THE WYSIWYG COMPACTIFICATION

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Abstract. We show that the partial compactification of a stratum of Abelian differentials previously considered by Mirzakhani and Wright is not an algebraic variety. Despite this, we use a combination of algebro-geometric and other methods to provide a short, unconditional proof of Mirzakhani and Wright’s formula for the tangent space to the boundary of a $GL^+(2, \mathbb{R})$ orbit closure, and clarify the structure of the boundary.

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1. Introduction

WYSIWYG. Let $\overline{\mathcal{H}}$ be the complement of the zero section in the Hodge bundle over the Deligne-Mumford moduli space $\mathcal{M}_g$ of stable genus $g$ Riemann surfaces.

For $(X, \omega) \in \overline{\mathcal{H}}$ with $X$ a stable Riemann surface and $\omega$ a stable differential, define $\pi(X, \omega)$ to be the result of deleting each component of $X$ on which $\omega$ is identically zero, deleting each node, and filling in any punctures thus created with marked points. Thus $\pi(X, \omega)$ is a union of Riemann surfaces with marked points, each of which has a non-zero Abelian differential which is holomorphic away from the marked points and has at worse simple poles at the marked points. Notice that $(X, \omega)$
and \( \pi(X, \omega) \) have finite area under the flat metric induced by \( \omega \) if and only if \( \omega \) has no simple pole at any node of \( X \).

Define an equivalence relation on \( \mathcal{H} \) by \((X, \omega) \sim (X', \omega')\) if \( \pi(X, \omega) = \pi(X', \omega') \), and define the What You See Is What You Get partial compactification \( \tilde{\mathcal{H}} \) to be \( \mathcal{H}/\sim \). This comes equipped with the quotient map \( \pi: \mathcal{H} \to \tilde{\mathcal{H}} \).

For any partition \( \kappa = (\kappa_1, \ldots, \kappa_s) \) of \( 2g - 2 \) into positive integers, define \( \mathcal{H}(\kappa) \) and \( \tilde{\mathcal{H}}(\kappa) \) to be the closures of the stratum \( \mathcal{H}(\kappa) \) in \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \). We also define the projectivizations \( \mathbb{P}\mathcal{H}, \mathbb{P}\tilde{\mathcal{H}}, \mathbb{P}\mathcal{H}(\kappa) \) and \( \mathbb{P}\tilde{\mathcal{H}}(\kappa) \) of these spaces to be the quotient by the \( \mathbb{C}^\ast \) action that rescales the differential.

The WYSIWYG compactification is so named because, when an Abelian differential is presented via polygons in the plane, and when these polygons degenerate in a reasonable way, the limit will be given by the resulting polygons; see \([\text{MW17}, \text{Definition 2.2, Proposition 9.8}]\).

A non-algebraic, non-analytic space. Our first result cements the impression that the WYSIWYG compactification is a creation of flat geometry rather than algebraic geometry; it is in fact not an algebraic variety at all.

**Theorem 1.1.** For \( g \geq 3 \) and every \( \kappa \), as well as \( g = 2 \) and \( \kappa = (1, 1) \), \( \mathbb{P}\tilde{\mathcal{H}}(\kappa) \) does not admit the structure of an algebraic variety in such a way that \( \pi: \mathbb{P}\mathcal{H}(\kappa) \to \mathbb{P}\tilde{\mathcal{H}}(\kappa) \) is a morphism of algebraic varieties. Moreover, \( \mathbb{P}\tilde{\mathcal{H}}(\kappa) \) is not even a complex analytic space in such a way that \( \pi \) is a morphism of complex analytic spaces. When \( \mathbb{P}\tilde{\mathcal{H}}(\kappa) \) is disconnected, the same is true for each connected component.

This result was inspired by the surprisingly recent result that the set of birational automorphisms of \( \mathbb{P}^n \) of degree \( \leq d \) is not a variety in such a way that the quotient map from the variety of formulas of degree \( \leq d \) birational maps is a morphism, even though the space of birational automorphisms of degree exactly equal to \( d \) is a variety \([\text{BF13}]\). In both cases we see that the space is the disjoint union of varieties, but in order to have any connection to moduli problems these varieties must be combined using a topology in such a way that the result cannot be a variety.

Another analogous situation occurs for the topological compactification \( K_1\mathcal{M}_{g,1} \) of the moduli space of one-pointed genus \( g \) Riemann surfaces \( \mathcal{M}_{g,1} \) used by Kontsevich to prove Witten’s conjecture \([\text{Kon92}]\). Keel showed that the quotient map \( \mathcal{M}_{g,1} \to K_1\mathcal{M}_{g,1} \) cannot be an algebraic morphism \([\text{Kee99}]\).
Despite Theorem 1.1 questions about the WYSIWYG compactification are amenable to algebraic methods; Theorem 1.1 simply indicates that some care may be required.

**Orbit closures.** The finite area locus $\tilde{H}_{<\infty}(\kappa)$ of $\tilde{H}(\kappa)$ was studied in [MW17] in order to facilitate the inductive study of $GL^+(2, \mathbb{R})$ orbit closures. This partial compactification allows one to pass to the boundary of an orbit closure while staying in the realm of finite area Abelian differentials (possibly with marked points and possibly with multiple components).

Recall that $GL^+(2, \mathbb{R})$ orbit closures may intersect themselves; at such atypical points there are finitely many branches of the orbit closure. The following result concerns orbit closures of connected (single component) surfaces.

**Theorem 1.2.** Let $\mathcal{M}$ be a $GL^+(2, \mathbb{R})$ orbit closure and let $\partial \mathcal{M}_{<\infty}$ be its boundary in $\tilde{H}_{<\infty}$. Assume that $(X_n, \omega_n) \in \mathcal{M}$ converge to $(X_\infty, \omega_\infty) \in \partial \mathcal{M}_{<\infty}$.

The intersection $\mathcal{M}'$ of $\partial \mathcal{M}_{<\infty}$ and the stratum of $(X_\infty, \omega_\infty)$ is an algebraic variety locally described by a finite union of linear subspaces in local period coordinates.

After removing finitely many terms, the sequence $(X_n, \omega_n)$ may be partitioned into finitely many subsequences such that for each subsequence the tangent space to a branch of $\partial \mathcal{M}_{<\infty}$ in the boundary stratum at $(X_\infty, \omega_\infty) \in \tilde{H}_{<\infty}$ is equal to the intersection of the tangent space of a branch of $\mathcal{M}$ at $(X_n, \omega_n)$ and the tangent space of the boundary stratum, for $n$ sufficiently large.

The tangent space to the boundary stratum naturally sits inside the tangent space to $\mathcal{H}$ at $(X_n, \omega_n)$ for $n$ sufficiently large; this is explained in Section 2 and [MW17]. Note that $X_\infty$ denotes a Riemann surface with a finite set of marked points.

Theorem 1.2 was largely proven in [MW17]. However when $X_\infty$ has multiple components, the proof was conditional on an anticipated but still unproven generalization of [EM18,EMM15] to multi-component surfaces. Here we give an unconditional proof using different techniques, thus giving the first complete proof in the multi-component case, which is crucial for applications.

The new proof is much simpler and completely different, although it will use a non-trivial foundational result on $\mathcal{H}$ from [MW17]. The main difference is that here we use Filip’s result that orbit closures are algebraic varieties [Fil16b]. In light of Filip’s work, we may use
“invariant subvariety” as a synonym for $GL^+(2, \mathbb{R})$ orbit closure. When we do so, it is implicit that the invariant subvariety is irreducible.

**Other points of view.** Theorem 1.2 has an interpretation in terms of flat geometry. An orbit closure $\mathcal{M}$ is locally cut out by linear equations on period coordinates of the ambient stratum, and period coordinates of an Abelian differential corresponds to edges of a polygonal presentation. If some edges of the polygon shrink to length zero, the equations on the limit surface can be obtained by replacing the variables for any edges that are collapsed with zero.

A more abstract point of view is also possible in terms of the linear equations locally defining $\mathcal{M}$. Each equation can be interpreted as a relative cohomology class. Theorem 1.2 gives that the linear equations defining the boundary orbit closure arise from pushing forward those defining $\mathcal{M}$ via a collapse map, which will be introduced in Section 2.

**The structure of the multi-component boundary.** Components of the boundary of $\mathcal{M}$ where the translation surfaces have multiple components have some especially interesting and useful properties. Up to finite covers, each such locus of multi-component surfaces can be written as a product of prime invariant subvarieties, where an invariant subvariety is called prime if it cannot be written as a product of invariant subvarieties (even after passing to a finite cover). The definition is clarified in Section 7, where we also give a simple linear algebraic characterization.

We provide the following result so that it can be used with Theorem 1.2 in forthcoming work towards the goal of classifying orbit closures.

**Theorem 1.3.** Let $\mathcal{N}$ be a prime invariant subvariety. Then

1. if $\pi_i$ is the projection to the $i$-th component, then $\pi_i(\mathcal{N})$ is of the form $\mathcal{N}_i - \mathcal{N}'_i$, where $\mathcal{N}_i$ is an irreducible invariant subvariety and $\mathcal{N}'_i$ is a union of finitely many irreducible invariant subvarieties properly contained in $\mathcal{N}_i$,
2. locally in $\mathcal{N}$, the absolute periods of any component of a multi-component surface determine the absolute periods of any other component,
3. each $\mathcal{N}_i$ has the same rank, and
4. certain natural factors of the Jacobians of the different components of any surface in $\mathcal{N}$ are isogenous.

Part of this theorem was anticipated in [MW17, Conjecture 2.10]. We anticipate that the second point will be the most useful for classification.
Additional context. Compactifications of strata are currently an area of great interest, see for example [Gen18, Che17, BCG+18, BCG+19b, CC19a, CC19b, FP16]. It would be interesting to understand the boundary of a $GL^+(2, \mathbb{R})$ orbit closure in the larger compactifications studied by these works.

The study of orbit closures via their boundary is already proving successful, see for example [MW18, Api18, AN].

A version of $\tilde{\mathcal{H}}$ appeared previously in the work of McMullen [McM13].

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2. The partial compactification

Lemma 2.1. A sequence $(X_n, \omega_n) \in \mathcal{H}$ converges to $(X, \omega)$ in $\tilde{\mathcal{H}}$ if and only if, for every subsequence $n_k$ such that $X_{n_k}$ converges in $\overline{\mathcal{M}}_g$, the sequence $(X_{n_k}, \omega_{n_k})$ breaks into a union of finitely many subsequences $(X_{n_k}^{(i)}, \omega_{n_k}^{(i)})$ such that each $(X_{n_k}^{(i)}, \omega_{n_k}^{(i)})$ converges in $\mathcal{H}$ and the limit $(X', \omega^{(i)}) \in \tilde{\mathcal{H}}$ satisfies that $\pi(X', \omega^{(i)}) = (X, \omega)$ for all $i$.

Before proving the lemma, we first state a corollary.

Corollary 2.2. $\tilde{\mathcal{H}}$ and $\mathbb{P}\tilde{\mathcal{H}}$ are Hausdorff topological spaces.

Proof of Corollary 2.2. Note that Lemma 2.1 also applies to nets, and a space is Hausdorff if and only if convergent nets have unique limits. For $\mathbb{P}\tilde{\mathcal{H}}$, the conclusion similarly follows from a projective version of Lemma 2.1. □

Remark 2.3. Once $\mathbb{P}\tilde{\mathcal{H}}$ is known to be Hausdorff, it follows that it is metrizable [Wil04, Corollary 23.2]. Hence $\tilde{\mathcal{H}}$ is also metrizable, since it is a $\mathbb{C}^*$ bundle over $\mathbb{P}\tilde{\mathcal{H}}$. In particular, using nets is not required.

An irreducible component $Y$ of $(X', \omega') \in \tilde{\mathcal{H}}$ has two possible types, depending on whether $\omega'$ is identically zero or not on $Y$. We call the former type a “collapsed component”. The idea of the proof of Lemma 2.1 is that the data of $(X', \omega')$ is the data of $\pi(X', \omega')$ plus the additional data of the collapsed components and how the components are glued together at nodes. This additional data is recorded in $\overline{\mathcal{M}}_g$. In other words, the images of $(X', \omega')$ in $\tilde{\mathcal{H}}$ and $\overline{\mathcal{M}}_g$ almost determine...
\((X', \omega') \in \mathcal{H}\) uniquely. The only situation in which \((X', \omega') \in \mathcal{H}\) is not determined uniquely is when there are multiple components of the limit in \(\mathcal{M}_g\) that are isomorphic. In this case, there may be finitely many \((X', \omega') \in \mathcal{H}\), corresponding to different permutations of non-zero differentials on isomorphic components of \(X\).

**Proof of Lemma 2.1.** For \((X, \omega) \in \tilde{\mathcal{H}}\), we first analyze its preimage in \(\mathcal{H}\). Suppose that \((X, \omega)\) consists of \((Y_1, \omega_1), \ldots, (Y_m, \omega_m)\), where \(Y_1, \ldots, Y_m\) are irreducible components of \(X\) and \(\omega_i\) is the restriction of \(\omega\) to \(Y_i\), which is non-zero. Then \(\pi^{-1}(X, \omega) \subset \mathcal{H}\) is a compact subset given by the union of certain products of \(\mathcal{M}_{g_i, n_i}\) parameterizing the collapsed components. In particular, all \((X', \omega') \in \pi^{-1}(X, \omega)\) contain the same set of non-collapsed components \((Y_i, \omega_i)\) and only the collapsed components can vary.

Suppose that \((X_n, \omega_n)\) converges to \((X, \omega)\) in \(\tilde{\mathcal{H}}\). Let \(n_k\) be a subsequence such that \(X_n\) converges to \(X'\) in \(\mathcal{M}_g\). Then \(X'\) and \(\omega\) determine finitely many possible limits \((X', \omega^{(i)}) \in \pi^{-1}(X, \omega)\), where \(\omega^{(i)} = \omega\) on the non-collapsed components, \(\omega^{(i)} = 0\) on the collapsed components, and the \(\omega^{(i)}\) differ from each other by permuting differentials on isomorphic non-collapsed components. Thus the sequence \((X_n, \omega_n)\) breaks into finitely many subsequences, each of which converges to one such limit \((X', \omega^{(i)})\).

Conversely, suppose for every subsequence \(n_k\) such that \(X_n\) converges in \(\mathcal{M}_g\), the sequence \((X_n, \omega_n)\) breaks into a finite union of subsequences \((X^{(i)}_{n_k}, \omega^{(i)}_{n_k})\) such that each \((X^{(i)}_{n_k}, \omega^{(i)}_{n_k})\) converges in \(\mathcal{H}\) and the limit \((X', \omega^{(i)}) \in \mathcal{H}\) satisfies \(\pi(X', \omega^{(i)}) = (X, \omega)\). Since \(\pi^{-1}(X, \omega)\) is compact, this implies that \((X_n, \omega_n)\) converges to \(\pi^{-1}(X, \omega)\), which gives the result. \(\square\)

The second criterion for convergence below forms part of our basic understanding of \(\tilde{\mathcal{H}}\). It says that a sequence converges if and only if the flat metrics converge away from the part of the surfaces that is shrinking.

**Theorem 2.4 (Mirzakhani-Wright).** \((X_n, \omega_n) \in \mathcal{H}\) converge to \((X, \omega)\) in \(\tilde{\mathcal{H}}\) if and only if there are decreasing neighborhoods \(U_n \subset X\) of the set of marked points \(S\) with \(\cap U_n = S\) and maps \(g_n : X \setminus U_n \to X_n\) that are diffeomorphisms onto their range, such that

1. \(g_n^*(\omega_n) \to \omega\) in the compact open topology on \(X \setminus S\), and
2. the injectivity radius of points not in the image of \(g_n\) goes to zero uniformly in \(n\).
The injectivity radius at a point is defined as the sup of all $\varepsilon > 0$ such that the ball of radius $\varepsilon$ in the flat metric is embedded and does not contain any marked points or singularities. For the proof, see [MW17, Definition 2.2, Proposition 9.8].

**Corollary 2.5.** The action of $GL^+(2, \mathbb{R})$ extends continuously to $\tilde{\mathcal{H}}$.

If $(X_n, \omega_n)$ converge to $(X, \omega)$, then there are natural collapse maps $f_n: X_n \to X'$, defined for $n$ sufficiently large. Here $X'$ is $X$ with some additional identification between marked points. (These are only well defined up to pre- and post-compositions of automorphisms. This is related to the fact that all the moduli spaces under consideration are only orbifolds.)

Note that if $\Sigma'$ is a finite subset of $X'$ containing the nodes, and $\Sigma$ is its pre-image on $X$, then $H_1(X, \Sigma) \cong H_1(X', \Sigma')$. This allows us to conflate $X$ and $X'$ when working on the level of relative (co)homology.

The maps $f_n$ can be chosen to map zeros of $\omega_n$ to zeros or marked points. If $\Sigma_n$ is the set of zeros of $\omega_n$, we obtain well defined maps

$$f_n^*: H^1(X, \Sigma) \to H^1(X_n, \Sigma_n) \quad \text{and} \quad (f_n)_*: H_1(X_n, \Sigma_n) \to H_1(X, \Sigma).$$

The first map is injective and the second surjective. Hence $f_n^*$ naturally identifies the tangent space $(X, \omega)$ in its stratum with the image of $f_n^*$, a subspace of the tangent space of the stratum of $(X_n, \omega_n)$.

This subspace can also be identified with the annihilator $\text{Ann}(V_n)$ of the subspace $V_n = \ker((f_n)_*)$ of relative cohomology. We call $V_n$ the space of vanishing cycles, although some care should be taken since this term might equally well be used to refer to a similar object in absolute homology.

Since so much collapsing can occur, we do not know of a nice way to describe the local structure of $\mathcal{H}$ near a boundary point. However we have the following lemma indicating that the situation is not too pathological from at least one point of view.

**Proposition 2.6.** Let $(X, \omega) \in \tilde{\mathcal{H}}_{<\infty} \text{Ann}(V)$ can be defined on a neighborhood of $(X, \omega)$ in $\mathcal{H}$ and is locally constant. There is a neighborhood of 0 in $\text{Ann}(V)$ such that if $\xi_n, \xi$ are in this neighborhood and $\xi_n \to \xi$, then $(X_n, \omega_n) + \xi_n$ is well-defined and converges to $(X, \omega) + \xi$.

Here we use the notation “$+\xi$” to denote the surface whose period coordinates differ from the original surface by $\xi$: this is only defined for $\xi$ small. This proposition, which appears as [MW17, Proposition 2.6], rules out the pathological possibility that the stratum could get skinnier and skinner as it approaches the boundary point $(X, \omega)$, so
that only increasingly tiny deformations in the $\text{Ann}(V)$ directions are well defined.

We end the section with some remarks.

**Remark 2.7.** We could allow $(X, \omega)$ to have marked points, i.e. zeros of order zero.

**Remark 2.8.** One could define a version of $\widetilde{\mathcal{H}}$ that remembered finitely much more information, namely which of the marked points were glued together after the collapse.

**Remark 2.9.** As we hinted at in the introduction, one could state Theorem 1.2 in a dual form, which would say that the space of equations defining a boundary invariant subvariety is the image under the collapse map $(f_n)_*$ of the set of equations defining the invariant subvariety.

### 3. Complex analytic spaces

We first recall the definition of a complex analytic space \cite{GR84}.

**Definition 3.1.** A $\mathbb{C}$-space is a pair $(X, \mathcal{O}_X)$, where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of $\mathbb{C}$-algebras on $X$. A morphism $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a continuous map $f : X \rightarrow Y$ together with a collection of $\mathbb{C}$-algebra maps $\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(f^{-1}(V))$ for each open subset $V \subset Y$ that commute with restriction maps. The morphism is called an isomorphism if $f$ is a homeomorphism and all the maps $\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(f^{-1}(V))$ are isomorphisms.

**Definition 3.2.** A $\mathbb{C}$-space $(X, \mathcal{O}_X)$ is a complex analytic space if it is locally isomorphic to the vanishing locus of a finite collection $f_1, \ldots, f_k$ of holomorphic functions defined on an open subset $U$ of some affine space $\mathbb{C}^n$, equipped with the sheaf of holomorphic functions on $U$ modulo the ideal $(f_1, \ldots, f_k)$. A morphism of complex analytic spaces is a morphism of their underlying $\mathbb{C}$-spaces where the maps on sheaves are given by pull back of functions.

**Example 3.3.** Every quasi-projective variety over $\mathbb{C}$ is in particular a complex analytic space. A morphism of quasi-projective varieties is in particular a morphism of complex analytic spaces.

**Lemma 3.4.** Let $M$ and $N$ be compact connected complex manifolds, and let $X$ be a complex analytic space. If a morphism $f : M \times N \rightarrow X$ maps $\{m\} \times N$ to a point for some $m \in M$, then $f$ factors through the projection $M \times N \rightarrow M$.

A version of this result, sometimes called the Rigidity Lemma, holds for algebraic varieties. Presumably Lemma 3.4 is also known, but we do not know a reference.
Proof. Let $S$ be the set of $m \in M$ such that $\{m\} \times N$ is mapped to a point. Note that $S$ is closed.

We assume that $X$ is not a point, for otherwise the claim is trivial. Suppose in order to find a contradiction that we can pick $m_0 \in \partial S$. Since $X$ is a positive dimensional complex analytic space, every point in $X$ has a neighborhood $U$ such that for any pair of distinct points $p, q \in U$ there exists $g \in \mathcal{O}_X(U)$ (i.e. a holomorphic function $g: U \to \mathbb{C}$) such that $g(p) \neq g(q)$. Let $U$ be such a neighborhood of the point $f(\{m_0\} \times N)$. The open set $f^{-1}(U)$ contains $\{m_0\} \times N$. Since $N$ is compact, we can find an open set $U_M$ of $M$ containing $m_0$ such that $U_M \times N \subset f^{-1}(U)$. Pick $m_1 \in U_M \setminus S$, and pick $n_1, n_2 \in N$ such that $f(m_1, n_1) \neq f(m_1, n_2)$. Choose $g \in \mathcal{O}_X(U)$ such that $g \circ f(m_1, n_1) \neq g \circ f(m_1, n_2)$. The function $g \circ f$ restricted to $\{m_1\} \times N$ is a non-constant holomorphic function on a connected compact complex manifold, which is a contradiction. □

Proof of Theorem 1.1. We first prove the claim for each stratum, regardless of whether it is connected or not, and then later we will address the individual connected components.

Let $\mathcal{H}(\kappa_1, \ldots, \kappa_s, -2, -2)$ be the stratum of meromorphic differentials that have zeros $z_i$ of type $\kappa$ and two double poles $p_1$ and $p_2$. As long as $\kappa \neq (2)$, by [GT] there exists a differential $(C, \xi)$ in this stratum such that $\xi$ has zero residue at $p_1$ and $p_2$. Attach $C$ to a fixed genus one curve $E_1$ at $p_1$, where $E_1$ carries a fixed non-trivial differential $\omega_1$, and attach $C$ to a pencil $B \simeq \mathbb{P}^1$ of plane cubics $E_b$ by identifying $p_2$ to a base point of the pencil, where $b \in B$ parametrizes the genus one curves in the pencil; see Figure 3.1. Note that the Hodge bundle

![Figure 3.1. The underlying nodal curves in the construction](image)

$\mathcal{H}$ restricted to $B$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(g-1)}$, as $C$ and $E_1$ do not vary and in the case of genus one the Hodge bundle on a pencil of plane cubics has degree one; see e.g. [HM98, Chapter 3 F]. Let $\sigma$ be a
non-trivial global section of the $O(1)$ part. Then $\sigma$ has only one zero. Let $\omega_b = \sigma(b)$ be the corresponding differential on $E_b$. It follows that $\omega_b$ is non-trivial on all $E_b$ but one (when $\sigma(b)$ is zero).

Let $[u, v]$ be the homogeneous coordinates of $\mathbb{P}^1$. For given $[u, v]$, consider the one-parameter family $B_{[u, v]} \cong B$ of stable differentials given by

$$(E_1, u\omega_1), (C, 0), (E_b, v\omega_b)$$

where $b$ varies in $B$. By the zero residue assumption at $p_1$ and $p_2$, we know that $B_{[u, v]}$ is contained in $\mathbb{P}\mathcal{H}(\kappa)$ according to $\text{[BCG+18]}$, as the global residue condition holds. Now varying $[u, v]$ as well, the union of these $B_{[u, v]}$ forms a complex two-dimensional family isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, where the base $\mathbb{P}^1$ (the former) has coordinates $[u, v]$ and the fiber $\mathbb{P}^1$ (the latter) over $[u, v]$ corresponds to $B_{[u, v]} \cong B$. Note that the fiber $B_{[u, v]}$ is contracted by $\pi$ iff $[u, v] = [1, 0]$, as it gets contracted iff the differentials on $E_b$ are identically zero. We thus conclude the proof by using Lemma 3.4, where in that context $M = \mathbb{P}^1$ (the base), $N = \mathbb{P}^1$ (the fiber), $X = \mathbb{P}\mathcal{H}(\kappa)$, $m = [1, 0] \in M$, and $B_{[1, 0]} = \{m\} \times N$ is the only fiber mapped to a point under $\pi$.

If $\mathcal{H}(\kappa)$ is disconnected, then the above proof also applies to each connected component of $\mathcal{H}(\kappa)$. For example, consider $\mathcal{H}(2g-2)$, which has three connected components $\mathcal{H}(2g-2)^{\text{hyp}}$, $\mathcal{H}(2g-2)^{\text{even}}$ and $\mathcal{H}(2g-2)^{\text{odd}}$ when $g > 3$. If the genus $g-2$ curve $C$ is hyperelliptic and if the unique zero $z$ and the two poles $p_1, p_2$ on $C$ are all Weierstrass points (or 2-torsion to each other if $C$ is of genus one), then the constructed family $B_{[u, v]}$ lies in $\mathbb{P}\mathcal{H}(2g-2)^{\text{hyp}}$. For the spin components, since the nodal curves in the family are of compact type, their spin parity will be determined by the parity of $C$ only, as the parity of the two elliptic tails does not vary. Hence one can take the underlying curve $(C, z, p_1, p_2)$ from a meromorphic differential in the respective spin components of $\mathcal{H}(2g-2, -2, -2)$ according to $\text{[KZ03, Boi15]}$. The other cases are similar and we omit the details.

□

Remark 3.5. When $\mathbb{P}\mathcal{H}(\mu)$ is disconnected, the closures of its connected components in the Hodge bundle or in the Deligne-Mumford moduli space may intersect each other in the boundary. See $\text{[Che17, Gen18]}$ for some examples. Nevertheless, one can add log structures to help distinguish the components in the boundary $\text{[CC19b]}$.

Remark 3.6. The above proof does not apply to the remaining case $\kappa = (2)$ in genus two, because there is no differential on $\mathbb{P}^1$ with a unique zero and at least two poles such that all residues are zero (see $\text{[BCG+18] Lemma 3.6}$ and $\text{[GT]}$).
Remark 3.7. Following the previous remark, indeed $\mathbb{P}\tilde{\mathcal{H}}(2)$ can be given a natural variety structure such that it is compatible with the other algebraic compactifications. To see this, we slightly abuse notation to denote by $\mathbb{P}\tilde{\mathcal{H}}(2)$ the compactification in $[\text{BCG}^{18}]$. Namely, we mark the unique zero $z$ and take the closure of $\mathbb{P}\tilde{\mathcal{H}}(2)$ in the projectivized Hodge bundle over $\mathcal{M}_{2,1}$. Note that $\mathbb{P}\tilde{\mathcal{H}}(2)$ can be identified via hyperelliptic (admissible) double covers with the moduli space $\mathcal{M}_{0,1,5}$ of stable rational curves with six marked branch points where one of the markings is distinguished and the others are unordered, and the distinguished marking $z_1$ corresponds to the image of the unique double zero $z$ in the domain of the covers.

If $(X, \omega, z)$ in $\mathbb{P}\tilde{\mathcal{H}}(2)$ has a genus one component $C$ containing $z$, then $\omega$ is identically zero on $C$. This is because $\omega$ restricted to $C$ must have a double zero (at $z$), and can have at most a single simple pole (at the node), and no such non-zero $\omega$ exists. Under the hyperelliptic (admissible) cover, the image of $C$ in the target is a rational tail that contains the distinguished marking $z_1$ and two other unordered markings. Hence forgetting the moduli of $C$ is the same as forgetting the moduli of the image rational tail that contains $z_1$ and two other markings (where the moduli corresponds to the cross ratio of the three markings together with the attaching node). By examining the boundary strata under the map $\pi: \mathbb{P}\tilde{\mathcal{H}}(2) \to \mathbb{P}\tilde{\mathcal{H}}(2)$, one can see that the above phenomenon is the only difference between the two spaces.

Let $\mathcal{M}_{0,A}$ be the Hassett weighted moduli space $\mathcal{M}_{0,A}$ $[\text{Has03}]$ where $A$ assigns weight $\epsilon$ to $z_1$ for $0 < \epsilon \ll 1$ and assigns weight $\frac{1}{2} - \epsilon$ to the other markings. The contraction morphism $f: \mathcal{M}_{0,1,5} \to \mathcal{M}_{0,A}$ exactly forgets the moduli of a rational tail that contains $z_1$ and two other markings, since the sum of their weights is less than one. Hence we can identify $\mathbb{P}\tilde{\mathcal{H}}(2)$ with $\mathcal{M}_{0,A}$ and identify the map $\pi$ with $f$, where $\mathcal{M}_{0,A}$ is an algebraic variety and $f$ is an algebraic morphism.

4. Cautionary examples

Our main cautionary example concerns a discontinuity in the behavior of periods for a certain straightforward degeneration of translation surfaces. Consider surfaces as in Figure 4.1 that result from gluing in two horizontal cylinders into slits on a surface in $\mathcal{H}(2)$. We will consider a family $(X_\varepsilon, \omega_\varepsilon)$ of such surfaces, parameterized by $\varepsilon \in (0, \varepsilon_0)$, such that

$$a = a' = b = b' = i\varepsilon$$
Figure 4.1. A surface with cylinders of large modulus

and

\[ x = \frac{1}{\epsilon}, \quad y - x = d - c. \]

Here we are using the label of the edge to also refer to the period coordinate of that edge. All other edges periods are constant along the family, and we assume \( c \neq d \).

Let \((X_0, \omega_0)\) denote the limit in \( \tilde{\mathcal{H}} \), and define the periods \( x, y \) etc on the limit by pushing forward the relative homology classes and then taking the period on the limit \((X_0, \omega_0)\).

All of these surfaces \((X_\epsilon, \omega_\epsilon)\) satisfy the equation on period coordinates \( y - x = d - c \). On the limit surface, the periods \( x \) and \( y \) are now zero, and since \( c \) and \( d \) are constant and \( c \neq d \) we have \( y - x \neq d - c \). That is, the limit surface does not satisfy the limit equation, even though the equation holds identically on the degenerating family \((X_\epsilon, \omega_\epsilon)\)!

In our situation, we worry that there might be some invariant subvariety \( \mathcal{M} \) containing all \((X_\epsilon, \omega_\epsilon)\), such that \( y - x = d - c \) is one of the equations defining \( \mathcal{M} \) in local period coordinates. If such an \( \mathcal{M} \) were to exist, then the tangent direction to the boundary orbit closure \( \mathcal{M}' \ni (X_0, \omega_0) \) resulting from scaling \( \omega_0 \) would not lie in \( T_{(X_\epsilon, \omega_\epsilon)}(\mathcal{M}) \), contradicting Theorem \[ 1.2 \]. However, it is easy to see that there is no such \( \mathcal{M} \) in this extremely particular situation, using the Cylinder Deformation Theorem of \[ \text{Wri15} \].

Our next cautionary example is even more basic. Consider the quasi-projective variety \( \mathcal{M} = \{(x, y, t) : y^2 = x^3 + t, t \neq 0\} \), whose closure in affine three-space is given by \( \mathcal{M} \cup \partial \mathcal{M} \), where \( \partial \mathcal{M} = \{(x, y, 0) : y^2 = x^3\} \). If one defines the tangent space as the space of derivatives of smooth maps from an interval in \( \mathbb{R} \) into the variety, than the tangent space to \( \partial \mathcal{M} \) at \((0, 0, 0)\) is zero dimensional; it does not contain the limit of tangent spaces to \( \mathcal{M} \) along a sequence of points converging
to $(0,0,0)$. In our situation, this relates to the possible worry that the tangent space to the boundary orbit closure might be smaller than expected. Keep in mind that we do not have any results on whether the closure of an orbit closure in any strata compactification is smooth. But in our situation we at least know that the relevant part of $\partial \mathcal{M}$ is a properly immersed smooth orbifold, which may account for why this possible worry is comparatively easier to rule out.

5. Cylinder deformations

Recall that two cylinders on $(X,\omega) \in \mathcal{M}$ with core curves $\gamma_1$ and $\gamma_2$, are said to be $\mathcal{M}$-parallel if there is some $c \in \mathbb{R}$ such that $\int_{\gamma_1} \omega = c \int_{\gamma_2} \omega$ holds on a neighborhood of $(X,\omega)$ in $\mathcal{M}$. We will call $c$ the ratio.

Given a cylinder $C$ on $(X,\omega)$ with an orientation of its core curve, define $u^C_t(X,\omega)$ (resp. $a^C_t(X,\omega)$) to be the result of rotating the surface so $C$ is horizontal and the period of the core curve is positive, applying $u_t$ (resp. $a_t$) just to $C$, and applying the inverse rotation. Here

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a_t = \begin{pmatrix} 1 & 0 \\ 0 & e^{t} \end{pmatrix}.$$ 

Define an orientation on a collection of parallel cylinders to be a choice of orientation on each core curve $\gamma_i$; say that the orientation is consistent if the integrals $\int_{\gamma_i} \omega$ are positive multiples of each other.

Given an equivalence class $\mathcal{C} = \{C_1, \ldots, C_n\}$ of consistently oriented $\mathcal{M}$-parallel cylinders, we define $u^C_t(X,\omega) = u^C_1 \circ \cdots \circ u^C_n(X,\omega)$, and similarly for $a^C_t$. The main result from [Wri15] gives the following.

**Theorem 5.1** (Cylinder Deformation Theorem). If $\mathcal{C}$ is an equivalence class of $\mathcal{M}$-parallel cylinders on $(X,\omega) \in \mathcal{M}$, then

$$a^C_s(u^C_t(X,\omega)) \in \mathcal{M}$$

for all $s,t \in \mathbb{R}$.

The rest of this section is devoted to the following theorem, which may be viewed as a black box. It will be used to avoid the situation of the first cautionary example.

---

1This notation works well if $\mathcal{M}$ is an embedded manifold, but in reality it may have self-crossings. In general $\mathcal{M}$ is a manifold which is immersed into the stratum, and it is an abuse of notation to write $(X,\omega) \in \mathcal{M}$. It would be more precise to write $(X,\omega,V) \in \mathcal{M}$, where $V \subset H^1(X,\Sigma,\mathbb{C})$ is the image of the tangent space to $\mathcal{M}$ in the stratum. Typically $V = T_{(X,\omega)}\mathcal{M}$ is determined by $(X,\omega)$, but at points of self-crossing there many be finitely many choices of $V$. The notion of $\mathcal{M}$-parallel of course depends on this choice. As is typical, we now continue with the abuse of notation.
Theorem 5.2. For each invariant subvariety $\mathcal{M}$ and each $M$ sufficiently large, there is some $M' > M$ such that for any $(X, \omega) \in \mathcal{M}$, there is a cylinder deformation of $(X, \omega)$ that remains in $\mathcal{M}$, only deforms disjoint cylinders of modulus greater than $M$, and leaves all the deformed cylinders with modulus in $[M, M']$.

This is closely related to the Cylinder Finiteness Theorem [MW17, Theorem 5.1], and in fact we use the following version of this result.

**Theorem 5.3** (Cylinder Finiteness Theorem). For any $\mathcal{M}$, the set of ratios of $\mathcal{M}$-parallel cylinders is finite. Furthermore, the set of equations which restrict the heights of $\mathcal{M}$-parallel cylinders is also finite.

We use the term “height” of a cylinder to denote the dimension transverse to the circumference. For any equivalence class $\mathcal{C} = \{C_1, \ldots, C_n\}$ of $\mathcal{M}$-parallel cylinders on any $(X, \omega) \in \mathcal{M}$ and any $t_1, \ldots, t_n \in \mathbb{R}$, we have

$$a_{t_1}^{C_1} \circ \cdots \circ a_{t_n}^{C_n}(X, \omega) \in \mathcal{M}$$

if and only if the heights of the $C_i$ on $a_{t_1}^{C_1} \circ \cdots \circ a_{t_n}^{C_n}(X, \omega)$ satisfy certain linear equations depending on $(X, \omega)$, $\mathcal{M}$ and $\mathcal{C}$. The second part of the Cylinder Finiteness Theorem says that only finitely many systems of linear equations arise in this way for each $\mathcal{M}$.

This statement of the Cylinder Finiteness Theorem is slightly stronger than [MW17, Theorem 5.1], but it follows from the same proof. We will nonetheless give an outline of a different proof here, using that $\mathcal{M}$ is an algebraic variety.

**Proof of Theorem 5.3.** Let $\mathcal{U}$ be a small neighborhood in $\mathcal{H}$ of the locus in $\overline{\mathcal{H}}$ where there are simple poles. We can pick $\mathcal{U}$ so that $\mathcal{M} \cap \mathcal{U}$ contains only finitely many components (compare to the arguments in Section 5). We can also pick it so that every $(X, \omega) \in \mathcal{U}$ has cylinders of enormous modulus corresponding to the simple poles of a nearby point of $\overline{\mathcal{H}}$.

For each equivalence class $\mathcal{C}$ on each $(X, \omega) \in \mathcal{M}$, consider

$$\lim_{t \to \infty} a_t^\mathcal{C}(X, \omega) \in \overline{\mathcal{H}}.$$ 

On this limit, each cylinder of $\mathcal{C}$ gives rise to a simple pole. (The ratios of the residues are exactly the ratios of the cylinders in $\mathcal{C}$.) Hence, for $t$ large enough, $a_t^\mathcal{C}(X, \omega)$ is in one of the finitely many components of $\mathcal{M} \cap \mathcal{U}$.

Each component has only finitely many large modulus cylinders, and there are only finitely many sets of linear equations determining which deformations of the large cylinders stay in $\mathcal{M}$. There are also
only finitely many equations fixing the ratios of these large modulus cylinders.

Remark 5.4. In the previous proof, we warn that \( \lim_{t \to \infty} a^t_i(X, \omega) \) may lie in a higher codimension subvariety of the boundary of \( \mathcal{M} \), so the relationship between finiteness of ratios and finiteness of the number of components of the boundary is less obvious than it may seem at first. A related warning is that, for an arbitrary point of the boundary of \( \mathcal{M} \), there may be infinite cylinders (simple poles) that are \( \mathcal{M} \)-parallel to finite cylinders. Neither phenomenon occur unless \( \mathcal{C} \) admits “non-standard” cylinder deformations (see [MW17] for the definition). The simplest relevant example is when \( \mathcal{M} \) is a stratum and \( \mathcal{C} \) is a pair of homologous cylinders.

Now we give an elementary lemma.

**Lemma 5.5.** For each linear subspace \( V \subset \mathbb{R}^n \), and each \( H > 1 \), there is some \( H' > H \) such that for every \( v \in V \cap [0, \infty)^n \), there exists \( v' \in V \) such that every coordinate of \( v \) less than \( H \) is equal to the corresponding coordinate of \( v' \), and every other coordinate of \( v' \) is in \([H, H']\).

**Proof.** Consider the set of all \( v \in V \cap [0, \infty)^n \) where a fixed subset of the coordinates are at most \( H \). For each such \( v \), let \( v' \) be a vector as above minimizing \( \|v'\|_\infty \). The function \( v \mapsto \|v'\|_\infty \) is easily seen to be semi-continuous. Since this function only depends on a fixed subset of the coordinates of \( v \), all of which are in \([0, H]\), it can be viewed as a function on a compact set, and is hence bounded. 

**Proof of Theorem 5.2.** Let \( R \) be the maximum ratio of two \( \mathcal{M} \)-parallel cylinders. Let \( H' \) be the maximum of the \( H' \) given by Lemma 5.5 for each of the finitely many systems of linear equations given by Theorem 5.3, with \( H = RM \). Finally, set \( M' = H'R \).

Let \( \mathcal{C} = \{C_1, \ldots, C_n\} \) be an equivalence class of \( \mathcal{M} \)-parallel cylinders that contains at least one cylinder of modulus at least \( M \), say \( C_1 \). Consider the vector \( \vec{h} \in \mathbb{R}^n \) of heights. Then consider the normalized vector \( v = \vec{h}/c(C_1) \), whose first entry is the modulus \( m(C_1) = h(C_1)/c(C_1) \) of \( C_1 \). Lemma 5.5, with \( H = RM \), applied to \( v \), gives a \( v' \) with all coordinates at most \( H' \) and all of the changed coordinates at least \( RM \). If we define \( \vec{h}' = c(C_1)v' \), then all entries of \( \vec{h} \) are at most \( H'c(C_1) \), and all the changed coordinates (compared to \( \vec{h} \)) are at least \( RMc(C_1) \). This shows that we can deform the cylinders in \( \mathcal{C} \) with height greater than \( Hc(C_1) = RMc(C_1) \) so that they have height in \([RMc(C_1), H'c(C_1)]\).

Any cylinder in \( \mathcal{C} \) with height greater than \( RMc(C_1) \) has, before the deformation, modulus at least \( M \). After the deformation, it has modulus in \([M, H'R]\).
Since cylinders of modulus greater than a universal constant are disjoint, this gives the result. □

6. The Boundary of an Orbit Closure

Before we discuss Theorem 1.2, we should clarify the basic context for our discussion. Note that, in the context of multi-component surfaces, it has not been proven that every $GL^+(2, \mathbb{R})$ orbit closure is a variety.

Lemma 6.1. The boundary of an invariant subvariety $\mathcal{M}$ in a stratum of $\tilde{\mathcal{H}}_{<\infty}$ is an invariant subvariety, and is defined by linear equations in local period coordinates in the boundary stratum.

Sketch of proof. We first see that it is an algebraic variety. Indeed, the closure of $\mathcal{M}$ in $\tilde{\mathcal{H}}$ is a variety, and hence so is its intersection with the locus of nodal surfaces with a fixed number of components and where the differential is required to be zero on certain components. Up to a finite cover, such a locus is a union of products. Given a subvariety of a product of varieties, its image under projection to a factor is also a variety.

The boundary is $GL^+(2, \mathbb{R})$ invariant since $\mathcal{M}$ is, and since the $GL^+(2, \mathbb{R})$ action extends continuously to the boundary. Hence a folklore observation of Kontsevich, recorded in [Möl08, Proposition 1.2], gives that the boundary is locally defined by linear equations in period coordinates. □

We start our proof of Theorem 1.2 by applying Theorem 5.2.

Lemma 6.2. Suppose that $(X_n, \omega_n) \in \mathcal{M}$ converge to $(X_\infty, \omega_\infty) \in \tilde{\mathcal{H}}_{<\infty}$. Let $M$ be strictly greater than the modulus of any cylinder on $(X_\infty, \omega_\infty)$, and let $M'$ be given by Theorem 5.2. Then the $(X'_n, \omega'_n)$ given by Theorem 5.2 also converge to $(X_\infty, \omega_\infty) \in \tilde{\mathcal{H}}$.

Proof. Let $\varepsilon_n$ be the largest circumference of a cylinder of modulus greater than $M$ on $(X_n, \omega_n)$. It is easy to see that $\varepsilon_n \to 0$; see [MW17, Section 2.4] for more details.

Since only disjoint cylinders of smaller and smaller circumference are modified in the passage from $(X_n, \omega_n)$ to $(X'_n, \omega'_n)$, the two sequences have the same limit in $\tilde{\mathcal{H}}$. □

Corollary 6.3. It suffices to prove Theorem 1.2 under the assumption that there is some $M'$ such that the $(X_n, \omega_n)$ have no cylinders of modulus greater than $M'$.

Proof. First note that, assuming $M$ is large enough, two cylinders of modulus at least $M$ cannot cross. Thus, passing from $(X_n, \omega_n)$ to
(\(X'_n, \omega'_n\)) removes the large modulus cylinders without creating any new ones, since even at the end of the deformation the large modulus cylinders have modulus at least \(M\).

Next, note that the cylinder deformation provides a path in \(\mathcal{M}\) from \((X_n, \omega_n)\) to \((X'_n, \omega'_n)\). Along this path, \(T(\mathcal{M})\) and \(T(\mathcal{M}) \cap \text{Ann}(V)\) are constant. □

We now give a result which, together with the above, will rule out the problem of the first cautionary example in Section 4.

**Theorem 6.4.** For any \((X_\infty, \omega_\infty)\) in the boundary of \(\bar{\mathcal{H}}(\kappa)\), there exist finitely many connected, simply connected sets \(S_i\) with maps \(S_i \rightarrow \mathcal{H}(\kappa)\) such that:

1. There is a family of translation surfaces over each \(S_i\) whose fiber over each point is the translation surface represented by the image of that point in \(\mathcal{H}(\kappa)\).
2. There is a locally constant map \(H_1(X, \Sigma) \rightarrow H_1(X_\infty, \Sigma_\infty)\) for each \(S_i\) from the relative homology of the fibers of this family to that of \((X_\infty, \omega_\infty)\), such that for any sequence in \(S_i\) converging to \((X_\infty, \omega_\infty)\), the collapse maps discussed in Section 2 can be chosen to induce this map on \(H_1\).
3. With respect to this map, the relative and absolute periods extend continuously from each \(S_i\) to \((X_\infty, \omega_\infty)\). (The relative cohomology of \((X_\infty, \omega_\infty)\) is a quotient of the relative cohomology of a translation surface in \(S_i\), so the period of each relative homology class can be defined at \((X_\infty, \omega_\infty)\).)
4. Let \(\mathcal{U}_M\) denote the subset of \(\mathcal{H}(\kappa)\) without cylinders of modulus greater than \(M\). Then for \(M\) sufficiently large, the union \(\cup S_i\) contains a neighborhood of \((X_\infty, \omega_\infty)\) in \(\mathcal{U}_M\).
5. For each \(i\), any subvariety \(\mathcal{M} \subset \mathcal{H}(\kappa)\) intersects the image in \(\mathcal{H}(\kappa)\) of \(S_i\) in at most finitely many connected components.

See Section 8 for an outline of the proof. The main purpose of all the above is to get the following result.

**Lemma 6.5.** If \((X_n, \omega_n) \in \mathcal{M}\) converge to \((X_\infty, \omega_\infty)\), then \(\omega_\infty \in T(\mathcal{M})\), using the identification between \(H^1(X_\infty, \Sigma_\infty)\) and \(\text{Ann}(V)\).

**Proof.** It suffices to assume that we are in the situation of Corollary 6.3. By partitioning into finitely many subsequences, it also suffices to assume that all \((X_n, \omega_n)\) are in one of the \(S_i\) of Theorem 6.4 and moreover in one of the components of \(\mathcal{M} \cap S_i\).

On this component, \(T(\mathcal{M})\) is constant, and is cut out by finitely many equations on periods. By Theorem 6.4, the periods extend continuously to \((X_\infty, \omega_\infty)\), so this gives the result. □
Remark 6.6. The previous lemma is treated rather indirectly in [MW17]. There, it is shown that certain tangent vectors to the orbit closure of $(X_\infty, \omega_\infty)$ are contained in $T(\mathcal{M})$. These tangent vectors span a space that contains $\omega$.

Lemma 6.7. In the situation of the previous lemma, for any sufficiently small $\xi \in T(\mathcal{M}) \cap \text{Ann}(V)$, we have that $(X_\infty, \omega_\infty) + \xi$ is contained in the boundary of $\mathcal{M}$.

Proof. This is handled in the same way as [MW17], by using Proposition 2.6 directly.

Proof of Theorem 1.2. This follows directly from the previous two lemmas. Indeed, the boundary is a variety and is $GL^+(2,\mathbb{R})$ invariant. The previous two lemmas show that its tangent space is “not too big” and “not too small” respectively.

7. The structure of the multi-component boundary

In this section we give some examples and then prove Theorem 1.3.

Lemma 7.1. An invariant subvariety $\mathcal{M}$ is not prime if and only if some (equivalently, any) $(X, \omega) \in \mathcal{M}$ can be written as $(X, \omega) = (X_1, \omega_1) \cup (X_2, \omega_2)$, where the $(X_i, \omega_i)$ are disjoint and are a non-empty union of components, and we have

$$T_{(X,\omega)} = (T_{(X,\omega)} \cap H^1(X_1, \Sigma_1)) \oplus (T_{(X,\omega)} \cap H^1(X_2, \Sigma_2)),$$

where $\Sigma_i$ is the set of zeros of $\omega_i$.

In this case, if $\mathcal{M}'$ is the minimal cover of $\mathcal{M}$ on which we can divide the components into two subsets consistent with $(X, \omega) = (X_1, \omega_1) \cup (X_2, \omega_2)$, then $\mathcal{M}' = \mathcal{M}_1 \times \mathcal{M}_2$ for invariant subvarieties $\mathcal{M}_1, \mathcal{M}_2$.

More concisely, $\mathcal{M}$ is not prime if and only if its tangent space is a direct sum in a way compatible with the decomposition into components, and it suffices to check this at a single point.

Proof. Suppose we have the direct sum decomposition of $T_{(X,\omega)}$. By definition of $\mathcal{M}'$, we have $\mathcal{M}' \subset \mathcal{H}_1 \times \mathcal{H}_2$, where $\mathcal{H}_i$ are strata of multi-component surfaces, and $(X_i, \omega_i) \in \mathcal{H}_i$.

Let $\pi_i : \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_i$ be the coordinate projections, and define $\mathcal{M}_i = \pi_i(\mathcal{M}')$. Thus $\mathcal{M}_i$ is locally described in period coordinates by $T_{(X,\omega)} \cap H^1(X_i, \Sigma_i)$ near $(X_i, \omega_i)$. By analytic continuation, we see that $\mathcal{M}_i$ is locally linear in period coordinates.

Since $\mathcal{M}'$ is locally linear, in a neighborhood of $(X, \omega)$ it is equal to $\mathcal{M}_1 \times \mathcal{M}_2$. Because both varieties are irreducible, $\mathcal{M}' = \mathcal{M}_1 \times \mathcal{M}_2$. \qed
Corollary 7.2. If $\mathcal{M}'$ is a lift of an invariant variety $\mathcal{M}$ to a product of strata of connected components, then $\mathcal{M}'$ is a product of prime invariant varieties and the product decomposition is unique.

Proof. The previous lemma gives that $\mathcal{M}'$ can be written as a product if it is not prime, and we can iterate this to get that $\mathcal{M}'$ is a product of prime invariant varieties. The uniqueness is an exercise in linear algebra. □

Example 7.3. Let $\mathcal{H}$ be a stratum. Then the diagonal embedding of $\mathcal{H}$ in $\mathcal{H} \times \mathcal{H}$ is prime.

Example 7.4. Let $\mathcal{H}$ be a stratum. Consider the locus $\mathcal{M}$ of pairs $((X_1, \omega_1), (X_2, \omega_2))$ for which there exists some $(X, \omega) \in \mathcal{H}$ such that $(X_1, \omega_1)$ is a triple cover of $(X, \omega)$ simply branched over four points and $(X_2, \omega_2)$ is a double cover of $(X, \omega)$ simply branched over two points. The branch points are allowed to be arbitrary points of $X$ that are not zeros of $\omega$. Then $\mathcal{M}$ is a prime invariant variety, but each of the two coordinate projections from $\mathcal{M}$ are infinite to one. (None of the choices in this example are important.)

We begin with the easiest part of Theorem 1.3.

Proof of Theorem 1.3, first statement. Suppose that $\mathcal{N} \subset \mathcal{H}_1 \times \cdots \times \mathcal{H}_k$, where each $\mathcal{H}_i$ is a stratum of single component surfaces.

Note that $\pi_i(\mathcal{N})$ is a constructible set, so it is open in its closure, and hence its Zariski and analytic closures are the same. Thus $\mathcal{N}_i = \pi_i(\mathcal{N})$ and $\mathcal{N}'_i = \mathcal{N}_i \setminus \pi_i(\mathcal{N})$ are both invariant subvarieties.

We now give an example to show that $\pi_i(\mathcal{N})$ need not be closed.

Example 7.5. Let $\mathcal{M}$ denote the set of pairs $((X, \omega, \{p_1\}), (X, \omega, \{p_2, p_3\})) \in \mathcal{H}(2,0) \times \mathcal{H}(2,0,0)$ such that $p_1$ is equal to $p_2$ or $p_3$, and $p_2$ and $p_3$ are interchanged by the hyperelliptic involution. Then the projection of $\mathcal{M}$ to the first factor is equal to $\mathcal{H}(2,0)$ minus the locus where the marked point $p_1$ is a Weierstrass point.

To proceed, we require the following result from [Wri14].

Theorem 7.6. Let $\mathcal{M}$ be an (irreducible) invariant subvariety of single component translation surfaces, and let $\mathcal{M}'$ be a proper invariant subvariety (possibly with several components). Then any proper flat subbundle of $T(\mathcal{M})$ defined over $\mathcal{M} \setminus \mathcal{M}'$ is contained in $\ker(p)$. 
The same statement holds if \( \mathcal{M} \) is a prime invariant subvariety of multi-component translation surfaces satisfying the second conclusion of Theorem 1.3.

Here "proper" just means that the subbundle is not all of \( T(\mathcal{M}) \), and \( p \) is the usual map from relative cohomology to absolute cohomology, both viewed as bundles over \( \mathcal{M} \). In [Wri14] the theorem is stated for bundles over \( \mathcal{M} \) rather than \( \mathcal{M} \setminus \mathcal{M}' \), but the proof gives the more general statement.

Similarly it is not stated for the prime (multi-component) case, but the proof is very similar. One needs the second conclusion of Theorem 1.3 to say that if a collection of pseudo-Anosovs \( (\phi_1, \ldots, \phi_k) \) in the affine group of \( (X, \omega) \) fixes \( T(X,\omega_{\mathcal{N}}) \), then its action on \( T(X,\omega_{\mathcal{N}}) \) has a unique largest and smallest eigenvalue. This is true because such eigenvalues must in fact come from the action on \( p(T(X,\omega_{\mathcal{N}})) \), which is isomorphic to each \( p(T\mathcal{N}_i) \). (This is all extremely closely related to the fact that the diagonal action of geodesic flow \( g_t \) on \( \mathcal{N} \) has a unique largest Lyapunov exponent \( \lambda_1 = 1 \). If \( \mathcal{N} \) were a product, each factor would contribute a copy of the largest Lyapunov exponent.)

**Proof of Theorem 1.3, second statement.** Suppose that \( \mathcal{N} \subset \mathcal{H}_1 \times \cdots \times \mathcal{H}_k \) is a counterexample. Assume \( k \) is as small as possible.

Let \( \pi_i : \mathcal{N} \to \mathcal{N}_i \) be the projection onto the \( i \)-th factor, and let \( \pi_i \) be the projection away from the \( i \)-th factor, so \( \pi_i(\mathcal{N}) \subset \prod_{j \neq i} \mathcal{H}_j \).

Note that \( \ker(\pi_i)_* \) is a subbundle of \( TN_i \). It cannot be equal to \( TN_i \), since this would imply that \( T(\mathcal{N}) = TN_i \oplus T\pi_i \mathcal{N} \), contradicting that \( \mathcal{N} \) is prime. Hence Theorem 7.6 gives that \( \ker(\pi_i)_* \subset \ker(p) \).

Our goal is to show that

\[
(p_i)_* : p(T\mathcal{N}) \to p(T\mathcal{N}_i)
\]

is an isomorphism for each \( i \), which is to say that \( \ker((\pi_i)_*) \subset \ker(p) \). If \( k = 2 \), then \( \pi_1 = \pi_2 \) and \( \pi_3 = \pi_1 \), so we already have the result.

So assume \( k > 2 \). Without loss of generality, assume that \( \ker((\pi_1)_*) \) is not contained in \( \ker(p) \). By minimality of \( k \), we can assume that

\[
\pi_1 \mathcal{N} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_s
\]

is a product (up to passing to a finite cover) of prime invariant subvarieties satisfying the second statement of Theorem 1.3. Here \( s \leq k - 1 \), and it might be that \( s = 1 \).

We now claim that \( \ker((\pi_1)_*) \) contains something in \( T\mathcal{L}_i \setminus \ker(p) \) for some \( i \). We will show that this is true for any \( i \) for which there exists \( v \in \ker((\pi_1)_*) \) such that \( (\pi_\mathcal{L}_i)_*(v) \notin \ker(p) \). (Here we write \( \pi_\mathcal{L}_i \) for the projection onto \( \mathcal{L}_i \), whereas we use \( \pi_i \) for the projection onto \( \mathcal{H}_i \).)
Indeed, given such a $v$, there is a loop in $\mathcal{L}_i$ whose monodromy does not fix $p((\pi_{\mathcal{L}_i})_*(v))$. (For example, this follows formally from Theorem 7.6, since otherwise the preimage under $p$ of the span of $p((\pi_{\mathcal{L}_i})_*(v))$ would define a flat subbundle.) We can lift this loop to a loop in $\mathcal{L}_1 \times \cdots \times \mathcal{L}_s$ by keeping all of the other coordinates constant. Considering $v$ minus its image under monodromy proves the claim.

For such an $i$, we can now consider the bundle $\ker(p_1) \cap T\mathcal{L}_i$.

By the above, this bundle is not contained in $\ker(p)$, so Theorem 7.6 gives that it is equal to $T\mathcal{L}_i$. Hence $T\mathcal{L}_i \subset T\mathcal{N}$. This shows that $\mathcal{L}_i$ is actually a factor of $\mathcal{N}$, showing that $\mathcal{N}$ is not prime. $\square$

We now turn to the third statement.

**Proof of Theorem 1.3, third statement.** There is an isomorphism from $p(T\mathcal{N})$ to $p(T\mathcal{N}_i)$ for each $i$. Since by definition rank is half the dimension of fibers of these bundles, this gives the result. $\square$

**Example 7.7.** Consider the locus $\mathcal{N} \subset \mathcal{H}(2)$ of pairs $((X, \omega), (X, \sqrt{2}\omega))$ for all $(X, \omega) \in \mathcal{H}(2)$. This $\mathcal{N}$ is locally cut out by linear equations in $\mathbb{Q}[\sqrt{2}]$, but both of its projections are a full stratum. This shows that the field $k(\mathcal{N})$ defined in [Wri14] does not have to be equal to the fields $k(\mathcal{N}_i)$ of the projections $\mathcal{N}_i$. It is however of course true that $k(\mathcal{N}_i) \subset k(\mathcal{N})$.

For the fourth statement, recall the decomposition

$$H^1 = \bigoplus p(TM)_\tau \oplus \mathbb{W},$$

where the sum is over the Galois conjugates of $p(TM)$, and $\mathbb{W}$ is the remaining part of the bundle $H^1$ of absolute cohomology. Filip showed that this direct sum decomposition is compatible with the Hodge decomposition $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$. For example, if we set $p(T_{(X,\omega)}\mathcal{M})^{1,0}_\tau = p(T_{(X,\omega)}\mathcal{M})_\tau \cap H^{1,0}(X)$ etc, we have

$$p(T_{(X,\omega)}\mathcal{M})_\tau = p(T_{(X,\omega)}\mathcal{M})^{1,0}_\tau \oplus p(T_{(X,\omega)}\mathcal{M})^{0,1}_\tau.$$ 

Filip’s result is stated for single component surfaces, and the difficulty in that context is that he does not assume the $GL(2, \mathbb{R})$ orbit closure is a subvariety. Prior to Filip, a similar result was known for any Variation of Hodge structure over a quasi-projective base, and this result applies to invariant sub-varieties of multi-component surfaces to give the same decomposition.
The natural factor of $\text{Jac}(X)$ referenced in Theorem 1.3 can be defined as

$$\text{Jac}_M(X, \omega) = \left( \bigoplus p(T_{(X, \omega)} M)_{1,0} \right)^* / \left( \bigoplus p(T_{(X, \omega)} M)_{\tau} \right)^*_Z,$$

where we define $\left( \bigoplus p(T_{(X, \omega)} M)_{\tau} \right)^*_Z$ to be the set of linear functionals on $\bigoplus p(T_{(X, \omega)} M)_{\tau}$ that take integer values on $H^1(X, \mathbb{Z}) \cap \bigoplus p(T_{(X, \omega)} M)_{\tau}$.

**Proof of Theorem 1.3, fourth statement.** By the second statement, we get an isomorphism from $p(T_N)_{1,0}$ to $p(T_N)_{1,0}$. Combining this with the Galois conjugate versions, we get the result. □

**8. Proof of Theorem 6.4**

Let $\hat{\mathcal{H}}(\kappa)$ be the strata compactification constructed in [BCG+19a], which is a smooth complex orbifold. Roughly speaking, $\mathcal{H}(\kappa)$ parameterizes twisted differentials compatible with a full level graph and with matchings of horizontal directions at the nodes, where the top level differentials are not projectivized (see also [BCG+18] for a detailed definition of twisted differentials). Note that $\mathcal{H}(\kappa)$ admits a continuous and surjective map to $\hat{\mathcal{H}}(\kappa)$ by forgetting lower level differentials (i.e. components of area zero). To prove Theorem 6.4, we will first prove an analogous result for $\hat{\mathcal{H}}(\kappa)$.

Take a boundary point $(X, \omega)$ in $\hat{\mathcal{H}}(\kappa)$ such that $X$ has $N$ levels $0, -1, \ldots, -N + 1$ and such that the subsurfaces $X^{(i)}$ in each level $i$ have no simple polar nodes (i.e. no horizontal edges in the level graph). Take a large enough $M$ such that the modulus of any cylinder in any subsurface $X^{(i)}$ of $X$ is smaller than $M$. Then there exists a small ball neighborhood $U \subset \hat{\mathcal{H}}(\kappa)$ of $(X, \omega)$ such that the modulus of any cylinder in any translation surface parameterized in $U$ is smaller than $M$. Let $t_i$ be the smoothing parameter for level $i$ of $(X, \omega)$ for $i = -1, \ldots, -N + 1$. Let $V \subset U$ be the complement of the union of hyperplanes defined by each equation $t_i = 0$. Said differently, $V$ parameterizes smooth translation surfaces near $(X, \omega)$ that have no cylinder of modulus $\geq M$. Consider a subset $S_j$ of $V$ such that the real and imaginary parts of all $t_i$ have prescribed signs for $j = 1, \ldots, 2^{N-1}$. Then $V$ is a disjoint union of these subsets $S_j$, and $V$ can be chosen so that the $S_j$ are connected and simply connected. Consequently, over each $S_j \cup \{(X, \omega)\}$ the relative homology groups of all surfaces in the universal family can be identified as one model. Take any one of these subsets $S_j$ and simply denote it by $S$.

\[^2\text{Here the notation } \hat{\mathcal{H}}(\kappa) \text{ is different from [BCG+19a] whose notation is a bit too heavy to carry over.}\]
Theorem 8.1. For any continuous family of smooth translation surfaces \((X_t, \omega_t)\) in \(S\) that converge to \((X, \omega)\) as \(t \to 0\), the periods of \((X_t, \omega_t)\) obtained by integrating \(\omega_t\) over the paths in the model relative homology group vary continuously and converge to that of \((X, \omega)\). Moreover if \((X, \omega)\) varies continuously in \(ı\) \(H(\kappa)\), these periods also vary continuously with \((X, \omega)\).

Proof. In the model relative homology group over \(S\), choose homology classes \(\gamma_k^{(i)}\) in each level \(i\) subsurface \(X^{(i)}\) for \(k = 1, \ldots, j_i\) such that their collection forms a basis of the relative homology group \(H_1(X^{(i)} \setminus P^{(i)}, Z^{(i)}, Z)\), where \(P^{(i)}\) is the set of polar nodes in \(X^{(i)}\) and \(Z^{(i)}\) is the set of non-polar nodes together with marked zeros in \(X^{(i)}\). Here a node is called polar if it is a pole of the (twisted) differential, and otherwise it is called non-polar. We remark that for the period of a path \(\gamma_k^{(i)}\) joining a non-polar node to be well-defined after plumbing, we choose a reference point near the node as the endpoint of \(\gamma_k^{(i)}\), which gives the so-called perturbed period coordinate constructed in [BCG+19a]. We may also label the subscripts of those \(\gamma_k^{(i)}\) such that \(\int_{\gamma_k^{(i)}} \omega_t \neq 0\) for all \(i\) and all \(t \neq 0\).

Denote by
\[
\lambda_k^{(i)}(t) = \frac{\int_{\gamma_k^{(i)}} \omega_t}{\int_{\gamma_1^{(i)}} \omega_t}
\]
which measures the relative ratios of level-\(i\) periods in the degeneration to \(X\) and which is well-defined also at \(t = 0\). Since there are no simple polar nodes, a local coordinate system of \(ı\) \(H(\kappa)\) at \((X, \omega)\) restricted to this family is given by
\[
\{(t_i(t))_{t_i=0}^{-N+1}; \{\lambda_k^{(i)}(t)\}_{k=2}^{j_i}\}.
\]
Denote by \(t_{[i]}(t) = t_0(t)t_{-1}(t) \cdots t_i(t)\), where one may regard \(t_{[i]}(t)\) as measuring \(\int_{\gamma_1^{(i)}} \omega_t\).

For an element \(\gamma\) in the model relative homology group, write it as the following linear combination
\[
\gamma = \sum_{i,k} c_k^{(i)} \gamma_k^{(i)} + \sum_{h,l} c^{(h,l)} \rho^{(h,l)}
\]

Technically speaking, if there are polar nodes of pole order bigger than two, then the \(t_i\) coordinates encode some extra finite data of matching horizontal directions when plumbing at such nodes. In that case one should use a suitable power \(t_j^{a_j}\) instead of \(t_j\) in the product \(t_{[i]}\), where the exponents \(a_j\) are determined by the level graph. Since this is only a matter of notation and does not affect the rest argument, we omit the distinction.
where those $c_k^{(i)}$ and $c^{(h,l)}$ are constant coefficients and where those $\rho^{(h,l)}$ are parts of (a path representative of) $\gamma$ that connect level $h$ to level $l$ for $h > l$. In other words, we decompose $\gamma$ into paths that are contained in a single level together with paths that cross two different levels through the corresponding (plumbed) node. Then the period of $(X_t, \omega_t)$ along $\gamma$ is

$$\lambda(t) = \sum_{k,i} c_k^{(i)} t_{[i]}(t) \lambda_k^{(i)}(t) + \sum_{h,l} c^{(h,l)} r^{(h,l)}(t)$$

where each $r^{(h,l)}(t)$ is a function that depends on $t_i(t)$ for $i$ between $h$ and $l$, depends on the choice of the reference points near the node for defining perturbed period coordinates of $\hat{\mathcal{H}}(\kappa)$, and depends on the way of gluing the plumbing fixture in the process of smoothing $(X,\omega)$. Despite these choices, the functions $r^{(h,l)}(t)$ vary continuously and converge to zero as $t \to 0$, which follows from the construction of $\hat{\mathcal{H}}(\kappa)$. In summary, all the quantities involved in the above expression of the period $\lambda(t)$ vary continuously and converge as $t \to 0$. We thus conclude that the same holds for $\lambda(t)$.

The last statement follows from continuity of perturbed period coordinates as local coordinates of $\hat{\mathcal{H}}(\kappa)$.

Now we can finish the proof of Theorem 6.4.

**Proof of Theorem 6.4.** Let $(X, \omega) \in \hat{\mathcal{H}}(\kappa)$ be a preimage of $(X_\infty, \omega_\infty) \in \tilde{\mathcal{H}}(\kappa)$. Let $S_j$ be the simply connected subsets described above for $(X, \omega)$. By their construction, each $S_j$ is simply connected and the universal family over $S_j$ has a model homology group for its fibers, thus satisfying (1) and (2). By Theorem 8.1, the absolute and relative periods extend continuously from each $S_j$ to $(X, \omega)$, hence the same conclusion holds for their images in $\hat{\mathcal{H}}(\kappa)$, since the map $\hat{\mathcal{H}}(\kappa) \to \tilde{\mathcal{H}}(\kappa)$ is continuous (which simply sets absolute and relative periods in lower levels to be zero), thus implying (3). To prove (4), denote by $W$ the total preimage of $(X_\infty, \omega_\infty)$ in $\hat{\mathcal{H}}(\kappa)$. Similarly as we argued in Section 2, $W$ is a compact subset. Hence we can find finitely many $(X_k, \omega_k)$ in $W$ and their neighborhoods $V_k \subset U_k$, with the desired decomposition $V_k = \bigcup S_{kj}$ as in the paragraph before Theorem 8.1, such that the union of these $U_k$ covers a neighborhood of $W$ in $\hat{\mathcal{H}}(\kappa)$. Consequently the union of the images of $U_k$ covers a neighborhood of $(X_\infty, \omega_\infty)$ in $\tilde{\mathcal{H}}(\kappa)$. Note that in the setting of $S_{kj} \subset U_k$, we have avoided the locus parameterizing translation surfaces with cylinders of modulus bigger than $M$, hence (4) also holds. Finally, (5) follows from the fact that
the intersection of $S_j$ with the smooth locus of $\mathcal{M}$ is a relatively compact semianalytic set (see e.g. [BM88, Corollary 2.7, Lemma 3.4]). □

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