

TEICHMÜLLER DYNAMICS IN THE EYES OF AN ALGEBRAIC GEOMETER

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ABSTRACT. This is an introduction to the algebraic aspect of Teichmüller dynamics, with a focus on its interplay with the geometry of moduli spaces of curves as well as recent advances in the field.

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1. INTRODUCTION

An Abelian differential defines a flat structure such that the underlying Riemann surface can be realized as a plane polygon whose edges are pairwise identified via translation. Varying the shape of the polygon by $GL_2^+(\mathbb{R})$ induces an action on the moduli space of Abelian differentials, called Teichmüller dynamics, see Figure 1.

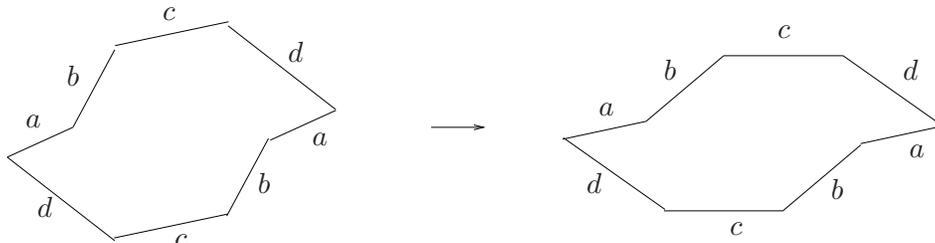


FIGURE 1. $GL_2^+(\mathbb{R})$ -action on a flat surface

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The corresponding $\mathrm{GL}_2^+(\mathbb{R})$ -orbit closures in the moduli space of Abelian differentials are now known as affine invariant submanifolds. A number of questions about surface geometry boil down to understanding the structures of affine invariant submanifolds. From the viewpoint of algebraic geometry affine invariant submanifolds are of an independent interest, which can provide special subvarieties in the moduli space of curves.

We aim to introduce Teichmüller dynamics from the viewpoint of algebraic geometry. In Section 2 we review background material, including translation surfaces, strata of Abelian differentials, the $\mathrm{GL}_2^+(\mathbb{R})$ -action, and affine invariant submanifolds. Section 3 focuses on Teichmüller curves formed by closed $\mathrm{GL}_2^+(\mathbb{R})$ -orbits, where we describe their properties, examples, classification, and invariants. In Section 4 we study general affine invariant submanifolds and survey recent breakthroughs about their structures, classification, and boundary behavior. Finally in Section 5 we discuss similar questions for meromorphic and higher order differentials.

This article is written in an expository style. We will often highlight motivations and minimize technical details. For further reading, we refer to [Z1, Mö5, Wr2, Wr3] for a number of excellent surveys on related topics.

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2. PRELIMINARIES

In this section we introduce basic background material that will be used later.

2.1. Abelian differentials and translation surfaces. A *translation surface* (also called a *flat surface*) is a closed, topological surface X together with a finite set $\Sigma \subset X$ such that:

- There exists an atlas of charts $X \setminus \Sigma \rightarrow \mathbb{C}$, where the transition functions are translation.
- For each $p \in \Sigma$, under the Euclidean metric of \mathbb{C} the total angle at p is $(k+1) \cdot (2\pi)$ for some $k \in \mathbb{Z}^+$.

We say that p is a *saddle point of cone angle* $(k+1) \cdot (2\pi)$. Locally one can glue $2k+2$ half-disks consecutively to form a cone of angle $2\pi \cdot (k+1)$, see Figure 2.

Equivalently, a translation surface is a closed Riemann surface X with an *Abelian differential* ω , not identically zero:

- The set of zeros of ω corresponds to Σ .
- If p is a zero of ω of order k , then the cone angle at p is $(k+1) \cdot (2\pi)$.

For example, take an octagon X with four pairs of parallel edges, see Figure 3. Identifying the edges with the same labels by translation, X becomes a closed surface. All vertices are glued as one point p . By the topological Euler characteristic formula, the genus of X is two. It is moreover a Riemann surface whose complex structure is

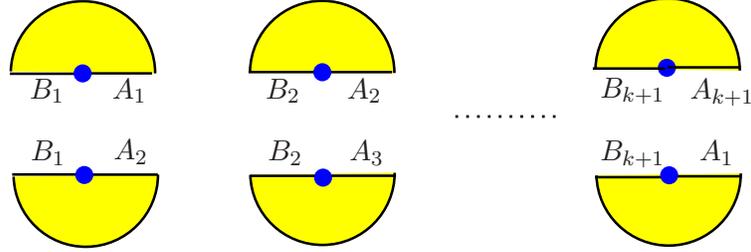


FIGURE 2. A saddle point of cone angle $(k + 1) \cdot (2\pi)$

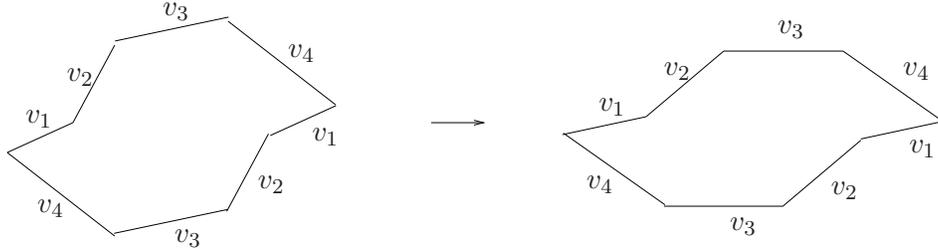


FIGURE 3. An octagon X with four pairs of parallel edges

induced from \mathbb{C} . Away from p it admits an atlas of charts with transition functions given by translation: $z' = z + \text{constant}$, see Figure 4.

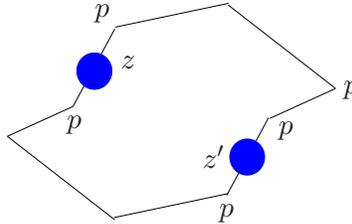


FIGURE 4. Translation structure on $X \setminus p$

The differential $\omega = dz$ is well-defined and nowhere vanishing on $X \setminus p$, which further extends to the entire X . The angle at p is $6\pi = 3 \cdot (2\pi)$, hence ω has a local expression $d(z^3) \sim z^2 dz$ at p . In summary, ω is an Abelian differential with a unique zero of order two on a Riemann surface of genus two.

The above example illustrates the equivalence between translation surfaces and Abelian differentials in general. Given a translation surface, away from its saddle points, differentiating local coordinates provides a globally defined Abelian differential. Conversely, integrating an Abelian differential away from its zeros provides an atlas of charts whose transition functions are translation, because antiderivatives differ by constants. In addition, a saddle point p has cone angle $(k + 1) \cdot (2\pi)$ if and only if $\omega = d(z^{k+1}) \sim z^k dz$ under a local coordinate z at p , namely, if and only if ω has a zero of order k at p .

Below we provide two more examples. Figure 5 represents a nowhere vanishing differential on a torus. Conversely every Abelian differential on a torus give rises to

such a parallelogram presentation. Figure 6 represents an Abelian differential with two simple zeros on a Riemann surface of genus two.

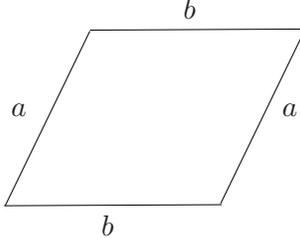


FIGURE 5. A flat torus

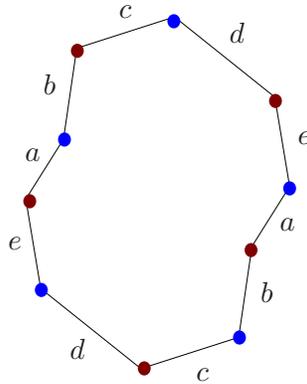


FIGURE 6. A flat surface with two simple zeros

Note that Abelian differentials are sections of the *canonical line bundle*, hence the study of translation surfaces is naturally connected to algebraic geometry.

2.2. Strata of Abelian differentials. We identify Riemann surfaces with smooth complex algebraic curves. Let \mathcal{M}_g be the *moduli space of genus g curves*. Let \mathcal{H} be the *Hodge bundle* over \mathcal{M}_g whose fibers parameterize Abelian differentials on a fixed genus g curve.

Let $\mu = (m_1, \dots, m_n)$ be a tuple of positive integers such that $\sum_{i=1}^n m_i = 2g - 2$. We say that μ is a *partition of $2g - 2$* .¹ Define a subset $\mathcal{H}(\mu)$ of \mathcal{H} that parameterizes pairs (X, ω) , where X is a Riemann surface of genus g and ω is an Abelian differential on X such that the zero divisor of ω is of type μ :

$$(\omega)_0 = m_1 p_1 + \dots + m_n p_n.$$

We say that $\mathcal{H}(\mu)$ is the *stratum of Abelian differentials of type μ* . Equivalently, $\mathcal{H}(\mu)$ parameterizes translation surfaces with n saddle points, each having cone angle $(m_i + 1) \cdot (2\pi)$. The union of $\mathcal{H}(\mu)$ over all partitions of $2g - 2$ is the Hodge bundle \mathcal{H} (with the zero section removed).

Take a basis $\gamma_1, \dots, \gamma_{2g+n-1}$ of the relative homology $H_1(X, p_1, \dots, p_n; \mathbb{Z})$. Integrating ω over each γ_i provides a local coordinate system for $\mathcal{H}(\mu)$, called the *period*

¹In Section 5 we will consider meromorphic differentials, where the entire m_i are allowed to be negative.

coordinates. For instance, the complex vectors v_1, v_2, v_3 , and v_4 in Figure 3 above are periods of a translation surface in $\mathcal{H}(2)$. Under the polygon presentation, locally deforming the periods preserves the number of saddle points, their cone angles, and the way of edge identification. Consequently $\mathcal{H}(\mu)$ is a $(2g + n - 1)$ -dimensional manifold.²

For special partitions μ , $\mathcal{H}(\mu)$ can be *disconnected*. Kontsevich-Zorich ([KonZor1]) classified connected components of $\mathcal{H}(\mu)$ for all μ , where extra components arise due to hyperelliptic and spin structures. If a translation surface (X, ω) satisfies that X is hyperelliptic, $(\omega)_0 = (2g-2)z$ or $(\omega)_0 = (g-1)(z_1+z_2)$, where z is a Weierstrass point of X in the former, or z_1 and z_2 are hyperelliptic conjugate in the latter, we say that (X, ω) is contained in the *hyperelliptic connected component* of the corresponding stratum. For other translation surfaces (X, ω) , if $(\omega)_0 = 2k_1z_1 + \cdots + 2k_nz_n$, then the line bundle $\mathcal{O}_X(\sum_{i=1}^n k_iz_i)$ is a square root of the canonical line bundle, namely, it is a *theta characteristic*. Define its *parity* by $h^0(X, \sum_{i=1}^n k_iz_i) \pmod{2}$, which is deformation invariant ([At, Mum, J]). A theta characteristic with its parity is called a *spin structure*. In general, $\mathcal{H}(\mu)$ can have *up to three* connected components, distinguished by these hyperelliptic and spin structures.

2.3. $\mathrm{GL}_2^+(\mathbb{R})$ -action and affine invariant submanifolds. Given $(X, \omega) \in \mathcal{H}$ and $A \in \mathrm{GL}_2^+(\mathbb{R})$, varying the polygon presentation of (X, ω) by A induces a $\mathrm{GL}_2^+(\mathbb{R})$ -action on \mathcal{H} , which is called *Teichmüller dynamics*. For example, the $\mathrm{GL}_2^+(\mathbb{R})$ -orbit of a flat torus consists of all parallelogram presentations, see Figure 7.

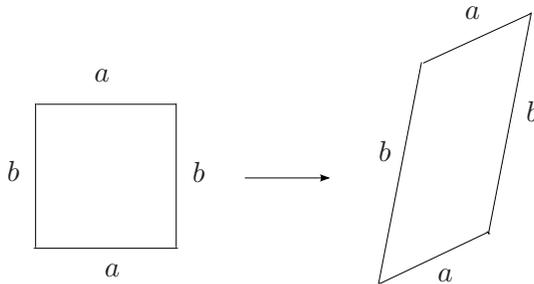


FIGURE 7. $\mathrm{GL}_2^+(\mathbb{R})$ -action on a flat torus

The number and cone angles of saddle points are preserved under this action, hence the $\mathrm{GL}_2^+(\mathbb{R})$ -action descends to each stratum $\mathcal{H}(\mu)$. Equip $\mathcal{H}(\mu)$ with the standard topology using its period coordinates. For almost all $(X, \omega) \in \mathcal{H}(\mu)$, Masur and Veech ([Ma, V1]) showed that its $\mathrm{GL}_2^+(\mathbb{R})$ -orbit is *equidistributed* in $\mathcal{H}(\mu)$, hence the orbit closure is the whole stratum (or a connected component if the stratum is disconnected). For special (X, ω) , however, its $\mathrm{GL}_2^+(\mathbb{R})$ -orbit closure can be a *proper* subset of $\mathcal{H}(\mu)$. Classifying $\mathrm{GL}_2^+(\mathbb{R})$ -orbit closures in $\mathcal{H}(\mu)$ is a central question in Teichmüller dynamics.

The recent breakthrough of Eskin-Mirzakhani-Mohammadi ([EMi, EMM]) showed that any $\mathrm{GL}_2^+(\mathbb{R})$ -orbit closure is an *affine invariant submanifold* in $\mathcal{H}(\mu)$, that is, it is a subspace of $\mathcal{H}(\mu)$ locally defined by real linear homogeneous equations of period

²More precisely it is an orbifold, because special translation surfaces can have extra automorphisms.

coordinates.³ Filip ([Fi1]) further showed that all affine invariant submanifolds are algebraic varieties defined over $\overline{\mathbb{Q}}$. In particular, it means that affine invariant submanifolds can be defined and characterized purely in terms of algebraic conditions on the Jacobian. We will elaborate on these results in Section 4.

2.4. Veech group. Let $(X, \omega) \in \mathcal{H}(\mu)$ be a translation surface. Suppose a matrix $A \in \mathrm{SL}_2(\mathbb{R})$ acts on (X, ω) . If the resulting translation surface $A \cdot (X, \omega)$ is isomorphic to (X, ω) , that is, if the polygon presentation of $A \cdot (X, \omega)$ can be cut into pieces and reassembled via translation to represent (X, ω) , we say that A is a *stabilizer* of (X, ω) . The subgroup of all stabilizers of (X, ω) in $\mathrm{SL}_2(\mathbb{R})$ is called the *Veech group*, and denoted by $\mathrm{SL}(X, \omega)$. For example, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is in the Veech group of the square torus, see Figure 8.

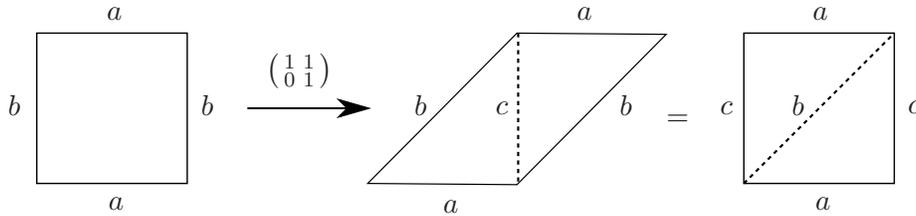


FIGURE 8. An element of the Veech group

A line segment under the flat metric that connects two zeros of ω (not necessarily distinct) is called a *saddle connection*. Since the set of saddle connections of (X, ω) is preserved by a stabilizer, it follows that $\mathrm{SL}(X, \omega)$ is *discrete*. Without loss of generality, suppose (X, ω) has a horizontal saddle connection. Let $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ act on (X, ω) . As $t \rightarrow \infty$, the horizontal saddle connection becomes arbitrarily long, hence the $\mathrm{SL}_2(\mathbb{R})$ -orbit of (X, ω) is *unbounded* in $\mathcal{H}(\mu)$. Consequently $\mathrm{SL}(X, \omega)$ is *not cocompact* in $\mathrm{SL}_2(\mathbb{R})$.

Adding the traces of all $A \in \mathrm{SL}(X, \omega)$ to \mathbb{Q} , we obtain a field extension of \mathbb{Q} , called the *trace field* of (X, ω) . The degree of the trace field over \mathbb{Q} is bounded by the genus of X (see [Mö5, Proposition 2.5]).

3. TEICHMÜLLER CURVES

The Hodge bundle \mathcal{H} maps to the moduli space \mathcal{M}_g of smooth genus g curves by forgetting the differentials: $(X, \omega) \mapsto X$. Note that the subgroup $\mathrm{SO}(2)$ acting on an Abelian differential amounts to rotating the corresponding flat surface, hence it does not change the underlying complex structure. Similarly scaling the size of a flat surface preserves the underlying complex structure. It follows that the projection of a $\mathrm{GL}_2^+(\mathbb{R})$ -orbit to \mathcal{M}_g factors through the upper half plane \mathbb{H} , and the induced map $\mathbb{H} \rightarrow \mathcal{M}_g$ (or simply its image) is called a *Teichmüller disk*. On rare occasions a Teichmüller disk forms an algebraic curve in \mathcal{M}_g . In that case we call it a *Teichmüller curve*.

³Here the term “affine” is different from what it usually means in algebraic geometry. It refers to the linear structure on \mathbb{C}^n . Moreover, the closure of an orbit is taken under the standard topology in the Hodge bundle over the interior of the moduli space parameterizing smooth curves. Finally, affine invariant submanifolds can have self crossings.

3.1. Properties of Teichmüller curves. Teichmüller curves are dimensionally minimal affine invariant submanifolds, which possess a number of fascinating properties. To name a few, a Teichmüller curve is a local *isometry* from a curve to \mathcal{M}_g under the Kobayashi/Teichmüller metric ([SW, V3]). The union of all Teichmüller curves is *dense* in moduli spaces ([EO1, Ch2]). McMullen ([Mc7]) proved that Teichmüller curves are *rigid*, hence they are defined over number fields.⁴ Conversely, Ellenberg-McReynolds ([EMc]) showed that every curve over a number field is birational to a Teichmüller curve over \mathbb{C} . If (X, ω) generates an algebraically primitive Teichmüller curve (see Section 3.3), Möller ([Mö2]) showed that the difference of any two zeros of ω is a torsion in the Jacobian of X . Möller ([Mö1]) also analyzed the variation of Hodge structures associated to a Teichmüller curve and deduced that it parameterizes curves whose Jacobians have *real multiplication*. These torsion and real multiplication conditions can in fact be used to define (algebraically primitive) Teichmüller curves in purely algebro-geometric terms (see e.g., [Wr2, Proposition 2.18]). Teichmüller curves are never complete in \mathcal{M}_g , but the closure of a Teichmüller curve only intersects certain boundary divisor of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ (see Section 3.4).⁵

3.2. Square-tiled surfaces. We first show that Teichmüller curves exist. One type of Teichmüller curves arises from certain *branched covering construction*. Let $\mu = (m_1, \dots, m_n)$ be a partition of $2g - 2$. Consider a branched cover $\pi : X \rightarrow E$ such that

- $\deg \pi = d$,
- $g(X) = g$,
- E is the square torus,
- π has a *unique* branch point $q \in E$,
- π has n ramification points p_1, \dots, p_n over q , each with ramification order m_i .

Let $\omega = \pi^*(dz)$, where z is the standard coordinate on E . Then by the Riemann-Hurwitz formula

$$(\omega)_0 = m_1 p_1 + \dots + m_n p_n,$$

hence $(X, \omega) \in \mathcal{H}(\mu)$. Such (X, ω) are called *square-tiled surfaces* (or *origami*). For example, Figure 9 exhibits a degree 5 and genus 2 branched cover of E with a unique ramification point of order 2. The resulting (X, ω) belongs to the stratum $\mathcal{H}(2)$.

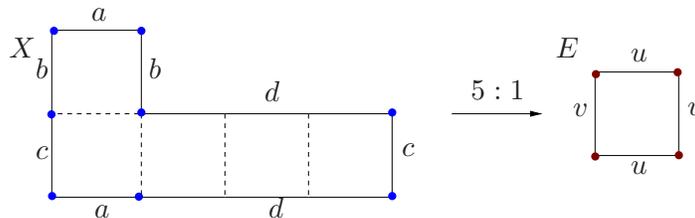


FIGURE 9. A square-tiled surface in $\mathcal{H}(2)$

⁴Here the rigidity means as a map it does not deform.

⁵Here we restrict to Teichmüller curves generated by Abelian differentials. Special quadratic differentials can also generate Teichmüller curves, see Section 5.1, but they may intersect other boundary divisors of $\overline{\mathcal{M}}_g$.

The $\mathrm{GL}_2^+(\mathbb{R})$ -action on a square-tiled surface amounts to varying the shape of the square, which is exchangeable with varying the shape of the flat torus first, see Figure 10 for an example. Therefore, the Teichmüller disk generated by a square-tiled surface corresponds to the one-dimensional *Hurwitz space* parameterizing degree d , genus g connected covers of all elliptic curves with a unique branch point of ramification type μ . Since Hurwitz spaces are algebraic varieties, it follows that the $\mathrm{GL}_2^+(\mathbb{R})$ -orbit of a square-tiled surface gives rise to a Teichmüller curve.⁶ Since d can be arbitrarily large, this way we indeed obtain *infinitely many* Teichmüller curves in $\mathcal{H}(\mu)$.

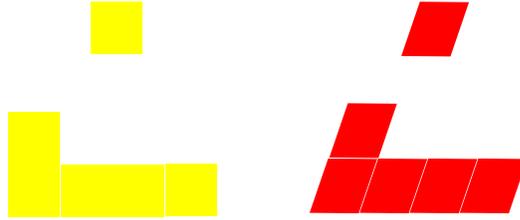


FIGURE 10. $\mathrm{GL}_2^+(\mathbb{R})$ -action on a square-tiled surface

Square-tiled surfaces correspond to *lattice points* under period coordinates of a stratum ([EO1, Lemma 3.1]), see Figure 11. If $\pi : X \rightarrow E$ is a square-tiled surface of type $\mu = (m_1, \dots, m_n)$, for any $\gamma \in H_1(X, p_1, \dots, p_n; \mathbb{Z})$, $\pi(\gamma)$ represents a closed loop in E , because p_1, \dots, p_n all map to the unique branch point. Therefore,

$$\int_{\gamma} \pi^*(dz) = \int_{\pi_*\gamma} dz \in \mathbb{Z} \oplus \mathbb{Z}[i].$$

Conversely if all relative periods of (X, ω) are lattice points, the map $\pi : X \rightarrow \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}[i]$ induced by

$$x \mapsto \int_b^x \omega$$

is well-defined, where b is a fixed base point, hence realizing (X, ω) as a square-tiled surface. This explains density of the union of Teichmüller curves in moduli spaces. It also provides an approach for analyzing *volume growths* of the strata of Abelian differentials by counting the number of such square-tiled surfaces ([EO1]).

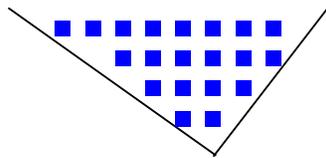


FIGURE 11. Square-tiled surfaces as lattice points in a stratum

Gutkin-Judge ([GJ]) showed that the Veech group of a translation surface is *commensurable with* $\mathrm{SL}(2, \mathbb{Z})$ if and only if the surface is tiled by parallelograms (see also

⁶When d , g , and μ are fixed, the Hurwitz space can still be disconnected. In that case each of its connected components gives a Teichmüller curve.

[Mc1, Theorem 9.8] for a short proof). In this sense Teichmüller curves generated by square-tiled surfaces are called *arithmetic* Teichmüller curves. However, there exist Teichmüller curves of other type that are *not* generated by square-tiled surfaces. In the next section we will survey known results on the classification of Teichmüller curves.

3.3. Classification of Teichmüller curves. By definition, the moduli space $\mathcal{M}_{1,1}$ of elliptic curves is a very first example of Teichmüller curves. Arithmetic Teichmüller curves generated by square-tiled surfaces are coverings of $\mathcal{M}_{1,1}$. On the contrary, a Teichmüller curve is called (geometrically) *primitive* if it does not arise from a curve in a moduli space of lower genus via such a covering construction.

It is a challenging task to find examples of primitive Teichmüller curves. In genus two, Calta and McMullen ([Ca, Mc2]) independently classified primitive Teichmüller curves in $\mathcal{H}(2)$. They found infinitely many primitive Teichmüller curves whose constructions have several incarnations. In terms of flat geometry, a translation surface (X, ω) generating a primitive Teichmüller curve in $\mathcal{H}(2)$ possesses an L -shaped polygon presentation whose edges satisfy relations in a real quadratic field. In terms of algebraic geometry, the Jacobian of X admits real multiplication, and the corresponding Teichmüller curve lies on a Hilbert modular surface. However for the other stratum $\mathcal{H}(1, 1)$ in genus two, McMullen ([Mc4]) proved that it contains a unique primitive Teichmüller curve generated by the regular decagon with parallel edges identified. McMullen ([Mc5]) further generalized the method of real multiplication by using Prym varieties and discovered infinitely many primitive Teichmüller curves in genus three and four. Using quotients of abelian covers of \mathbb{P}^1 by a finite group, in each genus Bouw-Möller ([BoM]) constructed (finitely many) primitive Teichmüller curves, generalizing earlier constructions of Veech and Ward ([V2, Wa]).

Besides the above results, to date only two sporadic examples of primitive Teichmüller curves are known (see e.g., [Wr1, p. 782] for an elaboration). It is natural to ask if there are only finitely many primitive Teichmüller curves in a given stratum $\mathcal{H}(\mu)$. There is a stronger notion of primitivity called *algebraically primitive*, if the trace field of a translation surface X in the Teichmüller curve has degree equal to the genus of X (see Section 2.4). For example, the trace field of a square-tiled surface is \mathbb{Q} , hence it is far from being algebraically primitive. Indeed, algebraically primitive Teichmüller curves are geometrically primitive, but the converse is not always true (see [Mö5, Section 5.1]).

Finiteness results of algebraically primitive Teichmüller curves have been established in various cases. Möller ([Mö3]) proved that the hyperelliptic component of $\mathcal{H}(g - 1, g - 1)$ contains finitely many algebraically primitive Teichmüller curves. The strategy is to track the degeneration of flat surfaces along an algebraically primitive Teichmüller curve in the horizontal and vertical directions and use it to bound the torsion order of the difference of the two zeros. By studying the boundary of the locus of curves with real multiplication, Bainbridge-Möller ([BaM]) showed finiteness of algebraically primitive Teichmüller curves in $\mathcal{H}(3, 1)$. Bainbridge-Habegger-Möller ([BHM]) further established finiteness of algebraically primitive Teichmüller curves for all strata in genus three by a mix of techniques, including the Harder-Narasimhan filtration of the Hodge bundle over Teichmüller curves and height bounds for the boundary points of Teichmüller

curves. Matheus-Wright ([MW]) proved finiteness of algebraically primitive Teichmüller curves in the minimal stratum $\mathcal{H}(2g - 2)$ for each prime genus $g \geq 3$. Their approach is to study orthogonality of Hodge-Teichmüller (real) planes in the Hodge bundle that respect the Hodge decomposition along a Teichmüller curve. Matheus-Nguyen-Wright ([NW]) further showed that there are at most finitely many non-arithmetic Teichmüller curves in the hyperelliptic component of $\mathcal{H}(4)$.

One can also study Teichmüller curves contained in an affine invariant submanifold of a stratum. Lanneau-Nguyen ([LN]) showed that there are at most finitely many closed $\mathrm{GL}_2^+(\mathbb{R})$ -orbits (including primitive Teichmüller curves) in certain Prym loci in genus three. Lanneau-Nguyen-Wright ([LNW]) further proved finiteness of closed $\mathrm{GL}_2^+(\mathbb{R})$ -orbits in each non-arithmetic rank 1 affine invariant submanifold (see Section 4.2). Apisa ([Ap]) showed finiteness of algebraically primitive Teichmüller curves in the hyperelliptic components of each minimal stratum in $g > 2$, as a byproduct of his classification of affine invariant submanifolds in the hyperelliptic components (see Section 4.2).

Although a complete classification of Teichmüller curves is still missing, the seminal work of Eskin-Mirzakhani-Mohammadi ([EMi, EMM]) on the structure of $\mathrm{GL}_2^+(\mathbb{R})$ -orbit closures provides us a powerful tool. Indeed some of the above results are built on their work. We hope the classification problem of Teichmüller curves (and in general affine invariant submanifolds) can be resolved in the next few decades.

3.4. Slope, Siegel-Veech constant, and Lyapunov exponent. In this section we discuss several invariants of Teichmüller curves and describe a relation between them.

First, the behavior of geodesics on translation surfaces is related to *billiards in polygons*. One can study various counting problems from this viewpoint. Recall that a saddle connection is a line segment connecting two saddle points. Consider counting saddle connections with bounded lengths. For instance, consider the standard torus formed by identifying parallel edges of the unit square and marked at the origin. The number of (non-primitive) saddle connections of length $< L$ (counting with direction) equals the number of lattice points in the disk of radius L , see Figure 12. It has

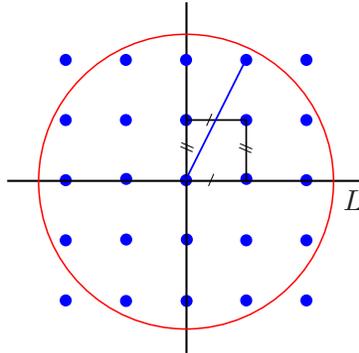


FIGURE 12. Saddle connections on the standard torus

asymptotically *quadratic growth* $\sim \pi L^2$. The leading term π is an example of so-called *Siegel-Veech constant*. In general, Siegel-Veech constants relate quadratic growth rates

of finite-length trajectories on a translation surface to the volume of the corresponding $\mathrm{SL}_2(\mathbb{R})$ -orbit. In what follows we will concentrate on one type of Siegel-Veech constants, called the *area* Siegel-Veech constant, which has a connection to dynamics as well as intersection theory on moduli space.

A real geodesic on a translation surface (X, ω) is called *regular*, if it does not pass through any saddle point, namely, it does not contain any zero of ω . Vary a closed regular geodesic in parallel until it hits a saddle point on both ends. The union of those geodesics fill a *cylinder* cyl , whose boundary circles contain saddle points. For example, the square-tiled surface in Figure 13 has two cylinders in the horizontal direction. Let

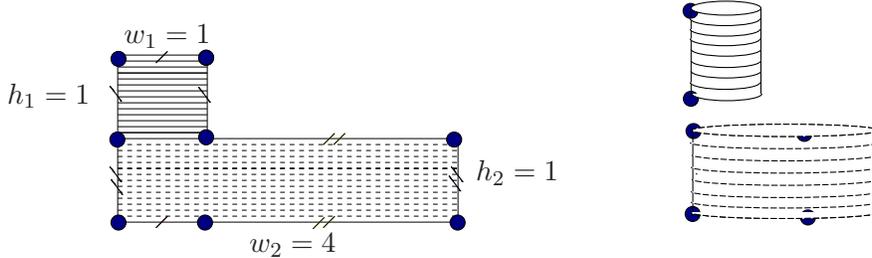


FIGURE 13. Two horizontal cylinders on a square-tiled surface

w and h be the *width* and *height* of a cylinder cyl , respectively. The *area* of the cylinder is $\mathrm{Area}(\mathrm{cyl}) = w \cdot h$.

For $L > 0$, define

$$N(X, L) = \frac{1}{\mathrm{Area}(X)} \cdot \sum_{\substack{\mathrm{cyl} \subset X \\ w(\mathrm{cyl}) < L}} \mathrm{Area}(\mathrm{cyl}).$$

Veech and Eskin-Masur ([V4, EMa]) showed that for any $\mathrm{GL}_2^+(\mathbb{R})$ -orbit closure \mathcal{N} , there exists a constant c such that

$$\frac{\pi}{3} \cdot \lim_{L \rightarrow \infty} \frac{N(X, L)}{L^2} = c$$

for *generic* $(X, \omega) \in \mathcal{N}$. The constant c is called the *area Siegel-Veech constant* of \mathcal{N} .⁷ Its value depends on \mathcal{N} .

If \mathcal{N} is the orbit of a square-tiled surface, namely, if its projection to \mathcal{M}_g is an arithmetic Teichmüller curve, there is a *combinatorial* way to calculate its area Siegel-Veech constant. Let N be the number of square-tiled surfaces in \mathcal{N} . In other words, N is the *Hurwitz number* that counts branched covers of a fixed torus with a unique branch point and prescribed ramification type. Take all square-tiled surfaces in \mathcal{N} . For each one, consider all of its horizontal cylinders. Sum up h/w (the *modulus* of a cylinder) over all of them. Denote the total sum by M . For example, the square-tiled surface in Figure 13 above contributes $\frac{h_1}{w_1} + \frac{h_2}{w_2} = \frac{1}{1} + \frac{1}{4}$ to the sum M .

⁷Here our definition of the Siegel-Veech constant differs from its usual definition by a scalar multiple of $\pi^2/3$. We choose to normalize it this way for the convenience of describing its relation with slope and Lyapunov exponent.

For an arithmetic Teichmüller curve generated by a square-tiled surface,

$$c = \frac{M}{N},$$

see [EKZ1, Appendix B]. The idea behind the formula is that c measures the average number of cylinders weighted by their moduli h/w in \mathcal{N} , where

$$\frac{h}{w} = \frac{hw}{w^2} = \frac{\text{Area}(\text{cyl})}{w^2},$$

hence it is related to the quadratic growth rate of $N(X, L)$ in this case.

For low degree d and low genus g , using monodromy of branched covers one can calculate N and M explicitly. Nevertheless, the enumeration of N as d and g increase is a highly non-trivial problem in symmetric group representations. Eskin-Okounkov ([EO1]) analyzed the asymptotic behavior of N and calculated the *volume* growth of strata of Abelian differentials. The enumeration of M is more complicated for large d and g . Joint with Möller and Zagier ([CMZ]) we are able to understand the asymptotic growth of M and hence c for arithmetic Teichmüller curves by using techniques of shifted symmetric functions and quasimodular forms.

Next we define an important index associated to a one-dimensional family of stable curves. The *boundary* Δ of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_g$ parameterizes *stable nodal curves*, where $\Delta = \bigcup_{i=0}^{[g/2]} \Delta_i$ is a union of irreducible *boundary divisors* Δ_i . General points of Δ_i parameterize nodal curves of a given topological type, and they can further degenerate, see Figure 14.

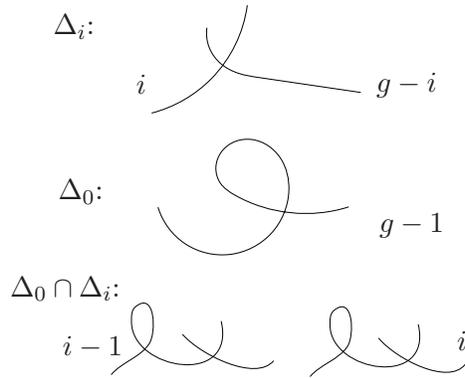


FIGURE 14. Curves in the boundary of $\overline{\mathcal{M}}_g$

The Hodge bundle \mathcal{H} over \mathcal{M}_g extends to a rank g bundle $\overline{\mathcal{H}}$ over $\overline{\mathcal{M}}_g$, whose fiber over a stable curve X parameterizes sections of the dualizing line bundle K_X . Geometrically speaking, the fiber of $\overline{\mathcal{H}}$ over X can be identified with the space of *stable differentials* that have at worst simple pole at each node of X with opposite residues on the two branches of a node ([HM, Chapter 3.A]). Denote by λ the first Chern class of $\overline{\mathcal{H}}$ over $\overline{\mathcal{M}}_g$. Given a one-dimensional family B of stable genus g curves, define its

slope by

$$s = \frac{\deg \Delta|_B}{\deg \lambda|_B}.$$

Morally speaking, $\deg \Delta|_B$ counts the number of nodes (with multiplicity) and $\deg \lambda|_B$ measures the variation of complex structures, see Figure 15.

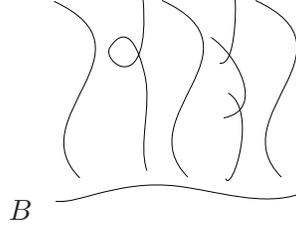


FIGURE 15. A one-dimensional family of stable curves

Let $\mu = (m_1, \dots, m_n)$ be a partition of $2g - 2$. Define

$$\kappa_\mu = \frac{1}{12} \sum_{i=1}^n \frac{m_i(m_i + 2)}{m_i + 1},$$

which depends on μ only. For a Teichmüller curve in $\mathcal{H}(\mu)$, despite that its area Siegel-Veech constant and slope are defined in different contexts, they determine each other ([Ch1, Ch2, CM1]):

$$s = \frac{12c}{c + \kappa_\mu}.$$

The upshot to prove the formula consists of the following. First, the constant κ_μ corresponds to the Miller-Morita-Mumford κ -class. Next, a cylinder with modulus h/w contributes h/w to the intersection of the Teichmüller curve with the boundary Δ , see Figure 16. Moreover, the numerical relation $12\lambda \equiv \kappa + \Delta$ holds on $\overline{\mathcal{M}}_g$.

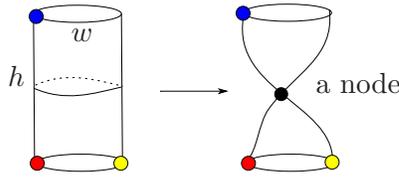


FIGURE 16. Shrinking the core curve of a cylinder

Finally we introduce an important dynamical invariant for an affine invariant submanifold. The diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ defines the *Teichmüller geodesic flow* on the Hodge bundle \mathcal{H} over an affine invariant submanifold \mathcal{N} . Since \mathcal{H} is of rank g , there are g nonnegative *Lyapunov exponents* $\lambda_1 \geq \dots \geq \lambda_g \geq 0$ as logarithms of mean eigenvalues of monodromy of \mathcal{H} along the flow on \mathcal{N} . Morally speaking, Lyapunov exponents measure the separation rates of infinitesimally closed trajectories (see [Z1] for more details). Denote the *sum* of the Lyapunov exponents by

$$L = \lambda_1 + \dots + \lambda_g.$$

In a seminal work, Eskin-Kontsevich-Zorich ([EKZ1]) proved that the sum of Lyapunov exponents and the area Siegel-Veech constant determine each other for *any* affine invariant submanifold:

$$L = c + \kappa_\mu.$$

As a corollary, we thus obtain that for a Teichmüller curve, any one of the three numbers s , c , and L determine the other two.

As an application, one can deduce a non-varying phenomenon of Teichmüller curves in low genus. By computer experiments, Kontsevich and Zorich observed that for many strata $\mathcal{H}(\mu)$ in low genus, *all* Teichmüller curves in the same stratum (component) have *non-varying* sums of Lyapunov exponents. They came up with a conjectural list of such strata:

- $g = 2$: $\mathcal{H}(1, 1)$, $\mathcal{H}(2)$.
- $g = 3$: $\mathcal{H}(4)$, $\mathcal{H}(3, 1)$, $\mathcal{H}(2, 2)$, $\mathcal{H}(2, 1, 1)$.
- $g = 4$: $\mathcal{H}(6)$, $\mathcal{H}(5, 1)$, $\mathcal{H}(4, 2)$, $\mathcal{H}(3, 3)$, $\mathcal{H}(3, 2, 1)$, $\mathcal{H}^{\text{odd}}(2, 2, 2)$.
- $g = 5$: $\mathcal{H}(8)$, $\mathcal{H}^{\text{odd}}(6, 2)$, $\mathcal{H}(5, 3)$, $\mathcal{H}^{\text{hyp}}(4, 4)$.

This phenomenon disappears when $g \geq 6$ except for the *hyperelliptic* strata.

Joint with Möller we proved Kontsevich-Zorich's conjecture ([CM1]). Let us use $\mathcal{H}(3, 1)$ as an example to explain our method. For $(X, \omega) \in \mathcal{H}(3, 1)$, it is easy to observe that X is *non-hyperelliptic*. The same holds for any nodal curve X contained in the *boundary* of a Teichmüller curve \mathcal{T} in $\mathcal{H}(3, 1)$. Let Hyp be the closure of the locus of hyperelliptic curves in $\overline{\mathcal{M}}_3$, which is a divisor with class

$$\text{Hyp} \equiv 9\lambda - \Delta_0 - 3\Delta_1,$$

see [HM, Chapter 3.H]. Since \mathcal{T} is disjoint with Hyp, we conclude that $\mathcal{T} \cdot \text{Hyp} = 0$. Moreover, $\mathcal{T} \cdot \Delta_i = 0$ for $i > 0$, because shrinking the core curve of a cylinder yields a node of type Δ_0 only, see Figure 16 above. By $\mathcal{T} \cdot (9\lambda - \Delta) = 0$, we see that the slope

$$s = \frac{\deg \Delta|_{\mathcal{T}}}{\deg \lambda|_{\mathcal{T}}} = 9.$$

The above calculation is independent of the choice of Teichmüller curves in $\mathcal{H}(3, 1)$. In summary, all Teichmüller curves in $\mathcal{H}(3, 1)$ have slope 9. By the slope, Siegel-Veech, and Lyapunov exponent formula, sums of Lyapunov exponents are the same for all Teichmüller curves in $\mathcal{H}(3, 1)$.

In general, the strategy is to find a geometrically meaningful *divisor* D in $\overline{\mathcal{M}}_g$ (or in $\overline{\mathcal{M}}_{g,n}$ by marking the zeros of ω) such that D is *disjoint* with all Teichmüller curves in $\mathcal{H}(\mu)$. The divisor class of D determines the invariants of those Teichmüller curves.

We remark that Yu-Zuo ([YZ]) gave another novel proof of Kontsevich-Zorich conjecture using the *Harder-Narasimhan filtration* of the Hodge bundle over Teichmüller curves. Their idea is the following. Suppose $f : \mathcal{X} \rightarrow \mathcal{T}$ is the universal curve over a Teichmüller curve. Let S_1, \dots, S_n be the disjoint sections of zeros of $(X, \omega) \in \mathcal{T}$. For a tuple of integers a_1, \dots, a_n , if $h^0(X, \sum_{i=1}^n a_i z_i) = k$ for all X , then the direct image sheaf $f_* \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^n a_i S_i)$ is a vector bundle of rank k on \mathcal{T} .

Use $\mathcal{H}(3, 1)$ as an example again. Since $3z_1 + z_2 \sim K_X$, one checks that $h^0(X, z_1 + z_2) = 1$ and $h^0(X, 2z_1 + z_2) = 2$ for all $X \in \mathcal{T}$ (including the boundary points). Hence

we obtain a filtration

$$f_*\mathcal{O}_{\mathcal{X}}(S_1 + S_2) \subset f_*\mathcal{O}_{\mathcal{X}}(2S_1 + S_2) \subset f_*\mathcal{O}_{\mathcal{X}}(3S_1 + S_2),$$

where the last term can be identified with the Hodge bundle twisted by the generating differential of \mathcal{T} . Then the sum L of Lyapunov exponents of \mathcal{T} follows from Chern class calculations of the subbundles in the filtration.

We also remark that the slopes of Teichmüller curves can provide information to understand the *cone of effective divisors* of $\overline{\mathcal{M}}_g$. The idea is that any effective divisor on $\overline{\mathcal{M}}_g$ cannot contain all arithmetic Teichmüller curves, because their union is *dense* in $\overline{\mathcal{M}}_g$. Hence the limit of slopes of arithmetic Teichmüller curves as the degree of the coverings approach infinity bounds the numerical class of the divisor. We refer to the survey ([CFM]) for more details on the effective cone of moduli spaces.

3.5. Miscellaneous. In this section we collect various results with a flavor in algebraic geometry that are related to the preceding discussions.

Consider the set of all arithmetic Teichmüller curves generated by square-tiled surfaces with a given number of squares and fixed ramification type. McMullen ([Mc3]) showed that in $\mathcal{H}(2)$ this set consists of either a single arithmetic Teichmüller curve, or two distinct arithmetic Teichmüller curves. The special case when the number of squares is prime was previously established by Hubert-Lelièvre ([HL]). For general strata, this is a wide open question. From the viewpoint of algebraic geometry, it amounts to classifying connected components of the Hurwitz space of torus coverings with a unique branch point and prescribed ramification type. Bainbridge ([Ba]) calculated Euler characteristics of Teichmüller curves in $\mathcal{H}(2)$. The idea is to calculate the fundamental classes of these Teichmüller curves in certain compactifications of Hilbert modular surfaces. Mukamel ([Muk]) further determined the number and type of orbifold points of these Teichmüller curves. Kumar-Mukamel ([KM]) described algebraic models of Teichmüller curves in genus two. Weiss ([We]) calculated the volume of certain twisted Teichmüller curves on Hilbert modular surfaces and partially classified their connected components.

Siegel-Veech constants and Lyapunov exponents can be defined similarly for any affine invariant submanifold, say the strata themselves. Eskin-Masur-Zorich ([EMZ]) analyzed the principal boundary of the moduli space of Abelian differentials and gave a recursive method to compute Siegel-Veech constants of the strata for any Siegel-Veech configuration. For the cylinder configuration, the limit of area Siegel-Veech constants of arithmetic Teichmüller curves in a stratum as the degree of the coverings approach infinity equals the area Siegel-Veech constant of the stratum ([Ch2, Appendix A]).

Previously we discussed the sum of Lyapunov exponents. For arithmetic Teichmüller curves generated by cyclic covers of \mathbb{P}^1 branched at four points, Eskin-Kontsevich-Zorich ([EKZ2]) calculated all individual Lyapunov exponents. Yu ([Y]) conjectured that the polygon of Lyapunov spectrum bounds the Harder-Narasimhan polygon on Teichmüller curves, and a proof of this conjecture is recently announced by Eskin-Kontsevich-Möller-Zorich. The fundamental work of Forni ([Fo1]) showed that for a stratum of Abelian differentials no Lyapunov exponent vanishes. The fundamental work of Avila and Viana ([AV]) further showed that for a stratum the Lyapunov spectrum

is simple, that is, the strict inequality $\lambda_i > \lambda_{i+1}$ holds for all i . For arithmetic Teichmüller curves generated by square-tiled surfaces, Matheus-Möller-Yoccoz ([MMY]) gave a Galois-theoretical criterion for the simplicity of their Lyapunov spectra.

The Lyapunov spectrum can degenerate for a special affine invariant submanifold. If $\lambda_2 = \dots = \lambda_g = 0$, we say that the Lyapunov spectrum is completely degenerate. Forni and Forni-Matheus-Zorich ([Fo2, FMZ]) found examples of Teichmüller curves with completely degenerate Lyapunov spectrum in genus three and four, respectively. By studying Teichmüller curves that are also Shimura curves, Möller ([Mö4]) showed that those examples are the only Teichmüller curves with completely degenerate Lyapunov spectrum with possible exceptions in genus five. Aulicino ([Au]) showed that an affine invariant submanifold with completely degenerate Lyapunov spectrum can only be an arithmetic Teichmüller curve in genus at most five, and he further established finiteness of Teichmüller curves with completely degenerate Lyapunov spectrum. Filip ([Fi2]) described all situations when an affine invariant submanifold can possess a zero Lyapunov exponent by analyzing monodromy of the corresponding Kontsevich-Zorich cocycle. As a higher dimensional analogue of families of curves, Filip ([Fi3]) considered families of K3 surfaces whose second cohomology groups form a local system, and showed that their top Lyapunov exponents are always rational.

To conclude this section we mention two problems that have broad connections to other fields. Kontsevich-Zorich ([KonZor2]) conjectured that every connected component of the strata is $K(\pi, 1)$, that is, it has a contractible universal cover and its fundamental group is commensurable with certain mapping class group. For all strata in genus three except $\mathcal{H}(1^4)$, Looijenga-Mondello ([LM]) determined their (orbifold) fundamental groups by analyzing geometry of canonical curves. Kontsevich-Soibelman ([KS]) speculated that moduli spaces of differentials can be identified with moduli spaces of certain stability conditions, where the $\mathrm{GL}_2^+(\mathbb{R})$ -action and saddle connections are analogues of the central charge in the theory of stability conditions. The seminal work of Bridgeland-Smith ([BS]) established this identification for moduli spaces of quadratic differentials with simple zeros by relating the finite-length trajectories of such quadratic differentials to the stable objects of the corresponding stability condition.

4. AFFINE INVARIANT SUBMANIFOLDS

In the preceding section we discussed Teichmüller curves as examples of affine invariant submanifolds. In this section we consider affine invariant submanifolds in general.

4.1. Structure of affine invariant submanifolds. The celebrated Ratner's orbit closure theorem in ergodic theory says that the closures of orbits of unipotent flows on the quotient of a Lie group by a lattice are homogeneous submanifolds. In the context of Teichmüller dynamics, the strata of Abelian differentials do not behave like homogeneous spaces. Hence it is unclear whether $\mathrm{GL}_2^+(\mathbb{R})$ -orbit closures can have nice geometric structures. Nevertheless, the recent breakthrough of Eskin-Mirzakhani-Mohammadi ([EMi, EMM]) showed that $\mathrm{GL}_2^+(\mathbb{R})$ -orbit closures are locally cut out by

homogeneous linear equations of period coordinates with real coefficients, thus justifying that $\mathrm{GL}_2^+(\mathbb{R})$ -orbit closures are affine invariant submanifolds.⁸ Previously this result was only proved in genus two by the fundamental work of McMullen ([Mc6]).

Recall that the period coordinates at $(X, \omega) \in \mathcal{H}(\mu)$ are given by integrating ω over a basis of the relative homology $H_1(X, \Sigma; \mathbb{Z})$, where Σ is the set of zeros of ω . Although period coordinates are not canonical, any two choices of period coordinates differ by a matrix in $\mathrm{GL}_n(\mathbb{Z})$, where $n = \dim \mathcal{H}(\mu)$, hence the above theorem does not depend on the choice of period coordinates. The proof of the theorem is remarkably long and technical, which involves many ideas from dynamics on homogeneous space, ergodic theory, and measure theory.

Since period coordinates are transcendental, affine invariant submanifolds are a priori only complex-analytic submanifolds. The hidden algebraic nature has been discovered by Filip ([Fil]), who proved that affine invariant submanifolds are algebraic subvarieties of $\mathcal{H}(\mu)$, defined over $\overline{\mathbb{Q}}$. In particular, Filip used tools from variations of Hodge structures and showed that affine invariant submanifolds (except those of full rank, see Section 4.2) parameterize curves with non-trivial endomorphisms, such as real multiplication on a factor of the Jacobians, which generalizes Möller’s earlier work ([Mö1, Mö2]) on torsion and real multiplication for Teichmüller curves. As a corollary, the closure of any Teichmüller disk (in the standard topology) in \mathcal{M}_g is a subvariety of \mathcal{M}_g .

4.2. Classification of affine invariant submanifolds. After the structure theorem of Eskin-Mirzakhani-Mohammadi, the classification of affine invariant submanifolds remains to be a central open question in the study of Teichmüller curves.

The tangent space of an affine invariant submanifold \mathcal{N} at (X, ω) can be identified with $H^1(X, \Sigma; \mathbb{C})$ under the period coordinates.⁹ Denote by $p : H^1(X, \Sigma; \mathbb{C}) \rightarrow H^1(X; \mathbb{C})$ the projection from the relative cohomology to the absolute cohomology. Avila-Eskin-Möller ([AEM]) proved that $p(T_{(X, \omega)}\mathcal{N})$ is a complex symplectic vector space, hence it has even dimension. Define the *rank* of \mathcal{N} to be $\frac{1}{2} \dim_{\mathbb{C}} p(T_{(X, \omega)}\mathcal{N})$, which is at most g by definition. If the rank of \mathcal{N} is bigger than one, we say that \mathcal{N} is of *higher rank*. If it is equal to g , we say that \mathcal{N} has *full rank*. For example, arithmetic Teichmüller curves generated by square-tiled surfaces have rank one.

By analyzing the boundary of affine invariant submanifolds, Mirzakhani-Wright ([MR]) proved that if an affine invariant submanifold is of full rank, then it is either a connected component of a stratum or the hyperelliptic locus in a connected component of a stratum. Apisa ([Ap]) further showed that all affine invariant submanifolds of higher rank in the hyperelliptic strata arise from covering constructions, which gives a coarse classification of affine invariant submanifolds in the hyperelliptic strata modulo finitely many non-arithmetic Teichmüller curves and their connected components. Based on previously known examples, Mirzakhani conjectured that higher rank affine invariant submanifolds are either connected components of the strata or arise from covering constructions. Recently counterexamples of Mirzakhani’s conjecture are announced by McMullen-Mukamel-Wright and Eskin-McMullen-Mukamel-Wright.

⁸Zorich ([Z2]) called it the “Magic Wand Theorem”, which is part of Mirzakhani’s Fields Medal work.

⁹In Teichmüller dynamics \mathcal{M} is commonly used to denote an affine invariant submanifold, but here we reserve \mathcal{M} for the moduli space.

4.3. Degeneration of Abelian differentials. Despite the analytic definition of Teichmüller dynamics, a profound algebro-geometric foundation behind the story has already been revealed by many of the preceding results. In order to apply ideas from algebraic geometry, one upshot is to understanding *degeneration* of Abelian differentials, or equivalently, describing a *compactification* of strata of Abelian differentials, analogous to the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of the moduli space of curves.

Recall that the Hodge bundle \mathcal{H} extends to a rank g bundle $\overline{\mathcal{H}}$ over $\overline{\mathcal{M}}_g$, parameterizing stable differentials that are sections of the dualizing line bundle. Hence it would be natural to compactify the stratum $\mathcal{H}(\mu)$ by taking its closure in $\overline{\mathcal{H}}$. Nevertheless, a disadvantage of this Hodge bundle compactification is that it can lose information of the limit positions of the zeros of ω , especially if ω vanishes entirely on a component of the underlying reducible curve. Alternatively, up to scaling an Abelian differential is determined by the associated canonical divisor. Hence one can consider the *stratum* $\mathcal{P}(\mu)$ of *canonical divisors of type μ* as the projectivization of $\mathcal{H}(\mu)$. Marking the n zeros of the divisors, $\mathcal{P}(\mu)$ can be regarded as a subvariety of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable genus g curves with n marked points, hence one can study degeneration of canonical divisors of type μ by taking the closure of $\mathcal{P}(\mu)$ in $\overline{\mathcal{M}}_{g,n}$.

Eisenbud-Harris ([EH1]) developed a theory of limit linear series that studies degeneration of line bundles and their sections from smooth curves to nodal curves of compact type. In our context the situation is slightly different, because the zero type μ of the sections is fixed and the underlying curves may fail to be of compact type. Nevertheless, the upshot of *twisting* line bundles by irreducible components of a nodal curve still works. More precisely, define a *twisted canonical divisor of type μ* on a nodal curve X to be a collection of (possibly meromorphic) canonical divisors D_j on each irreducible component X_j of X such that the following conditions hold:

- (1) The support of D_j is contained in the set of marked points and the nodes lying in X_j . Moreover, if p_i is a marked point contained in X_j , then $\text{ord}_{p_i}(D_j) = m_i$.
- (2) If q is a node of X lying in two irreducible components X_1 and X_2 , then $\text{ord}_q(D_1) + \text{ord}_q(D_2) = -2$.
- (3) If q is a node of X lying in two irreducible components X_1 and X_2 such that $\text{ord}_q(D_1) = \text{ord}_q(D_2) = -1$, then for any node $q' \in X_1 \cap X_2$, $\text{ord}_{q'}(D_1) = \text{ord}_{q'}(D_2) = -1$. In this case we write $X_1 \sim X_2$.
- (4) If q is a node of X lying in two irreducible components X_1 and X_2 such that $\text{ord}_q(D_1) > \text{ord}_q(D_2)$, then for any node $q' \in X_1 \cap X_2$, $\text{ord}_{q'}(D_1) > \text{ord}_{q'}(D_2)$. In this case we write $X_1 \succ X_2$.
- (5) There does not exist a directed loop $X_1 \succeq X_2 \succeq \cdots \succeq X_k \succeq X_1$ unless all the relations are \sim , where \succeq means \sim or \succ .

We briefly explain the motivation behind these conditions. Since the vanishing order along each zero section remains unchanged in a family, it implies condition (1). Since $K_X|_{X_i}$ is locally generated by differentials with a simple pole at a node $q \in X_1 \cap X_2$ for $i = 1, 2$, when twisting by X_i only, the zero or pole order increases by one on one branch of q and decrease by one on the other branch, hence the sum of the vanishing orders does not vary, which implies condition (2). If there is no twist at all, it is the case corresponding to condition (3). Note that twisting by $X_1 + X_2$ does nothing to the

nodes lying in their intersection. Hence one can consider twisting, say by a multiple of X_1 only. Then the vanishing orders at all nodes between X_1 and X_2 increase or decrease simultaneously, hence condition (4) follows. By the same token, $X_1 \succeq X_2$ means the twisting coefficient of X_1 is bigger than or equal to that of X_2 , which implies the last condition.

By an analytic approach, Gendron ([Ge]) implicitly derived the above conditions and used them to study the Kodaira dimension of strata of twisted canonical divisors. Motivated by the theory of limit linear series, the author ([Ch4]) considered these conditions for curves of compact type and used them to study Weierstrass point behavior for general elements in the strata. Farkas and Pandharipande ([FP]) imposed explicitly these conditions and showed that the corresponding closures in $\overline{\mathcal{M}}_{g,n}$ are reducible in general, containing extra boundary components of dimension one less compared to the main component. It is thus natural to ask what extra conditions can distinguish the main component from the other boundary components in the closure.

Joint with Bainbridge, Gendron, Grushevsky, and Möller ([BCGGM1]), we have found the missing condition that arises from a *residue* constraint. Consider the following example. Suppose a family \mathcal{X} of Abelian differentials (X_t, ω_t) degenerate to a nodal curve X at $t = 0$. Suppose X has a separating node q joining two components Y and Z . Without loss of generality, suppose $\lim_{t \rightarrow 0} \omega_t|_Y = \eta_Y$ is a holomorphic differential on Y , and $\lim_{t \rightarrow 0} (t^{-\ell} \omega_t)|_Z = \eta_Z$ is a meromorphic differential on Z , where $\ell \in \mathbb{Z}^+$. In other words, the twisting at q is given by $\mathcal{O}_{\mathcal{X}}(-\ell Z)$ from the viewpoint of limit linear series. Let v_t be the vanishing cycle on $X_t = Y_t \cup Z_t$ that shrinks to the node q , where $Y_t \rightarrow Y$ and $Z_t \rightarrow Z$ as $t \rightarrow 0$, see Figure 17.

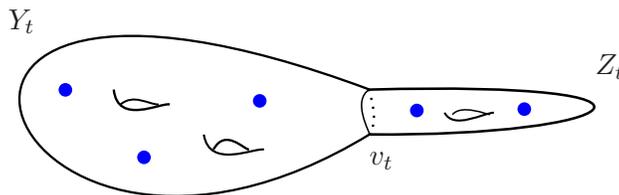


FIGURE 17. A surface with a separating vanishing cycle

Since q is a separating node, $v_t = 0 \in H_1(X_t; \mathbb{Z})$ for t nearby 0. It follows that $\int_{v_t} \omega_t = 0$ for $t \neq 0$ and hence $\int_{v_t} t^{-\ell} \omega_t = 0$. Taking the limit as $t \rightarrow 0$ and restricting to the component Z , we conclude that $\text{Res}_q(\eta_Z) = 0$. Conversely, once such a residue condition holds, along with conditions (1)–(4) we are able to prove that the limit differential is smoothable by plumbing techniques in complex-analytic geometry as well as by constructions of flat surfaces.

4.4. Cycle classes of strata of Abelian differentials. As mentioned before, affine invariant submanifolds can provide special subvarieties in the moduli space of curves. It is natural to ask if one can calculate their *cycle classes* in the Chow ring of the moduli space. The first step is to calculate the cycle classes of the strata of Abelian differentials. Tarasca and the author ([CT]) calculated the cycle classes of several minimal strata in low genus. Mullane ([Mul]) obtained a closed formula for the classes of

all strata whose projections are effective divisors in $\overline{\mathcal{M}}_g$. Both results rely on classical intersection theory on the moduli space of curves. Modulo a conjectural relation to Pixton's formula of the double ramification cycle, Janda-Pandharipande-Pixton-Zvonkine ([FP, Appendix]) obtained a recursive method to compute the cycle classes of all strata in $\overline{\mathcal{M}}_{g,n}$, which suggests that the strata classes are *tautological*. Their approach relies on analyzing the closure of a stratum of twisted canonical divisors in $\overline{\mathcal{M}}_{g,n}$ by imposing conditions (1)–(4) in the preceding section. Although the closure may have extra components contained in the boundary, those extra components are products of simpler strata of (possibly meromorphic) differentials, hence one can calculate the class of the main component recursively.

One can also consider the cycle class calculation in the Hodge bundle compactification. Korotkin-Zograf ([KorZog]) applied Tau functions to compute the divisor class of the closure of the stratum $\mathcal{P}(2, 1^{2g-4})$ in the projectivized Hodge bundle over $\overline{\mathcal{M}}_g$. The author ([Ch3]) gave another proof of this divisor class using intersection theory. Recently Sauvaget-Zvonkine have announced that they are able to compute the cycle classes of all strata closures in the Hodge bundle compactification.

5. MEROMORPHIC AND HIGHER ORDER DIFFERENTIALS

In this section we generalize the discussion of Abelian differentials to higher order differentials possibly with poles.

5.1. Quadratic differentials. First, if q is a quadratic differential, that is, a section of $K^{\otimes 2}$, then it also induces a flat structure, if one allows *rotation by π* in addition to translation as transition functions. The flat structure can be defined locally by taking a square root ω of q , which is up to \pm , and that is the reason why rotation by π needs to be part of the transition functions. Moreover if q has at worst simple poles, integrating ω along a path always provides finite length, hence the corresponding flat surface has finite area. In general, quadratic differentials with at worst simple poles are called *half-translation surfaces*. For example, Figure 18 presents a quadratic differential with four simple poles on \mathbb{P}^1 as a pillowcase.

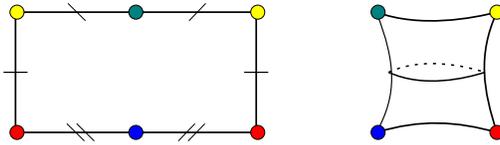


FIGURE 18. A quadratic differential with four simple poles on \mathbb{P}^1

Suppose (X, q) is a half-translation surface. One can take the unique *canonical double cover* $\pi : \hat{X} \rightarrow X$ branched at the odd singularities of q (zeros of odd order and simple poles). Then there exists a global Abelian differential $\hat{\omega}$ on \hat{X} such that $\pi^*q = \hat{\omega}^2$. From this viewpoint, all questions for Abelian differentials can be similarly asked for quadratic differentials, and indeed many of them can be similarly answered.

In what follows we mention several results analogous to the case of Abelian differentials. Fix a partition ν of $4g - 4$ such that all entries of ν are ≥ -1 . Let $\mathcal{Q}(\nu)$ be

the stratum of quadratic differentials of type ν that are not global squares of Abelian differentials. Laneeau ([L2]) classified the connected components of $\mathcal{Q}(\nu)$ for all ν . In $g \geq 5$, it can have at most two connected components, where extra components are caused by hyperelliptic structures. In particular, the spin parity in the Abelian case does not give rise to additional components in the quadratic case ([L1]). In genus three and four, there are several *exceptional* disconnected strata. Joint with Möller ([CM2]) we found an algebraic parity arising from geometry of canonical curves that distinguishes these exceptional components. We further discovered a number of strata $\mathcal{Q}(\nu)$ in low genus whose Teichmüller curves have non-varying sums of Lyapunov exponents by similar techniques as in the Abelian case. The relation between the area Siegel-Veech constant and the sum of Lyapunov exponents for affine invariant submanifolds in $\mathcal{Q}(\nu)$ also holds by the seminal work of Eskin-Kontsevich-Zorich ([EKZ1]). Eskin-Okounkov ([EO2]) analyzed the volume growth of $\mathcal{Q}(\nu)$ by enumerating covers of \mathbb{P}^1 with certain ramification profile determined by ν . Athreya-Eskin-Zorich ([AEZ]) proved an explicit formula for the volume of $\mathcal{Q}(\nu)$ in genus zero. Goujard ([Go2]) obtained explicit values for the volume of $\mathcal{Q}(\nu)$ in all low dimensions. Masur-Zorich ([MZ]) studied the principal boundary of $\mathcal{Q}(\nu)$, aiming at a recursive way to calculate Siegel-Veech constants of saddle connections. Goujard ([Go1]) proved an explicit formula that relates volumes of $\mathcal{Q}(\nu)$ and Siegel-Veech constants. Grivaux-Hubert ([GH]) constructed explicit affine invariant submanifolds in $\mathcal{Q}(\nu)$ of arbitrarily large dimension with completely degenerate Lyapunov spectrum.

5.2. Differentials with poles. Previously when we talked about an Abelian differential ω , we assumed that ω is holomorphic, so the corresponding translation surface has finite area. In many cases it would be useful to consider *meromorphic* differentials, in particular when we study the boundary structure of strata of Abelian differentials.

First, we describe the local flat geometry around a pole. Suppose p is a *simple* pole of ω . Then the flat neighborhood of p can be viewed as a *half-infinite cylinder*. The *width* of the cylinder corresponds to the *residue* of ω at p . Figure 19 exhibits a meromorphic differential that has two simple poles with opposite residues, where the poles locate at the positive infinity and negative infinity.

If p is a pole of order $m \geq 2$, Boissy ([Bo]) showed that one can glue $2m - 2$ *broken half-planes* consecutively to form a flat-geometric presentation for a neighborhood of p . The boundary of each broken half-plane consists of a half-line to the left and a half-line to the right, which are connected by finitely many broken line segments (saddle connections). The idea behind Boissy's description is the following. If one glues $2m - 2$ half-disks consecutively to form a zero of order $m - 2$, the local expression of the differential would be $z^{m-2}dz$. Now change coordinates by $z = 1/w$. Then the differential becomes $\sim w^{-m}dw$, which has a pole of order m , and the half-disks turn into half-planes. In particular, the residue of ω at p is determined by the complex lengths of the boundary line segments and the gluing pattern of the broken half-planes.

The lower right flat surface of infinite area in Figure 20 below shows the flat-geometric presentation of a meromorphic differential with a double zero and a double pole on a torus, where the pole locates at infinity. It is constructed by removing the interior of a parallelogram Z from the Euclidean plane and then identifying parallel edges by

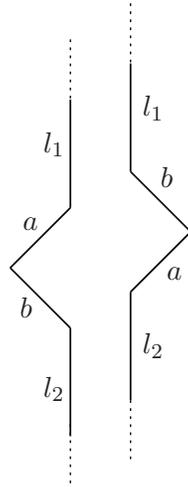


FIGURE 19. A flat surface with two simple poles

translation. If we slit the plane along a diagonal of Z , we thus recover a pair of broken half-planes as in Boissy's description. In particular, the double pole has no residue.

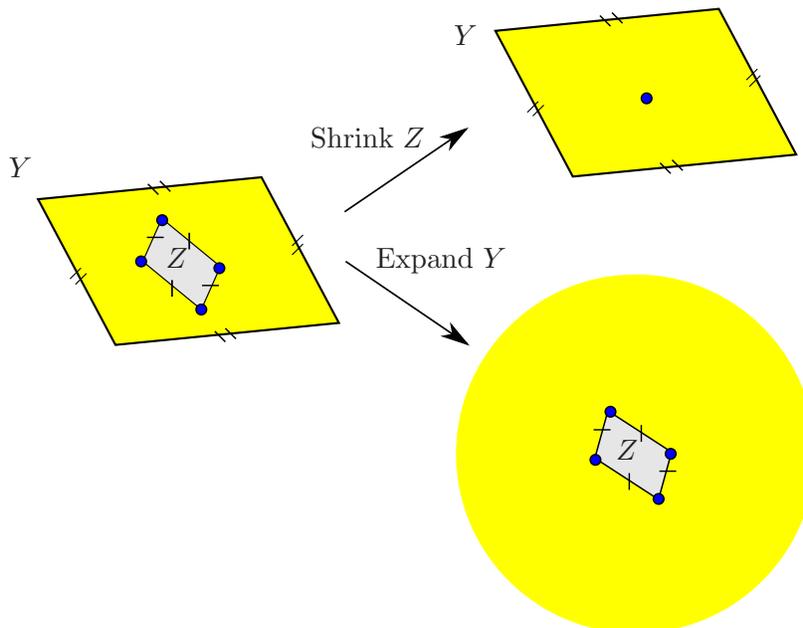
Building on earlier work of Kontsevich-Zorich and Lanneau ([KonZor1, L2]), Boissy ([Bo]) classified the connected components of strata of meromorphic differentials with prescribed numbers and multiplicities of zeros and poles. Similarly to the holomorphic case, the strata of meromorphic differentials can have at most three connected components, distinguished by hyperelliptic and spin structures.

Let us illustrate an interesting viewpoint using flat geometry of meromorphic differentials to study the boundary of strata of Abelian differentials. The flat surface on the left side of Figure 20 lies in $\mathcal{H}(2)$, which is constructed by removing a parallelogram Z from the interior of a parallelogram Y and identifying parallel edges. If we shrink Z to a point, we obtain a holomorphic differential $(Y, \eta_Y) \in \mathcal{H}(0)$, where the marked point encodes the limit position of the inner square. Alternatively, modulo scaling this procedure amounts to expanding Y to be arbitrarily large, hence the limit object represents a meromorphic differential $(Z, \eta_Z) \in \mathcal{H}(2, -2)$.

Note that these two perspectives correspond to exactly the two *aspects* in the theory of limit linear series, which applied to this case says that a family of curves of genus two with a marked double zero of a canonical divisor (Weierstrass point) degenerates to two elliptic curves joined at one node, where the limit marked point is 2-torsion respect to the node.

From the viewpoint of $\mathrm{GL}_2^+(\mathbb{R})$ -action and Teichmüller dynamics, the strata of meromorphic differentials somehow display different properties compared to the case of Abelian differentials (see [Bo, Appendix]).

5.3. Higher order differentials. We conclude the survey by describing some future directions. From the viewpoint of algebraic geometry, an Abelian differential is a section of the canonical line bundle K , and a quadratic differential is a section of $K^{\otimes 2}$. Therefore, it is natural to consider *higher order differentials* arising from sections of

FIGURE 20. Shrinking Z versus expanding Y

$K^{\otimes k}$ for a fixed positive integer k . Suppose $\mu = (k_1, \dots, k_n)$ is a partition of $k(2g - 2)$, where we allow k_i to be possibly negative. In other words, we want to take meromorphic differentials into account. Let $\mathcal{H}^k(\mu)$ be the *stratum of k -differentials of type μ* , which parameterizes (possibly meromorphic) sections of $K^{\otimes k}$ with zeros and poles of type μ on genus g Riemann surfaces. Since $\mathrm{GL}_2^+(\mathbb{R})$ does not preserve angles in general, there is no natural $\mathrm{GL}_2^+(\mathbb{R})$ -action on the strata of k -differentials for $k > 2$. Nevertheless, many other questions regarding Abelian and quadratic differentials can still be asked for k -differentials. In particular, what are the dimension, connected components, compactification, and cycle class of $\mathcal{H}^k(\mu)$? In a forthcoming work ([BCGGM2]), we will treat these questions systematically.

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