-plane flag $(q \in \Lambda_2)$ that defines F. Suppose Λ_2 intersects Λ_{n-2} at a point r. The line $\overline{qr}$ intersects a unique codimension two subspace in the pencil. As above, one checks that B_1 and F intersect transversely, so $B_1.\, F = 1$. □

Proof of Theorem 1.3 (1). Since H_n is a smooth, rationally connected variety, we know that $h^1(\mathcal{O}_{H_n}) = 0$, so that $\mathrm{Pic}(H_n)$ is finitely generated and free. This implies that linear equivalence and numerical equivalence for divisors coincide.

Recall that N is the exceptional divisor of the blow-up of $\mathrm{Sym}^2\mathbb{G}(n-2,n)$ along the diagonal. Then $H_n \backslash N$ is isomorphic to $\mathrm{Sym}^2\mathbb{G}(n-2,n)\backslash\Delta$, whose divisor class group has rank one. Hence, $\mathrm{Pic}(H_n)$ is a rank two free $\mathbb{Z}$ module. By the intersection numbers $B_2.\, M = 1, B_2.\, F = 0$ and $B_4.\, M = 0, B_4.\, F = 1$, cf. Lemma 3.2, the divisor classes M and F generate $\mathrm{Pic}(H_n)$.

Using the test curves B_1 and B_2, we get $N = 2M - 2F$. Using B_1 and B_3, we get $E = M - N = 2F - M$. □

These divisors decompose the effective cone of H_n as follows:



If a divisor D has negative intersection with an irreducible curve, then its stable base locus $\mathbf{B}(\mathrm{D})$ necessarily contains the locus swept out by the deformations of this curve.

Proof of Theorem 1.3 (2). By definition, M and F are clearly base-point-free. For instance, for a point in the locus (IV) corresponding to a double Λ_{n-2} with an embedded Λ_{n-3}, one can take a point-plane flag $(p \in \Lambda_2)$ defining F such that the intersection between Λ_2 and the double Λ_{n-2} is a double point not collinear with p. Hence, F has no base point in the locus (IV). In the same way, we can perturb the defining lines and flags of M and F to avoid any point in H_n. So (F, M) is the ample cone.

The divisor N parameterizes the loci (II) ∪ (IV), which is the exceptional divisor of the blow-up of $\mathrm{Sym}^2\mathbb{G}(n-2,n)$ along the diagonal. Hence, N spans an extremal ray of the effective cone. Note that $B_3.\, E = 0$ and B_3 is a moving curve in H_n. Since $\mathrm{Pic}(H_n)$ is of rank two, E spans another extremal ray of the effective cone.

Note that $B_4.\, M = 0$ and $B_4.\, N = B_4.\,(2M-2F) = -2$, which implies that B_4 is contained in $\mathbf{B}(\mathrm{D})$ for a divisor D in the chamber $(M, N]$. Since B_4 is a moving curve in N and M is base-point-free, we know $\mathbf{B}(\mathrm{D})$ consists of N.

For a divisor class D in the chamber $[E, F)$, since $B_2.\, F = 0$ and $B_2.\, E = B_2.\,(M-N) = -1$, we have $B_2.\, D < 0$. Since B_2 sweeps out the loci (III) ∪ (IV), $\mathbf{B}(\mathrm{D})$ contains (III) ∪ (IV). Note that F is base-point-free. For a subscheme X not parameterized in (II) ∪ (III) ∪ (IV), we can choose the defining $\mathbb{P}^3$ of E away from the intersection of the two subspaces of X. Since (II) is a divisorial locus parameterized by N and any two subschemes in (II) are projectively equivalent, it implies the base locus of E does not intersect (II). Hence, $\mathbf{B}$(D) consists of (III) $\cup$ (IV) for D in the chamber $[E, F)$.

The moving cone of H_n follows from the description of the base loci of divisors in each chamber. Note that the locus (III) is divisorial iff $n = 3$, hence the moving cone of H_3 is different from that of H_n for $n \geq 4$. □

Let us calculate the canonical class of H_n.

Proposition 3.3. *Let $n \geq 3$ be an integer. The canonical divisor K_{H_n} has class $-(n+1)M+(n-2)N$. In particular, H_n is Fano iff $n = 3$ or 4.*

Proof. Let $Y = \mathbb{G}(n-2, n) \times \mathbb{G}(n-2, n)$ and Y' be the blow-up of Y along its diagonal. Let E_0 denote the exceptional divisor of the blow-up. Since $H_n \cong \mathrm{Bl}_\Delta \mathrm{Sym}^2\mathbb{G}(n-2, n)$, we calculate the canonical class K_{H_n} via the following commutative diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{g} & \mathrm{Bl}_\Delta \mathrm{Sym}^2\mathbb{G}(n-2, n) \\ \phi\big\downarrow & & \big\downarrow\rho \\ Y & \xrightarrow{f} & \mathrm{Sym}^2\mathbb{G}(n-2, n) \end{array}$$

Note that g is a double cover branched along N, and E_0 is the ramification divisor. By the Riemann-Hurwitz formula, $K_{Y'} = g^*K_{H_n} + E_0$. By the blow-up formula, $K_{Y'} = \phi^*K_Y + (2n-3)E_0$. Hence, $g^*K_{H_n} = \phi^*K_Y + (2n-4)E_0$. The canonical class K_Y is equivalent to $\mathcal{O}_Y(-n-1, -n-1)$. Moreover, $\phi^*\mathcal{O}_Y(1, 1) = g^*M$ and $g^*N = 2E_0$. Suppose $K_{H_n} = aM + bN$. Then $ag^*M + 2bE_0 = -(n+1)g^*M + (2n-4)E_0$, which implies $a = -(n+1)$ and $b = n-2$. Therefore, $-K_{H_n} = (n+1)M - (n-2)N = 2(n-2)F + (5-n)M$, which lies in the ample cone of H_n iff $2 < n < 5$, cf. Theorem 1.3 (2). □

Now we consider the Proj model $P(D)$ induced by a divisor D. The following result will be used frequently, cf. e.g., [La, 2.1.B].

Lemma 3.4. *Let $f\colon X \to Y$ be a birational morphism between two normal varieties. Let D be an ample divisor on Y. Then f^*D is semi-ample on X and the Proj model $P(f^*D)$ is isomorphic to Y.*

As mentioned in Theorem 1.6, the case $n = 3$ is slightly different from $n \geq 4$, since the locus (III) is divisorial iff $n = 3$. Therefore, we first study H_3. This may also help the reader get a feel for the models.

Proof of Theorem 1.6 for $n = 3$. Let $\mathrm{Sym}^2\mathbb{G}(1, 3)$ be the Chow variety parameterizing cycles $[L_1+L_2]$, where L_1 and L_2 are two lines in $\mathbb{P}^3$. Let M_0 be the divisor class parameterizing cycles in the Chow variety that intersect a fixed line. Then M_0 yields the defining ample line bundle for the Chow variety, cf. [H, 1.a]. The Hilbert-Chow morphism $H_3 \to \mathrm{Sym}^2\mathbb{G}(1, 3)$ pulls M_0 back to M. Moreover, $\mathrm{Sym}^2\mathbb{G}(1, 3)$ has finite quotient singularities, hence is normal. By Lemma 3.4, the model $P(M)$ is isomorphic to $\mathrm{Sym}^2\mathbb{G}(1, 3)$. The locus of double lines supported on a reduced line L gets contracted to a point parameterizing the cycle $[2L]$ in $\mathrm{Sym}^2\mathbb{G}(1, 3)$.

Consider the Plücker embedding of $\mathbb{G}(1, 3)$. The image is a smooth quadric 4-fold Q in $\mathbb{P}^5$. Recall in Definition 1.5 that $\Phi_3 \cong \mathbb{G}(3, 5)$ parameterizes codimension two linear sections of Q. Define a morphism $f\colon H_3 \to \mathbb{G}(3, 5)$ by sending a subscheme X to the locus of lines whose intersections with X have length ≥ 2. Let us check that this locus is a codimension two linear section of Q.

If X is a pair of skew lines $L_1 \cup L_2$, the space of lines in $\mathbb{P}^3$ that intersect both L_1 and L_2 is a smooth quadric surface contained in Q, which is cut out by a general 3-dimensional linear subspace of $\mathbb{P}^5$. If X is a double line without embedded point, its Zariski tangent space T_qX at a point q is

2-dimensional. Consider a line passing through q whose schematic intersection with X has length at least two. This line has to be contained in T_qX. The space of such lines forms a 2-dimensional quadric cone in X. The cone point parameterizes the line X_{red}. A ruling through the cone point parameterizes lines passing through q and contained in T_qX. If X consists of two lines L_1 and L_2 contained in a plane Λ with an embedded point at their intersection p, the space of lines in $\mathbb{P}^3$ that intersect L_1 and L_2 is a union of two planes contained in Q. One plane is the Schubert variety $\Sigma_{1,1}$ parameterizing lines contained in Λ and the other is the Schubert variety Σ_2 parameterizing lines passing through p. The two planes intersect along a 1-dimensional linear subspace parameterizing lines contained in Λ and passing through p. Note that these two planes are determined by the flag $(p \in \Lambda)$ and independent of the two lines L_1, L_2. Such two planes or quadric cones in Q are cut out by special 3-dimensional linear subspaces of $\mathbb{P}^5$. Therefore, f is well-defined and is a surjective morphism.

The family B_2 in Lemma 3.2 is a moving curve in E and has zero intersection with F. For a subscheme X parameterized by B_2, we have seen that the space of lines in $\mathbb{P}^3$ whose intersections with X have length at least two does not depend on X. It is only determined by the embedded point and the plane containing X_{red}. Hence, the morphism f contracts B_2 to the point in $\mathbb{G}(3,5)$ corresponding to the linear section $\Sigma_{1,1} \cup \Sigma_2$. Let σ_1 be the hyperplane class of $\mathbb{G}(1,3)$. Using the test curve B_1 in Lemma 3.2, we have $B_1 . f^*\sigma_1 = f_*B_1 . \sigma_1 = 1$. Since $F . B_1 = 1, F . B_2 = 0$ and the Picard number of H_3 is two, we get $F = f^*\sigma_1$, which implies that the model $P(F)$ is isomorphic to $\mathbb{G}(3,5)$. □

Recall in Corollary 2.8, there is a smooth Hilbert component W_n whose general point parameterizes a pair of skew lines in $\mathbb{P}^n$. The geometry of $H_3 \cong W_3$ serves as a prototype for that of W_n. We can similarly define effective divisors on W_n as follows.

Definition 3.5. Let $n \geq 4$ be an integer.

Let M' denote the divisor class parameterizing the locus of subschemes whose supports intersect a fixed codimension two linear subspace.

Let N' denote the divisor class parameterizing the locus of double lines.

Let E' denote the divisor class parameterizing the locus of two coplanar lines with a spatial embedded point at their intersection.

Let R' denote the divisor class parameterizing the locus of subschemes such that the 3-dimensional linear subspaces they span intersect a fixed codimension four linear subspace.

Fix a flag $\Lambda_{n-3} \subset \Lambda_{n-1} \subset \mathbb{P}^n$. For a pair of general lines, let p, q denote their intersection points with Λ_{n-1}. Consider the locus of two lines such that p, q and Λ_{n-3} only span a codimension two linear subspace. Denote by F' the divisor class parameterizing the closure of this locus.

Remark 3.6. There is a rational map $W_n \dashrightarrow H_3$ by projecting a subscheme from a codimension four linear subspace to a linear subspace $\mathbb{P}^3 \subset \mathbb{P}^n$. Then M', N', E', F' on W_n are equivalent to the pull-backs of M, N, E, F from H_3, respectively. Since H_3 naturally embeds into W_n via the inclusion $\mathbb{P}^3 \subset \mathbb{P}^n$, the divisors M', N', E', F' restrict to the corresponding divisors on H_3. In particular, their intersections with $B_1, \ldots, B_4$ in Definition 3.1 are the same as in H_3. Hence, we can adapt the test curves $B_1, \ldots, B_4$ and their intersection numbers in Lemma 3.2 to W_n for $n \geq 4$.

Two more test curves are needed as follows.

Definition 3.7. Let $n \geq 4$ be an integer.

Take a line L away from a fixed plane in $\mathbb{P}^n$. Let B_5 denote a pencil of lines on that plane union L as a one parameter family in W_n.

Fix two lines L_1 and L_2 meeting at a point p on a plane Λ. Take a line L away from Λ. For each point $q \in L$, there is a unique spatial embedded structure at p such that the 3-dimensional linear subspace it spans with Λ contains q. Varying q, we get a one parameter family B_6 in W_n.

Now we study the stable base locus decomposition of the effective cone of W_n.

Proposition 3.8. *Let $n \geq 4$ be an integer. On W_n, we have $E' = 2F' - M' - R'$ and $N' = 2M' - 2F'$. The effective cone of W_n is generated by R', E', N' and the semi-ample cone of W_n is generated by R', F', M'. For a divisor D in the chambers $\langle E', F', R'\rangle \cup \langle E', F', M'\rangle$, $\mathbf{B}(D)$ consists of E'. For D in the chamber $\langle R', M', N'\rangle$, $\mathbf{B}(D)$ consists of N'. For D in the chamber $\langle E', M', N'\rangle$, $\mathbf{B}(D)$ consists of E' and N'.*

Proof. Since W_n admits an H_3 fibration over $\mathbb{G}(3, n)$, its Picard number equals three. Using B_1, B_2 and B_3, we get $E' = 2F' - M' + aR'$ and $N' = 2M' - 2F' + bR'$, since $B_i.\, R' = 0$ for $1 \leq i \leq 3$. Using the curve B_5, we have $B_5.\, M' = B_5.\, F' = B_5.\, R' = 1$ and $B_5.\, N' = B_5.\, E' = 0$. Therefore, we get $E' = 2F' - M' - R'$ and $N' = 2M' - 2F'$.

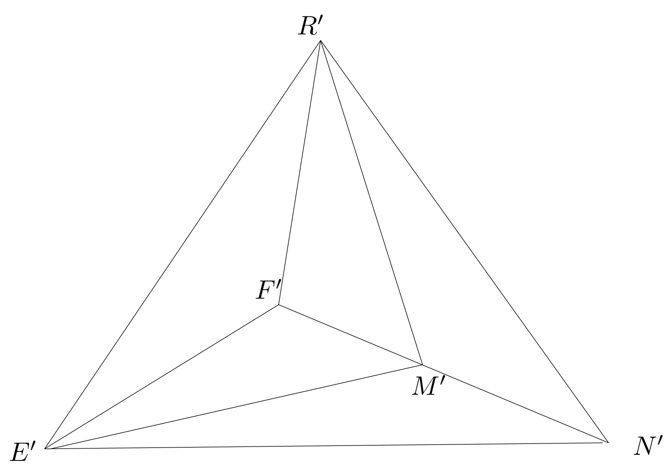
M', F' and R' are base-point-free by their definitions, since we can vary their defining linear subspaces or flags to make their loci avoid any point in W_n. So the semi-ample cone contains $\langle M', F', R'\rangle$. On the other hand, we have $B_2.\, F' = B_2.\, R' = 0$. It implies that no divisor in $\langle F', R'\rangle$ is ample, hence $\langle F', R'\rangle$ lies in the boundary of the semi-ample cone. The same holds for $\langle R', M'\rangle$ and $\langle F', M'\rangle$. Therefore, we conclude that the semi-ample cone of W_n equals $\langle M', F', R'\rangle$.

Since $B_2.\, E' = -1$, $B_2.\, F' = B_2.\, R' = 0$ and B_2 is a moving curve in E', E' spans an extremal ray of the effective cone and it is the stable base locus of a divisor in the chamber $\langle E', F', R'\rangle$. We have $B_6.\, M' = B_6.\, F' = 0$, $B_6.\, R' = 1$, so $B_6.\, E' = -1$. This implies E' is the stable base locus for a divisor in the chamber $\langle E', F', M'\rangle$.

Since $B_4.\, N' = -2$, $B_4.\, R' = B_4.\, M' = 0$ and B_4 is a moving curve in N', N' spans an extremal ray of the effective cone and it is the stable base locus of a divisor in the chamber $\langle M', N', R'\rangle$.

Finally, $B_6.\, E' = -1$ and $B_6.\, M' = B_6.\, N' = 0$ imply that a divisor D in the chamber $\langle E', N', M'\rangle$ contains E' in its stable base locus. After removing E', since $B_4.\, M' = 0$ and $B_4.\, N' = -2$, N' is also contained in the base locus of D. Since M' is base-point-free, it implies that the stable base locus of D consists of $E' \cup N'$. □

The picture below describes this stable base locus decomposition:



Now we study the morphism $\psi_D\colon W_n \to P(D)$ induced by a divisor D in the semi-ample cone $\langle R', F', M'\rangle$. Recall the spaces Ψ_n and Θ_n in Definition 1.5.

Proposition 3.9. *Recall that $H_n \cong Bl_{\Delta} Sym^2 \mathbb{G}(1, n)$. Let a, b denote two positive integers and $n \geq 4$ denote an integer.*

(1) For $D_1 = aF' + bM'$, the morphism ψ_{D_1} contracts E' and $P(D_1)$ is isomorphic to $Bl_{\Delta} Sym^2 \mathbb{G}(1, n)$.

(2) For $D_2 = aF' + bR'$, the morphism ψ_{D_2} contracts E' and $P(D_2)$ is isomorphic to Ψ_n.

(3) For $D_3 = aM' + bR'$, the morphism ψ_{D_3} contracts N' and $P(D_3)$ admits a $Sym^2\mathbb{G}(1,3)$ fibration over $\mathbb{G}(3,n)$, which is the relative Chow variety parameterizing two lines in a 3-dimensional linear subspace.

(4) The morphism $\psi_{F'}$ contracts E' and $P(F')$ is isomorphic to Θ_n. Moreover, $P(D_1) \dashrightarrow P(D_2)$ is a flip over $P(F')$.

(5) The morphism $\psi_{M'}$ contracts E' and N'. The model $P(M')$ is isomorphic to $Sym^2\mathbb{G}(1,n)$.

(6) The morphism $\psi_{R'}$ is the H_3 fibration $W_n \to \mathbb{G}(3,n)$ and the model $P(R')$ is isomorphic to $\mathbb{G}(3,n)$.

Proof. (1) On the one hand, by $B_6.\,F' = B_6.\,M' = 0$, the curve class B_6 is contracted by ψ_{D_1}. Since B_6 sweeps out E', ψ_{D_1} contracts E'.

On the other hand, suppose an effective curve C in W_n satisfying $C.\,M' = 0$. It implies that the subschemes parameterized by C have the same support. Otherwise we can choose a defining codimension two linear subspace for M' to intersect with finitely many points of C and $C.\,M'$ would be non-zero, a contradiction. Now, suppose C does not intersect F'. If a general subscheme parameterized by C consists of two skew lines or a non-planar double line, we can always choose a defining flag $\Lambda_{n-3} \subset \Lambda_{n-1}$ of F' such that F' intersects finitely many members of C. If a general subscheme parameterized by C consists of two incident lines but the plane they span varies within the family, one can also choose a defining flag of F' such that F' only intersects finitely many members of C. Moreover, if the support of the spatial embedded point moves within C, one can still choose a defining flag of F' such that F' intersects finitely many members of C. In all these cases, $C.\,F'$ would be non-zero, a contradiction. So $C.\,F' = 0$ implies C is a family of two incident lines contained in a common plane and both passing through a common point where the embedded points arise. Hence, if $C.\,M' = C.\,F' = 0$, the subschemes parameterized by C have the same planar support and their embedded points also have the same support at a point p. Only the spatial embedded structure pointing outward the plane varies at p in order to get the family C.

Note that $\mathrm{Bl}_\Delta\mathrm{Sym}^2\mathbb{G}(1,n)$ parameterizes a pair of skew lines in $\mathbb{P}^n$, double lines of arithmetic genus -1 and two incident lines without specifying the embedded structure at their intersection point. By forgetting the embedded point of a subscheme parameterized by E', the map $W_n \to \mathrm{Bl}_\Delta\mathrm{Sym}^2\mathbb{G}(1,n)$ contracts the locus E', and B_6 spans the contracted curve class. Pulling back an ample divisor from $\mathrm{Bl}_\Delta\mathrm{Sym}^2\mathbb{G}(1,n)$, we get a semi-ample divisor on W_n whose intersection with B_6 is zero. So this divisor is of type $aF' + bM'$. Since $\mathrm{Bl}_\Delta\mathrm{Sym}^2\mathbb{G}(1,n)$ is smooth, by Lemma 3.4, it is isomorphic to the model $P(D_1)$.

(2) By $B_2.\,F' = B_2.\,R' = 0$, the curve class B_2 is contracted by ψ_{D_2}. Since B_2 sweeps out E', ψ_{D_2} contracts E'.

Now suppose an effective curve C in W_n satisfying $C.\,R' = 0$. Those subschemes parameterized by C must span the same $\mathbb{P}^3$. Otherwise we can choose a defining codimension four linear subspace of R' such that R' intersects finitely many points of C. Then $C.\,R'$ would be non-zero, a contradiction. If $C.\,F' = 0$, in (1), we analyzed that the 1-dimensional parts of subschemes parameterized by C span the same plane, and the embedded points of those subschemes have the same support. Hence, if $C.\,F' = C.\,R' = 0$, then C parameterizes a family of two incident lines in a common plane with the same spatial embedded structure supported on a common point. Only the 1-dimensional part of the two lines varies in that plane to get the family C.

For a subscheme X parameterized by W_n, it spans a unique $\mathbb{P}^3$. Associate to X the closure of locus in $\mathbb{G}(1,3)$ of lines in that $\mathbb{P}^3$ whose intersection with X is a length-2 zero dimensional subscheme. By Theorem 1.6 (4), we get a morphism from W_n to Ψ_n, which is a $\mathbb{G}(3,5)$ bundle over $\mathbb{G}(3,n)$. This morphism restricted to each fiber H_3 only contracts the curve class B_2. Pulling back an ample divisor from the target, we get a semi-ample divisor on W_n whose intersection with B_2 is zero. So this divisor is of type $aF' + bR'$ and its Proj model is isomorphic to Ψ_n.

(3) By $B_4. M' = B_4. R' = 0$, the curve class B_4 is contracted by ψ_{D_3}. Since B_4 sweeps out N', ψ_{D_3} contracts N'.

For an effective curve C in W_n, by the analysis in (1) and (2), $C. M' = C. R' = 0$ implies that the subschemes parameterized by C have the same support and span the same $\mathbb{P}^3$. Hence, they are double lines supported on a common line with the double structure varying to get the family C.

For a subscheme X parameterized by W_n, it spans a unique $\mathbb{P}^3$. Associate to X its support as a cycle in that $\mathbb{P}^3$. By Theorem 1.6 (2) for $n = 3$, we get a morphism from W_n to a $\mathrm{Sym}^2\mathbb{G}(1,3)$ bundle over $\mathbb{G}(3,n)$, which parameterizes a pair of linear cycles in a 3-dimensional linear subspace. This morphism restricted to each fiber H_3 of W_n only contracts the curve class B_4. Pulling back an ample divisor from the target, we get a semi-ample divisor on W_n whose intersection with B_4 is zero. So this divisor is of type $aM' + bR'$ and its Proj model is the relative Chow variety parameterizing two lines in a 3-dimensional linear subspaces.

(4) Since $B_2. F' = B_6. F' = 0$ and the two curve classes sweep out E', the map $\psi_{F'}$ contracts E'. In (1), we have seen that $C. F' = 0$ for an effective curve C implies it is a family of two incident lines contained in a common plane Λ and both passing through a common point p where the embedded points arise. Namely, $\psi_{F'}$ forgets the lines and embedded structures within the family but only remembers the common flag $(p \in \Lambda)$.

Define a morphism $f\colon W_n \to \Theta_n$ by sending a subscheme X to the locus in $\mathbb{G}(n-2,n)$ parameterizing codimension two linear subspaces whose intersections with X have length ≥ 2. Let us check that this locus corresponds to a codimension two linear section of the Plücker embedding of $\mathbb{G}(n-2,n)$ of type $\Sigma_1 \cap \Sigma_1'$ or its degenerations.

If X consists of two skew lines L and L', this locus is the subvariety $\Sigma_1 \cap \Sigma_1'$ in $\mathbb{G}(n-2,n)$, where Σ_1 and Σ_1' are Schubert varieties corresponding to L and L', respectively. If X is a pure double line of genus -1, the locus in $\mathbb{G}(n-2,n)$ of codimension two linear subspaces whose intersections with X have length ≥ 2 is a subvariety of $\mathbb{G}(n-2,n)$ with cycle class Σ_1^2, which corresponds to the limit case when Σ_1' approaches Σ_1. If X consists of two incident lines or a double line contained in a plane Λ with an embedded point at p, the corresponding locus is the subvariety $\Sigma_{1,1} \cup \Sigma_2$, where $\Sigma_{1,1}$ parameterizes codimension two linear subspaces intersecting Λ and Σ_2 parameterizes those containing p. Therefore, f is a well-defined surjective morphism.

Note that E' is the exceptional locus of f. A contracted fiber over a point in $f(E')$ parameterizes two incident lines with the same point-plane flag. So this fiber is isomorphic to $\mathbb{P}^{n-3} \times \mathbb{P}^2$, where $\mathbb{P}^{n-3}$ specifies the spatial embedded point pointing outward Λ and $\mathbb{P}^2$ specifies the two lines passing through p and contained in Λ. The curve classes B_2 and B_6 generate the cone of curves of this contracted fiber. Since $\psi_{F'}$ contracts the curve classes in the same way, it can be identified as f and the model $P(F')$ is isomorphic to Θ_n.

Since the curve classes contracted by ψ_{D_1} and ψ_{D_2} are also contracted by $\psi_{F'}$, the morphism $\psi_{F'}$ factors through $P(D_1)$ and $P(D_2)$, respectively. Moreover, ψ_{D_1} and ψ_{D_2} both contract E' and the image of E' is of codimension ≥ 2 in each target. So $P(D_1)$ and $P(D_2)$ are isomorphic in codimension one. By the formal definition of flips, $P(D_1) \dashrightarrow P(D_2)$ is a flip over $P(F')$ with respect to the divisor F'.

(5) By $B_4. M' = B_6. M' = 0$, we know the morphism $\psi_{M'}$ contracts N' and E', since B_4 sweeps out N' and B_6 sweeps out E'.

Consider the Hilbert-Chow morphism from W_n to $\mathrm{Sym}^2\mathbb{G}(1,n)$ by sending a subscheme to its 1-dimensional support with multiplicity. This map contracts N' and E' by forgetting double structures and embedded structures, respectively. An ample divisor on $\mathrm{Sym}^2\mathbb{G}(1,n)$ can be defined as the locus of cycles intersecting a fixed codimension two linear subspace. Note that this divisor pulls back to M' on W_n. Since $\mathrm{Sym}^2\mathbb{G}(1,n)$ is normal, by Lemma 3.4, it is isomorphic to the model $P(M')$.

(6) W_n admits an H_3 fibration over $\mathbb{G}(3,n)$. By its definition, R' is equivalent to the pull-back of σ_1 from $\mathbb{G}(3,n)$. Hence, the morphism $\psi_{R'}$ contracts each fiber H_3 and the model $P(R')$ is isomorphic to $\mathbb{G}(3,n)$. □

Remark 3.10. There is another way to interpret the model $P(F') \cong \Theta_n$ in Proposition 3.9 (4). Consider the Hilbert scheme of quadric surfaces with class $\sigma_{n-1,n-3}+\sigma_{n-2,n-2}$ in the Plücker embedding of $\mathbb{G}(1,n)$. Lines parameterized by such a quadric Q span a $\mathbb{P}^3$ in $\mathbb{P}^n$, which induces an inclusion $\mathbb{G}(1,3) \subset \mathbb{G}(1,n)$. Then Q is uniquely determined by a codimension two linear section of $\mathbb{G}(1,3)$. Hence, this Hilbert scheme of quadrics is isomorphic to the model $P(D_2) \cong \Psi_n$ in Proposition 3.9 (2), which is a $\mathbb{G}(3,5)$ bundle over $\mathbb{G}(3,n)$.

Associate to Q the maximal subvariety Σ in $\mathbb{G}(n-2,n)$, where a linear subspace parameterized by Σ contains some line parameterized by Q. Since Q is a quadric surface in the Plücker embedding of $\mathbb{G}(1,n)$, Σ is a codimension two linear section of the Plücker embedding of $\mathbb{G}(n-2,n)$. In other words, this association maps $P(D_2)$ to the space Ξ_n of codimension two linear sections of $\mathbb{G}(n-2,n)$ such that the linear subspaces parameterized by a section Σ contain some line parameterized by the corresponding quadric Q in $\mathbb{G}(1,n)$. We claim that the space Ξ_n is isomorphic to $P(F') \cong \Theta_n$.

When Q is smooth, the lines parameterized by Q all intersect two skew lines L and L'. The corresponding subvariety in $\mathbb{G}(n-2,n)$ parameterizes linear subspaces that intersect both L and L'. Hence, that codimension two linear section is the intersection of two Schubert varieties $\Sigma_1 \cap \Sigma_1'$.

When Q is singular but irreducible, by the proof of Theorem 1.6 for $n=3$, Q parameterizes lines that intersect a fixed double line of genus -1 with length ≥ 2. The corresponding codimension two linear section of $\mathbb{G}(n-2,n)$ parameterizes linear subspaces that intersect a fixed double line with length ≥ 2.

When Q is reducible, it is a union of two planes that are determined by a flag $\{p \in \Lambda_2 \subset \Lambda_3\}$, where p is a point, Λ_2 is a plane and Λ_3 is a $\mathbb{P}^3$. The two planes of Q are determined by lines in Λ_3 passing through p or contained in Λ_2, respectively. The locus of codimension two linear subspaces that contain some line parameterized by Q is reducible. It contains the Schubert variety $\Sigma_{1,1}$ of codimension two linear subspaces containing a line in Λ_2 and the Schubert variety Σ_2 parameterizing those linear subspaces containing p. In this case, no matter what Q is, the image point in Ξ_n only depends on the flag $p \in \Lambda_2$.

Hence, we conclude that this association map $P(D_2) \to \Xi_n$ can be identified as $P(D_2) \to P(F')$, cf. the proof of Proposition 3.9 (4).

Let us calculate the canonical class of W_n.

Proposition 3.11. *Let $n \geq 3$ be an integer. The canonical divisor K_{W_n} has class $-(n+1)M' + (n-2)N' + (n-3)E'$. In particular, W_n is a Fano variety.*

Proof. By Proposition 3.9 (1), we know $\pi\colon W_n \to \mathrm{Bl}_\Delta \mathrm{Sym}^2\mathbb{G}(1,n) \cong H_n$ contracts E'. The divisor E' admits a $\mathbb{P}^{n-3}$ fibration over Γ, where Γ parameterizes a pair of coplanar lines with a point p at their intersection and the fiber specifies the embedded structure at p pointing outward the plane. From this description, Γ is a $\mathbb{P}^2$ fibration over the flag variety $\{p \in \Lambda_2 \subset \mathbb{P}^n\}$, hence a smooth variety of dimension $3n-2$. From the dual viewpoint, in H_n the subvariety Γ consists of the loci (III) and (IV). The smoothness of Γ also follows from the fact that it is the transverse intersection of the smooth components H_n and H_n'.

Each fiber $\mathbb{P}^{n-3}$ of E' gets contracted under π to the base Γ. Therefore, we get $K_{W_n} = \pi^*K_{H_n} + (n-3)E' = -(n+1)M' + (n-2)N' + (n-3)E'$.

Since $N' = 2M' - 2F'$ and $E' = 2F' - M' - R'$ on W_n, we have $-K_{W_n} = 2M' + 2F' + (n-3)R'$, which is ample, cf. Proposition 3.8. Hence, W_n is a Fano variety. □

Note that $H_n \cong \mathrm{Bl}_\Delta \mathrm{Sym}^2 \mathbb{G}(n-2,n) \cong \mathrm{Bl}_\Delta \mathrm{Sym}^2 \mathbb{G}(1,n)$ appears as an intermediate model of W_n. Using the duality between $\mathbb{G}(1,n)$ and $\mathbb{G}(n-2,n)$, the above results provide a recipe for analyzing the models induced by divisors on H_n.

Proof of Theorem 1.6 for $n \geq 4$. Part (1) is obvious because (F, M) is the ample cone of H_n.

For (2), since N is the exceptional divisor of the blow-up and it is contained in the base locus of a divisor D in $[M, N)$, after removing N, the model $P(D)$ is isomorphic to $P(M)$, which is the Chow variety $\mathrm{Sym}^2 \mathbb{G}(n-2,n)$ parameterizing a pair of codimension two linear cycles in $\mathbb{P}^n$.

Since $\mathbb{G}(n-2,n) \cong \mathbb{G}(1,n)$, we can adapt the models obtained from W_n to H_n. Note that H_n is isomorphic to the model $P(D_1) \cong \mathrm{Bl}_\Delta \mathrm{Sym}^2 \mathbb{G}(1,n)$ in Proposition 3.9 (1). A pair of general codimension two linear subspaces corresponds to a pair of general lines. A double codimension two linear subspace corresponds to a double line. A pair of codimension two linear subspaces that span a hyperplane with an embedded component supported on their intersection corresponds to a pair of incident lines without specifying the embedded point, since the morphism $W_n \to P(D_1) \cong H_n$ is induced by forgetting the spatial embedded structure of a subscheme. Via this translation, Theorem 1.6 (3) and (5) for $n \geq 4$ can be verified as follows.

For (3), the morphism $H_n \to \Theta_n$ sends a subscheme X to the locus of lines in $\mathbb{G}(1,n)$ that intersect X with length ≥ 2. This locus is a codimension two linear section of the Plucker embedding of $\mathbb{G}(1,n)$ as follows. If X is a subscheme of type (I), the corresponding locus in $\mathbb{G}(1,n)$ is $\Sigma_1 \cap \Sigma_1'$ parameterizing lines that intersect both components of X. For X of type (II), the locus parameterizes lines whose intersections with X contain double points. It is a subvariety of $\mathbb{G}(1,n)$ with cycle class Σ_1^2 corresponding to the limit case when Σ_1' approaches Σ_1. For X of type (III) or (IV), its Cohen-Macaulay part is contained in a hyperplane Λ_{n-1} and its embedded component is supported on a subspace Λ_{n-3}. The corresponding locus in $\mathbb{G}(1,n)$ consists of two Schubert varieties $\Sigma_{1,1} \cup \Sigma_2$, where $\Sigma_{1,1}$ parameterizes lines contained in Λ_{n-1} and Σ_2 parameterizes lines intersecting Λ_{n-3}. Therefore, the morphism $H_n \to \Theta_n$ is well-defined. Furthermore, it is a small contraction by forgetting the components of a subscheme X of type (III) or (IV) and only remembering the flag $(\Lambda_{n-3} \subset \Lambda_{n-1})$ determined by X. The contracted curve classes are the same as those contracted by the map ψ_F, hence the model $P(F)$ is isomorphic to Θ_n.

For (5), we blow up H_n along the locus (III) and the new space is isomorphic to W_n, which is an H_3 fibration over $\mathbb{G}(n-4,n) \cong \mathbb{G}(3,n)$. The blow-up corresponds to specifying a subspace $\mathbb{P}^{n-4}$ in the embedded $\mathbb{P}^{n-3}$ of a subscheme X of type (III). By Proposition 3.9 (4), we can contract the exceptional divisor of the blow-up in a different way using the morphism ψ_D induced by a divisor D in the chamber (E, F). The resulting model $P(D) \cong \Psi_n$ is a $\mathbb{G}(3,5)$ bundle over $\mathbb{G}(n-4,n)$, which is the desired flipping space. After the flip, the birational transform E on Ψ_n is equivalent to the pull-back of σ_1 from the base $\mathbb{G}(n-4,n)$. Hence, the map induced by E contracts the $\mathbb{G}(3,5)$ bundle structure to the base $\mathbb{G}(n-4,n) \cong \mathbb{G}(3,n)$. □

References

[BCHM] Caucher Birkar, Paolo Cascini, Christopher D. Hacon and James M^{c}Kernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. 23 (2010), 405–468.

[G] Alexander Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique IV*, Les schémas de Hilbert, Séminaire Bourbaki, Vol. 6, Exp. No. 221, 249–276, Soc. Math. France, Paris, 1995.

[H] Joe Harris, *Curves in projective space*, Presses de l'Université de Montréal, Montreal, Que., 1982.

[Ha] Robin Hartshorne, *Connectedness of the Hilbert scheme*, Inst. Hautes Études Sci. Publ. Math. no. 29 (1966), 5–48.

[HK] Yi Hu and Sean Keel, *Mori dream spaces and GIT*, Michigan Math. J. 48 (2000), 331–348.

[La] Robert Lazarsfeld, *Positivity in algebraic geometry I, classical setting: line bundles and linear series*, Springer-Verlag, 2004.

[L] Yoon-Ho Alex Lee, *The Hilbert schemes of curves in* $\mathbb{P}^3$, Senior Thesis, Harvard University, 2000.

[M] Juan Migliore, *On linking double lines*, Trans. Amer. Math. Soc. 294 (1986), no. 1, 177–185.

[N] Scott Nollet, *The Hilbert schemes of degree three curves*, Ann. Sci. École Norm. Sup. (4) 30 (1997), no. 3, 367–384.

[PS] Ragni Piene and Michael Schlessinger, *On the Hilbert scheme compactification of the space of twisted cubics*, Amer. J. Math. 107 (1985), no. 4, 761–774.

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607

E-mail address: dwchen@math.uic.edu

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607

E-mail address: coskun@math.uic.edu

Department of Mathematics, Texas Christian University, Fort Worth, TX 76129

E-mail address: s.nollet@tcu.edu