STRATA OF ABELIAN DIFFERENTIALS AND THE TEICHMÜLLER DYNAMICS

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Abstract. This paper focuses on the interplay between the intersection theory and the Teichmüller dynamics on the moduli space of curves. As applications, we study the cycle class of strata of the Hodge bundle, present an algebraic method to calculate the class of the divisor parameterizing abelian differentials with a non-simple zero, and verify a number of extremal effective divisors on the moduli space of pointed curves in low genus.

1. Introduction

Let $\mathcal{H}$ be the moduli space of abelian differentials parameterizing pairs (C, ω), where C is a smooth, connected, complete complex curve of genus g and ω is an abelian differential on C, i.e. ω is a holomorphic one-form, or equivalently speaking, a section of the canonical line bundle K of C. Since the space of abelian differentials on C is g-dimensional, $\mathcal{H}$ forms naturally a vector bundle of rank g on the moduli space $\mathcal{M}_g$ of genus g curves. The fiber of $\mathcal{H}$ over $[C] \in \mathcal{M}_g$ is identified with $H^0(C, K)$. In this sense $\mathcal{H}$ is also called the Hodge bundle in algebraic geometry.

Let $\mu = (m_1, \ldots, m_n)$ be a partition of $2g-2$, i.e. m_i's are unordered positive integers whose sum is equal to $2g-2$. Denote by $\mathcal{H}(\mu)$ the locus of (C, ω) in $\mathcal{H}$ such that the associated zero divisor of ω is of type $(\omega)_0 = \sum_{i=1}^n m_i p_i$ for distinct points $p_1, \ldots, p_n$ in C. Take a basis $\gamma_1, \ldots, \gamma_{2g+n-1}$ of the relative homology $H_1(C, p_1, \ldots, p_n; \mathbb{Z})$. Integrating ω along $\gamma_1, \ldots, \gamma_{2g+n-1}$ provides a local coordinate system for $\mathcal{H}(\mu)$, called the relative period coordinates, and hence $\mathcal{H}(\mu)$ is a $(2g+n-1)$-dimensional submanifold of $\mathcal{H}$, see [Kon97]. We call $\mathcal{H}(\mu)$ a stratum of $\mathcal{H}$ with signature μ. Kontsevich and Zorich [KZ03] classified completely the connected components of all strata. Note that $\mathcal{H}(\mu)$ may have up to three connected components, due to some additional hyperelliptic, odd or even spin structures associated to the abelian differentials.

An abelian differential ω defines a flat structure on the underlying Riemann surface C, such that C can be realized as a plane polygon whose edges are vectors in the Eucliean plane given by the relative period coordinates and are glued appropriately under parallel translation, see [Zor06, Figure 12] for an illustration. Changing the shape of the plane polygon induces an $\mathrm{SL}_2(\mathbb{R})$-action on $\mathcal{H}$, called the Teichmüller dynamics. A central question in the study of Teichmüller dynamics is to understand the structure of its orbit closures. What are their dimensions? Do they possess a manifold structure? What are the associated dynamical quantities, such as Lyapunov exponents and Siegel-Veech constants? We refer to [EMZ03, KZ03, Zor06, EKZ11] for a comprehensive introduction to these subjects.

Although the questions are analytic in nature, recently there have been some attempts using tools in algebraic geometry to study them. For instance, if the projection of an orbit

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forms an algebraic curve in $\mathcal{M}_g$, we call it a Teichmüller curve. Based on a limited number of computer experiments, Kontsevich and Zorich came up with a list of strata of abelian differentials in low genus and conjectured that for every stratum (component) on the list the sum of Lyapunov exponents is the same for all Teichmüller curves contained in that stratum. Marking the zeros of an abelian differential, one can lift a Teichmüller curve to the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable nodal genus g curves with n ordered marked points. In [CM12a] the conjecture was proved by calculating the intersection of Teichmüller curves with divisor classes on $\overline{\mathcal{M}}_{g,n}$. Later on Yu and Zuo [YZar] found another proof using certain filtration of the Hodge bundle. An analogous result regarding Teichmüller curves generated by quadratic differentials was established in [CM12b]. Furthermore in [CMZ] the authors consider a special type of higher dimensional orbit closures given by torus coverings. The upshot also relies on certain intersection calculation on the Hurwitz space compactified by admissible covers, see [HM98, Chapter 3.G].

The recent breakthrough work of Eskin and Mirzakhani [EM13] shows that any orbit closure has an affine invariant submanifold structure. Nevertheless, a complete classification of the orbit closures is still missing. Note that the $\mathrm{SL}_2(\mathbb{R})$-action preserves every stratum $\mathcal{H}(\mu)$, hence it is already interesting to study these strata from the viewpoint of intersection theory. To set it up algebraically, we projectivize the Hodge bundle $\mathcal{H}$ as a projective bundle $\mathbb{P}\mathcal{H}$ with fiber $\mathbb{P}^{g-1}$ on $\mathcal{M}_g$. In other words, the projectivization $\mathbb{P}\mathcal{H}$ parameterizes canonical divisors $(\omega)_0$ instead of abelian differentials ω on a genus g curve. If we label the zeros of a differential in an ordered way as n marked points, we can lift $\mathbb{P}\mathcal{H}(\mu)$ to $\mathcal{M}_{g,n}$. Then the first step of the framework is to understand the cycle class of the lifting in the Chow ring of $\mathcal{M}_{g,n}$. In Section 2 we compute this class using Porteous' formula (Proposition 2.3).

We remark that in the above calculation the n zeros as marked points remain distinct. In other words, we have not taken the boundary of $\overline{\mathcal{M}}_{g,n}$ into account. However, $\mathbb{P}\mathcal{H}$ does extend to the boundary of $\overline{\mathcal{M}}_g$ as a projective bundle $\mathbb{P}\overline{\mathcal{H}}$. Therefore, one can further take the closure $\mathbb{P}\overline{\mathcal{H}}(\mu)$ of a stratum $\mathbb{P}\mathcal{H}(\mu)$ in $\mathbb{P}\overline{\mathcal{H}}$ and study its cycle class. If we can describe geometrically the boundary of the stratum closure, we may apply standard techniques in algebraic geometry like intersecting with a test family to study its cycle class, see [HM98, Chapter 3.F] for some examples. Unfortunately to the author's best knowledge, a precise geometric description for the boundary of $\mathbb{P}\overline{\mathcal{H}}(\mu)$ is still unknown in general, partially because we do not have a good control of degenerating abelian differentials from smooth curves to an arbitrary nodal curve.

However for codimension-one degeneration, i.e. for $\mu = (2, 1^{2g-4})$, the stratum $\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})$ is a divisor in $\mathbb{P}\overline{\mathcal{H}}$. In Section 3 we are able to calculate its class (Proposition 3.1 and Theorem 3.2). Understanding this divisor is useful from a number of aspects, e.g. in [Ham12] it was used to detect signatures of surface bundles. We remark that its divisor class was first calculated in [KZ11, Theorem 2] by an analytic approach using the Tau function, see also [KKZ11, vdGK11] for some relevant earlier work in the setting of Hurwitz spaces. Here our method is purely algebraic, hence it provides additional information regarding the birational geometry of $\mathbb{P}\overline{\mathcal{H}}$. For instance, it is often useful but difficult to find an extremal effective divisor intersecting the interior of a moduli space. The existence of such a divisor can provide crucial information for the birational type of the moduli space. As a by-product of our intersection theoretical approach, we show that the divisor class of $\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})$ lies on the boundary of the pseudo-effective cone of $\mathbb{P}\overline{\mathcal{H}}$ (Proposition 3.5).

In Section 4 we reverse our engine, using the Teichmüller dynamics to study effective divisors on the moduli space of curves. The history of studying effective divisors on $\overline{\mathcal{M}}_g$ dates back to [HM82], where Harris and Mumford used the Brill-Noether divisor parameterizing curves with exceptional linear series to show that $\overline{\mathcal{M}}_g$ is of general type for large g. Later on Logan studied a series of pointed Brill-Noether divisors on $\overline{\mathcal{M}}_{g,n}$ [Log03]. As mentioned above, we would like to understand whether those divisors are extremal. By checking their intersections with various Teichmüller curves, we prove the extremality for a number of pointed Brill-Noether divisors in low genus (Theorem 4.3).

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2. Class of the strata

The calculation in this section is standard to an algebraic geometer. But we still write down everything in detail for the readers who are only familiar with the dynamical side of the story. Here the main tool is Porteous' formula, which expresses the class of the locus where the rank of a map between vector bundles is less than or equal to a given bound. In what follows we briefly review this formula, see e.g. [ACGH85, Chapter II §4 (iii)] and [HM98, Chapter 3.E] for more details.

For a vector bundle E, let

$$c_t(E) = \sum_i c_i(E)t^i$$

be its Chern polynomial, where $c_i(E)$ is the ith Chern class of E and t is a formal variable. For any integer p and any positive integer q, define a $q \times q$ matrix $M_{p,q}(c_t)$ whose (i, j)th entry is c_{p+j-i} for a formal power series $c_t = \sum_i c_i t^i$. Define its determinant as

$$\Delta_{p,q}(c_t) = \det(M_{p,q}(c_t)).$$

Theorem 2.1 (Porteous' Formula)**.** *Let $\phi : E \to F$ be a homomorphism between vector bundles of ranks m and n, respectively, on a complex manifold X. Consider the degenerate locus*

$$D_k = \{x \in X \mid \operatorname{rank}(\phi_x) \le k\}$$

and let $[D_k]$ be its cycle class in the Chow ring of X. If D_k is either empty or of the expected codimension $(m - k)(n - k)$, then

$$[D_k] = \Delta_{n-k,m-k}((c_t(F)/c_t(E)).$$

Even if D_k has dimension higher than expected, we can still compute the virtual class of D_k using Porteous' formula. If D_k is of the expected dimension, the virtual class thus equals its actual class.

We want to realize the locus of abelian differentials with a given type of zeros as certain degeneration locus. Then we can calculate its class by Porteous' formula. Let us first consider a more general setting. Denote by $\mu = (m_1, \ldots, m_n)$ a partition of a positive integer d. Let BN^r_μ be the locus of $(C, p_1, \ldots, p_n)$ in the moduli space $\mathcal{M}_{g,n}$ of genus g curves with n ordered marked points such that the divisor $D = \sum_{i=1}^n m_i p_i$ in C satisfies

$$h^0(C, D) \ge r + 1.$$

Here the notation BN stands for the Brill-Noether divisors, see [ACGH85, Chapter V] for a comprehensive introduction to the Brill-Noether theory. Our goal is to calculate the class of BN^r_μ in the Chow ring of $\mathcal{M}_{g,n}$. We remark that in the case of codimension one, i.e. when BN^r_μ is an effective divisor, the calculation dates back to Harris and Mumford [HM82] and they used its divisor class to study the Kodaira dimension of $\overline{\mathcal{M}}_g$. Later on Cukierman [Cuk89] calculated the class of the divisor of Weierstrass points in $\overline{\mathcal{M}}_{g,1}$. The calculation of the Brill-Noether divisor with more marked points was completed by Logan [Log03] and he obtained similar results regarding the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$.

For a flat family of curves $f : S \to B$ over a base manifold B, define the relative dualizing sheaf Ω (the vertical cotangent line bundle) of f as the dual of the quotient line bundle T_S/f^*T_B, where T_S and T_B are the tangent bundles of S and B, respectively. Geometrically speaking, Ω restricted to a fiber curve C is thus the canonical line bundle (the cotangent line bundle) of C, see e.g. [HM98, Chapter 3.A] for more details.

Let $\mathcal{C}^n$ denote the n-fold fiber product of the universal curve $\mathcal{C} = \mathcal{M}_{g,1}$ over $\mathcal{M}_g$. In other words, the fiber of $\mathcal{C}^n$ over $[C] \in \mathcal{M}_g$ is the nth direct product of C. Note that $\mathcal{M}_{g,n}$ is the complement of the big diagonal in $\mathcal{C}^n$. Let Ω be the relative dualizing sheaf associated to $\mathcal{M}_{g,1} \to \mathcal{M}_g$. Define the forgetful map $f_i : \mathcal{C}^n \to \mathcal{C}$ by forgetting all but the ith factor. Define $\Omega_i = f^*\Omega$ and denote its first Chern class by ω_i. Let $\pi : \mathcal{C}^{n+1} \to \mathcal{C}^n$ be the projection forgetting the last factor. Define the divisor Δ_μ in $\mathcal{C}^{n+1}$ as

$$\Delta_\mu = \sum_{i=1}^{n} m_i \Delta_{i,n+1},$$

where $\Delta_{i,j}$ is the diagonal corresponding to $p_i = p_j$.

Define a sheaf $\mathbb{F}_\mu$ on $\mathcal{C}^n$ as

$$\mathbb{F}_\mu = \pi_*(\mathcal{O}_{\Delta_\mu} \otimes \Omega_{n+1}),$$

where π_* means taking the direct image sheaf. More precisely, the stalk of $\mathbb{F}_\mu$ at a point $(C, p_1, \ldots, p_n)$ can be identified with $H^0(C, K/K(-D))$, where K is the canonical line bundle of C and $D = \sum_{i=1}^n m_i p_i$. Consider the exact sequence

$$0 \to K(-D) \to K \to \mathcal{O}_D(K) \to 0.$$

We have

$$H^0(C, K/K(-D)) \cong H^0(\mathcal{O}_D(K)) \cong H^0(\mathcal{O}_D) \cong \mathbb{C}^d,$$

since D is a divisor of degree d in C. Consequently $\mathbb{F}_\mu$ is a vector bundle of rank d.

Let $\mathbb{E}$ be the pullback of the Hodge bundle $\mathcal{H}$ from $\mathcal{M}_g$ to $\mathcal{C}^n$, i.e. its fiber at $(C, p_1, \ldots, p_n)$ is canonically identified with $H^0(C, K)$. We have an evaluation map

$$\phi : \mathbb{E} \to \mathbb{F}_\mu.$$

Restricting to the fibers over $(C, p_1, \ldots, p_n) \in \mathcal{C}^n$, ϕ is given by the map

$$H^0(C, K) \to H^0(C, K/K(-D))$$

associated to the above exact sequence. If $\mathrm{rank}(\phi) \leq d - r$, then using the exact sequence we read off

$$h^0(C, K(-D)) \geq g - d + r.$$

Hence by the Riemann-Roch formula we have

$$h^0(C, D) \geq 1 - g + d + (g - d + r) = r + 1.$$

In summary, the locus where

$$\operatorname{rank}(\phi) \leq d - r$$

parameterizes $(C, p_1, \ldots, p_n)$ satisfying $h^0(C, D) \geq r + 1$.

Recall in the setting of Porteous' formula,

$$\Delta_{p,q}\Big(\sum_{i=0}^{\infty} c_i t_i\Big) = \det \begin{pmatrix} c_p & c_{p+1} & \cdots & c_{p+q-1} \\ c_{p-1} & c_p & \cdots & c_{p+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_p \end{pmatrix}.$$

We thus conclude that the virtual class of BN^r_μ in the Chow ring of $\mathcal{M}_{g,n}$ is given by

$$[BN^r_\mu] = \Delta_{r,g-d+r}(c(\mathbb{F}_\mu)/c(\mathbb{E})). \tag{1}$$

Let $\mu' = (m_1 - 1, m_2, \ldots, m_n)$ and similarly define $\mathbb{F}_{\mu'}$. We have a filtration

$$0 \to \mathbb{F}_1 \to \mathbb{F}_\mu \to \mathbb{F}_{\mu'} \to 0$$

where the fiber of $\mathbb{F}_1$ at $(C, p_1, \ldots, p_n)$ is

$$H^0(C, K(-D + p_1)/K(-D)),$$

namely, $\mathbb{F}_1$ is isomorphic to $\Omega_1^{\otimes m_1}$, because the multiplicity of p_1 in D is m_1. Using the filtration sequences by subtracting 1 successively from the m_i's, we can reduce the signature μ to $(1, 0, \ldots, 0)$, and consequently we obtain that

$$\begin{aligned} c(\mathbb{F}_\mu) &= \prod_{i=1}^{n}\prod_{j=1}^{m_i}(1 + j\omega_i) \\ &= 1 + \Big(\sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}\omega_i\Big) + \cdots \end{aligned}$$

Let $\lambda_i = c_i(\mathbb{E})$ be the ith Chern class of the Hodge bundle. We have

$$\frac{1}{c(\mathbb{E})} = 1 - \lambda_1 + (\lambda_1^2 - \lambda_2) + \cdots.$$

Indeed all the λ_i classes can be expressed as polynomials of the Miller-Morita-Mumford classes $\kappa_i = \pi_*(\omega^{i+1})$, where ω is the first Chern class of the dualizing sheaf of $\pi : \mathcal{C} \to \mathcal{M}_g$. For instance, we have $\lambda_1 = \frac{1}{12}\kappa_1$ and $\lambda_2 = \frac{\kappa_1^2}{288}$. Note that in general these polynomial expressions are not simple powers of κ_1, but they can be worked out explicitly in any given case, see e.g. [HM98, Chapter 3.E] for more details.

Example 2.2. Let $\mu = (m_1, \ldots, m_n)$ be a partition of g. Consider the divisor BN^1_μ in $\mathcal{M}_{g,n}$. In this case $r = 1$ and $d = g$, hence using (1) we conclude that the class of BN^1_μ in $\mathcal{M}_{g,n}$ is

$$c_1(\mathbb{F}) - c_1(\mathbb{E}) = -\lambda_1 + \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}\omega_i.$$

The class of the closure of BN^1_μ in $\overline{\mathcal{M}}_{g,n}$ was calculated in [Log03, Theorem 5.4]. As we see the results are the same modulo boundary classes.

Now we specialize to the strata of abelian differentials. Let $\mu = (m_1, \ldots, m_n)$ be a partition of $2g-2$. Although the zeros of a differential are not ordered, we choose an order between them and mark the zeros. In this way we can lift $\mathbb{P}\mathcal{H}(\mu)$ to $\mathcal{M}_{g,n}$. Note that this is only a mild modification. For instance, if $m_1, \ldots, m_n$ are all distinct, then we still get an embedding of $\mathbb{P}\mathcal{H}(\mu)$ into $\mathcal{M}_{g,n}$. In general, one can calculate the class in $\mathcal{M}_{g,n}$ first, and then push forward the class via the finite morphism $\mathcal{M}_{g,n} \to \mathcal{M}_{g,[n]}$, where $\mathcal{M}_{g,[n]} = \mathcal{M}_{g,n}/\mathfrak{S}_n$ is the moduli space of genus g curves with n unordered marked points.

Denote by $[\mathbb{P}\mathcal{H}(\mu)]_{g,n}$ the class of the lift of $\mathbb{P}\mathcal{H}(\mu)$ in the Chow ring of $\mathcal{M}_{g,n}$. In order to calculate this class, in Porteous' formula we set $r = g-1$ and $d = 2g-2$. Moreover, we know $\dim \mathbb{P}\mathcal{H}(\mu) = 2g-2+n$, hence it has codimension $g-1$ in $\mathcal{M}_{g,n}$, which is equal to the expected codimension predicted by Porteous' formula. Therefore, we thus obtain the following result.

Proposition 2.3. *The class $[\mathbb{P}\mathcal{H}(\mu)]_{g,n}$ in $\mathcal{M}_{g,n}$ is given by*

$$\Delta_{g-1,1}(c(\mathbb{F}_\mu)/c(\mathbb{E})) = [c(\mathbb{F}_\mu)/c(\mathbb{E})]_{g-1}.$$

More precisely, we have

$$c(\mathbb{F}_\mu) = \prod_{i=1}^{n}\prod_{j=1}^{m_i}(1 + j\omega_i),$$

$$\frac{c(\mathbb{F}_\mu)}{c(\mathbb{E})} = \Big(\prod_{i=1}^{n}\prod_{j=1}^{m_i}(1 + j\omega_i)\Big) \cdot \Big(1 - \lambda_1 + (\lambda_1^2 - \lambda_2) + \cdots\Big),$$

and the class $[\mathbb{P}\mathcal{H}(\mu)]_{g,n}$ equals the term of degree $g-1$ in the expansion.

Finally let us comment on the limitation of Porteous' formula. It is tempting to generalize the above formal calculation to the boundary of the moduli space. Nevertheless, the vector bundles $\mathbb{E}$ and $\mathbb{F}$ used above may not have a vector bundle structure along the boundary, hence Porteous' formula does not apply. For a concrete example of this issue as well as how to fix it, see [Dia04, Tho12] for a calculation of Porteous' type for the divisor class of the locus of hyperelliptic curves in $\overline{\mathcal{M}}_3$. However, the method there is ad hoc and seems very hard to generalize. Hence it would be useful to establish a generalized Porteous' formula that deals with not only vector bundles but also coherent sheaves. We leave it as an interesting question to the reader.

3. Abelian differentials with a non-simple zero

In this section we consider the locus $\mathbb{P}\mathcal{H}(2, 1^{2g-4})$ parameterizing canonical divisors with a zero of multiplicity ≥ 2. First, take its closure in $\mathbb{P}\mathcal{H}$ and denote it by $\mathbb{P}\widetilde{\mathcal{H}}(2, 1^{2g-4})$. Then $\mathbb{P}\widetilde{\mathcal{H}}(2, 1^{2g-4})$ forms a divisor in $\mathbb{P}\mathcal{H}$. Note that in $\mathbb{P}\widetilde{\mathcal{H}}(2, 1^{2g-4})$ we allow further coincidences among the zeros, e.g. it contains $\mathbb{P}\mathcal{H}(3, 1^{2g-5})$ by pinching the double zero with a simple zero, but the underlying curve remains to be smooth.

The rational Picard group $\mathrm{Pic}(\mathbb{P}\mathcal{H}) \otimes \mathbb{Q}$ (i.e. over rational coefficients instead of integer) is generated by λ and ψ, where λ is the pullback of the Hodge class λ_1 from $\mathcal{M}_g$ and ψ is the class of the universal line bundle $\mathcal{O}(1)$, see e.g. [HM98, Chapter 3.D] for an introduction to the Picard group of the moduli space. We emphasize that here the projectivization $\mathbb{P}V$ of a vector space V parameterizes lines instead of hyperplanes in V. Moreover, ψ represents the universal line bundle instead of the tautological line bundle $\mathcal{O}(-1)$. Consequently the divisor classes λ and ψ in our setting are the opposites to the corresponding classes in [KZ11].

Proposition 3.1. *The divisor class of* $\mathbb{P}\widetilde{\mathcal{H}}(2,1^{2g-4})$ *in* $\mathrm{Pic}(\mathbb{P}\mathcal{H})\otimes\mathbb{Q}$ *is given by*

$$[\mathbb{P}\widetilde{\mathcal{H}}(2,1^{2g-4})] = (6g-6)\psi - 24\lambda.$$

Proof. Suppose the class is

$$[\mathbb{P}\widetilde{\mathcal{H}}(2,1^{2g-4})] = a\psi + b\lambda. \tag{2}$$

We will use the method of test curves to compute the coefficients a and b. Let us explain this method first. Suppose B is a non-trivial one-dimensional family of canonical divisors parameterized in $\mathbb{P}\mathcal{H}$. Then B itself can be regarded as a curve in $\mathbb{P}\mathcal{H}$. Consider the intersection between curve classes and divisor classes in the Chow ring of $\mathbb{P}\mathcal{H}$. We conclude that B has the same intersection number with the left hand side and the right hand side of (2). By calculating directly the intersection numbers $B\cdot\mathbb{P}\widetilde{\mathcal{H}}(2,1^{2g-4})$, $B\cdot\psi$ and $B\cdot\lambda$, we get a relation between a and b. For some examples of using test curves, see [HM98, Chapter 3.F].

Take a general curve C of genus g and consider its canonical embedding in $\mathbb{P}H^0(C,K)\cong\mathbb{P}^{g-1}$. A hyperplane section of C gives rise to a canonical divisor. Now fix a linear subspace $\Lambda=\mathbb{P}^{g-3}$. There is a one-dimensional family B of hyperplanes in $\mathbb{P}^{g-1}$ that contain Λ. It is easy to see that $B\cong\mathbb{P}^1$. For each hyperplane parameterized in B, its section in C is a canonical divisor. Hence in this way B can also be regarded as a one-dimensional family of canonical divisors.

Project C from Λ to a line L in $\mathbb{P}^{g-1}$. It induces a branched covering map $\pi: C\to\mathbb{P}^1$. Take a point p in L. The pre-image π^*p consists of the intersection of C with the hyperplane spanned by p and Λ. Since the degree of C is $2g-2$, we obtain that π is of degree $2g-2$. Note that if a hyperplane is tangent to C, then the corresponding section gives rise to a canonical divisor with a non-simple zero at the ramification point. Therefore, we conclude that the number of simple ramification points of π is equal to the intersection number $B\cdot\mathbb{P}\mathcal{H}(2,1^{2g-4})$. By the Riemann-Hurwitz formula, π has $2g-2+2(2g-2)=6g-6$ simple ramification points, hence we obtain that

$$B\cdot\mathbb{P}\widetilde{\mathcal{H}}(2,1^{2g-4}) = 6g-6.$$

Moreover, the family B by its construction is a one-dimensional linear subspace in the fiber $\mathbb{P}^{g-1}$ of $\mathbb{P}\mathcal{H}$ over $[C]\in\mathcal{M}_g$. Since ψ restricted to each fiber is the universal line bundle class $\mathcal{O}(1)$, we obtain that

$$B\cdot\psi = 1.$$

Finally, the underlying curve C remains the same for all canonical divisors parameterized in B. In other words, B maps to the point $[C]$ under the projection $\mathbb{P}\mathcal{H}\to\mathcal{M}_g$. Since λ is the pullback of λ_1 from $\mathcal{M}_g$, by the projection formula we conclude that

$$B\cdot\lambda = 0.$$

Now use B to intersect every class in (2). We obtain that

$$6g-6 = a\cdot 1 + b\cdot 0,$$

$$a = 6g-6.$$

In order to calculate b, we use the relation

$$\lambda = \kappa_\mu\cdot\psi$$

restricted to a stratum $\mathbb{P}\mathcal{H}(\mu)$, see [EKZ11, Section 3.4] and [CM12a, Section 4] for more details. Here for a partition $\mu = (m_1, \ldots, m_n)$, κ_μ is defined as

$$\kappa_\mu = \frac{1}{12}\sum_{i=1}^{n}\frac{m_i(m_i+2)}{m_i+1}.$$

The above relation comes from Noether's formula $12\lambda_1 = \kappa_1 + \delta$ on $\overline{\mathcal{M}}_g$ modulo the boundary δ, see e.g. [HM98, Chapter 3.E]. In particular, we have

$$\kappa_{(1^{2g-2})} = \frac{g-1}{4}.$$

Since the complement of the principal stratum $\mathbb{P}\mathcal{H}(1^{2g-2})$ in $\mathbb{P}\mathcal{H}$ consists of the divisorial stratum $\mathbb{P}\mathcal{H}(2, 1^{2g-4})$ union strata of higher codimension, we conclude that in $\mathrm{Pic}(\mathbb{P}\mathcal{H}) \otimes \mathbb{Q}$

$$\lambda = \frac{g-1}{4}\psi + c \cdot [\mathbb{P}\widetilde{\mathcal{H}}(2, 1^{2g-4})]$$

with c unknown.

Using the test curve B again for the above equality, we have

$$\frac{g-1}{4} + (6g-6)c = 0,$$

$$c = -\frac{1}{24}.$$

Therefore, we conclude that

$$[\mathbb{P}\widetilde{\mathcal{H}}(2, 1^{2g-4})] = -24\lambda + (6g-6)\psi.$$

□

Next we take the further closure $\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})$ of $\mathbb{P}\mathcal{H}(2, 1^{2g-4})$ in the projective bundle $\mathbb{P}\overline{\mathcal{H}}$ over $\overline{\mathcal{M}}_g$. Still use δ_i to denote the pullback of the boundary divisor δ_i from $\overline{\mathcal{M}}_g$. Then the rational Picard group $\mathrm{Pic}(\mathbb{P}\overline{\mathcal{H}}) \otimes \mathbb{Q}$ is generated by λ, ψ and $\delta_0, \ldots, \delta_{[g/2]}$. In order to define these boundary divisors, we call a node of a curve a separating node if its removal disconnects the curve. Otherwise we call it a non-separating node. For $i > 0$, δ_i is the closure of the locus of curves consisting of two connected components of genus i and $g-i$, respectively, intersecting at a separating node. Moreover, δ_0 is the locus of curves that possess a non-separating node. These boundary divisors may intersect each other or self intersect, corresponding to curves with more nodes. In particular, all the boundary divisors are closed loci in $\overline{\mathcal{M}}_g$ and in $\mathbb{P}\overline{\mathcal{H}}$. We also use $\delta = \sum_{i=0}^{[g/2]}$ to denote the total boundary class.

Now we can calculate the full class of $\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})$ including the boundary divisors. The calculation still replies on using test curves and dealing with nodal degenerations. For the reader who is unfamiliar with nodal curves, consider a simple example, say, a family of plane cubics C_t defined by $y^2 = x(x-1)(x-t)$. Note that for $t = 0$, C_0 is given by $y^2 = x^2(x-1)$, hence it has a nodal singularity at the origin, while for t small enough but not equal to zero, C_t is smooth. Therefore, nodal curves arise naturally in a complete family of curves even if its general member parameterizes a smooth curve.

Theorem 3.2. *In* $\mathrm{Pic}(\mathbb{P}\overline{\mathcal{H}}) \otimes \mathbb{Q}$, *we have*

$$[\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})] = (6g-6)\psi - 24\lambda + 2\delta_0 + 3\sum_{i=1}^{[g/2]}\delta_i.$$

Proof. In the proof of Proposition 3.1, the relation $\lambda = \kappa_\mu \cdot \psi$ restricted to a stratum $\mathbb{P}\mathcal{H}(\mu)$ arises from Noether's formula $12\lambda = \kappa_1 + \delta$ modulo boundary, where κ_1 is the first Miller-Morita-Mumford class and δ is the total boundary class. Note that a general degeneration from the principal stratum $\mathbb{P}\mathcal{H}(1^{2g-2})$ to the boundary δ_0 keep the $2g-2$ sections of simple zeros away from the non-separating node. Then we can regard the above formula as $12\lambda = \kappa_1 + \delta_0$ restricted to $\mathbb{P}\mathcal{H}(1^{2g-2})$ union δ_0. By Proposition 3.1 we can rewrite this relation as

$$[\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})] = (6g-6)\psi - 24\lambda + 2\delta_0$$

in $\mathrm{Pic}(\mathbb{P}\overline{\mathcal{H}}) \otimes \mathbb{Q}$ modulo higher boundary classes δ_i for $i > 0$.

Next, take a general one-dimensional family Z of genus g curves with $2g-2$ sections such that in a generic fiber the sum of the sections yields a canonical divisor in $\mathbb{P}\mathcal{H}(1^{2g-2})$. In other words, each section parameterizes a zero of the canonical divisor varying in this family Z. Moreover, suppose there are k special fibers C in Z that consist of two components C_1 and C_2 joined at a separating node t, and C_1 and C_2 have respectively genus i and $g-i$. Note that the canonical line bundle K of C satisfies $K|_{C_j} = K_{C_j}(t)$ for $j = 1, 2$. Hence it implies that t is a simple base point of $K|_{C_j}$, namely, every canonical divisor of C restricted to C_1 and C_2 contains t as a zero. Reformulating differently, as the $2g-2$ distinct zeros vary from a general fiber nearby C into the special fiber C, $2i-2$ of them remain in the smooth locus of C_1, $2(g-i)-2$ of them remain in the smooth locus of C_2, and the rest two of them go to the node t. Therefore, without loss of generality we can assume that the first two sections pass through t, the next $2i-2$ sections meet C_1 away from t and the last $2(g-i)-2$ sections meet C_2 away from t.

Blow up the family at these k nodes and let $\pi : \mathcal{C} \to Z$ be the resulting family. Use $S_1, \ldots, S_{2g-2}$ to denote the proper transforms of the $2g-2$ sections, and $E_1, \ldots, E_k$ the k exceptional curves. Note that the corresponding E_l is a rational curve between C_1 and C_2 for each of the k special fibers C. Moreover, E_l intersects C_j at the pre-image of the node t in C_j for $j = 1, 2$. By our setting in the preceding paragraph, S_1 and S_2 intersect E_l transversally while the other sections are disjoint with E_l, see Figure 1.

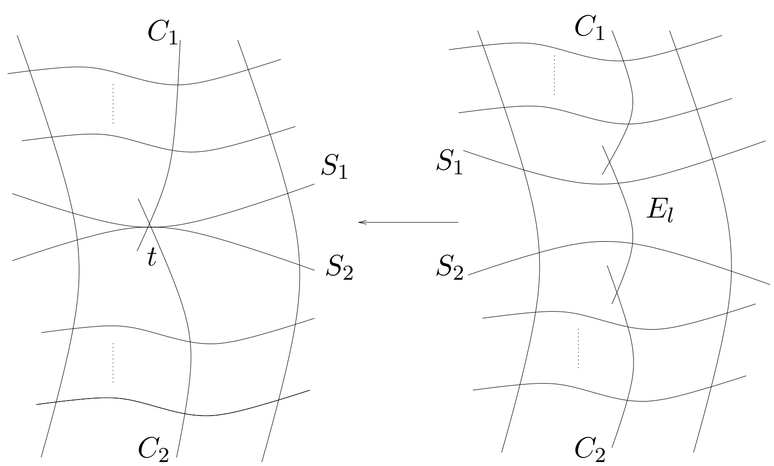

FIGURE 1. Blow up at the node t.

Since all fibers represent the same numerical class in $\mathcal{C}$, we have the intersection number

$$E_l \cdot (E_l + C_1 + C_2) = 0$$

by deforming the special fiber to a nearby fiber disjoint with E_l. Since $E_l \cdot C_i = 1$ for $i = 1, 2$, we conclude that the self-intersection $E_l^2 = -2$ for $l = 1, \ldots, k$.

As an analogue of the exact sequence in [CM12a, Proof of Proposition 4.8], we have

$$0 \to \pi^*\mathcal{O}(1) \to \Omega \otimes \mathcal{O}_{\mathcal{C}}\Big(-\sum_{l=1}^{k} E_l\Big) \to \sum_{i=1}^{2g-2} \mathcal{O}_{S_i}(S_i) \to 0.$$

Here $\mathcal{O}(1)$ is the universal line bundle whose first Chern class is ψ and Ω is the relative dualizing sheaf associated to π. The middle term restricted to C_i is the canonical line bundle of C_i and it is trivial restricted to E_l. Let $\omega = c_1(\Omega)$. Then we obtain that

$$\omega = \pi^*\psi + \sum_{l=1}^{k} E_l + \sum_{i=1}^{2g-2} S_i. \tag{3}$$

Moreover, based on the above analysis we have

$$E_l \cdot S_1 = E_l \cdot S_2 = 1,$$

$$E_l \cdot S_j = 0,\ j > 2,$$

$$E_l^2 = -2,\ E_l \cdot E_j = 0,\ l \neq j.$$

Then for $i \neq 1, 2$ we see that

$$\pi_*(S_i^2) = -\pi_*(\omega \cdot S_i) = -\psi|_Z - \pi_*(S_i^2).$$

Here the first equality comes from the adjunction formula, i.e. the self-intersection of S_i in the surface $\mathcal{C}$ equals the dual of the vertical cotangent line bundle restricted to S_i, the second equality comes from intersecting the right hand side of (3) with S_i and applying the projection formula to π_*, and $\psi|_Z = \psi \cdot Z$ is the degree of the ψ class restricted to the one-dimensional family Z. As a result, we read off

$$\pi_*(S_i^2) = -\frac{1}{2}\psi|_Z$$

for $i \neq 1, 2$.

By a similar calculation as above we obtain that

$$\pi_*(S_1^2) = \pi_*(S_2^2) = -\frac{1}{2}\psi|_Z - \frac{k}{2}.$$

Taking the square of (3), pushing forward and using the above calculation, we obtain that

$$\pi_*(\omega^2) = (3g-3)\psi|_Z + k.$$

Note that $E_l^2 = -2$, and hence the family Z intersects δ_i with multiplicity two at every point corresponding to a nodal fiber in Z. We thus have $Z \cdot \delta_i = 2k$. By adjusting i from 1 to $[g/2]$ in the setting of Z, we then conclude that as divisor class

$$\pi_*(\omega^2) = (3g-3)\psi + \frac{1}{2}\sum_{i=1}^{[g/2]} \delta_i$$

for an arbitrary family of canonical divisors.

Finally by Noether's formula, we have

$$\begin{aligned}\lambda &= \frac{\pi_*(\omega^2)+\delta}{12}\\ &= \frac{1}{12}\Big((3g-3)\psi + \frac{1}{2}\sum_{i=1}^{[g/2]}\delta_i + \sum_{i=0}^{[g/2]}\delta_i\Big)\\ &= \frac{g-1}{4}\psi + \frac{1}{12}\delta_0 + \frac{1}{8}\sum_{i=1}^{[g/2]}\delta_i.\end{aligned}$$

In other words, in $\mathrm{Pic}(\mathbb{P}\overline{\mathcal{H}})\otimes\mathbb{Q}$ we have

$$\lambda = \frac{g-1}{4}\psi + \frac{1}{12}\delta_0 + \frac{1}{8}\sum_{i=1}^{[g/2]}\delta_i + c\cdot[\mathbb{P}\overline{\mathcal{H}}(2,1^{2g-4})].$$

But we have seen that $c=-1/24$ in the proof of Proposition 3.1. Therefore, we thus conclude the desired divisor class. □

Remark 3.3. The divisor class of $\mathbb{P}\mathcal{H}(2,1^{2g-4})$ was first calculated by Korotkin and Zograf [KZ11, Theorem 2] using the Tau function. After finishing the paper the author also learnt from Zograf that another calculation of the divisor class was recently discovered by Zvonkine [Zvo]. Comparing Theorem 3.2 with [KZ11], we see that the corresponding coefficients are opposite to each other. This sign issue is due to different conventions when taking projectivization of a vector space V. In other words, here we consider $\mathbb{P}V$ parameterizing lines in V, while in [KZ11] it parameterizes hyperplanes. Consequently in our setting the divisor classes are the opposites to those in [KZ11].

Below we carry out a cross-check for Theorem 3.2 in the case $g=3$.

Example 3.4. A non-hyperelliptic genus three curve has canonical embedding as a quartic curve in $\mathbb{P}^2$. Take two general homogeneous polynomials F and G of degree four in three variables. For every point $[a,b]\in B\cong\mathbb{P}^1$, the vanishing locus $aF+bG$ is a plane quartic. In other words, B is a general one-dimensional family of plane quartics. Let L be a general line in $\mathbb{P}^2$. For every curve C parameterized in B, the intersection of L with C is a canonical divisor in C. Therefore, we can regard B as a one-dimensional family of canonical divisors in $\mathbb{P}\overline{\mathcal{H}}$.

We have the following intersection numbers:

$$B\cdot\lambda = 3,$$

$$B\cdot\delta_0 = 27,$$

$$B\cdot\delta_i = 0,\ i>0,$$

see [HM98, Chapter 3.F] for the relevant calculation. The universal canonical divisor over B has class $(1,4)$ in $B\times L\cong\mathbb{P}^1\times\mathbb{P}^1$. Projecting it to B induces a degree 4 covering map. If along L a simple ramification point occurs, it gives rise to a canonical divisor with a zero of multiplicity two. By the Riemann-Hurwitz formula, the number of ramification points of this projection map is equal to 6, hence we obtain that

$$B\cdot\mathbb{P}\overline{\mathcal{H}}(2,1,1) = 6.$$

The Hodge bundle $\overline{\mathcal{H}}$ restricted to B is isomorphic to $\mathcal{O}(1)^{\oplus 3}$. Hence its projectivization $\mathbb{P}\overline{\mathcal{H}}$ restricted to B is trivial, but the universal line bundle is isomorphic to $\mathcal{O}(2)$ (instead of $\mathcal{O}(1)$) due to the twist $\mathcal{O}(1)$ in the direct sum $\overline{\mathcal{H}}|_B$, see [Laz04, Appendix A]. Then we thus conclude that

$$B \cdot \psi = 2.$$

Using B as a test curve, one checks that these intersection numbers satisfy the relation in Theorem 3.2.

We say that a divisor class D in X is big if it lies in the interior of the cone of pseudo-effective divisors. There is another equivalent definition for a big divisor class. For a line bundle L, suppose it has n holomorphic sections $\sigma_1, \ldots, \sigma_n$. Then we have an induced map from X to $\mathbb{P}^{n-1}$ by sending x to $[\sigma_1(x), \ldots, \sigma_n(x)]$. This map is well-defined away from the base locus of L, i.e. the locus of points where all the sections vanish simultaneously. Moreover, its image might have smaller dimension compared to X. Under this setting, a divisor class D is big if there exists a positive integer m such that the map associated to the line bundle $\mathcal{O}_X(mD)$ is birational, i.e. its image has the same dimension as that of X, see [Laz04, Chapter 2.2] for more details.

Proposition 3.5. *The divisor class of $\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})$ lies on the boundary of the pseudo-effective cone of $\mathbb{P}\overline{\mathcal{H}}$.*

Proof. If $\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})$ is big, we can write it as $N + A$, where N is effective and A is ample. Consider Teichmüller curves $\mathcal{T}$ in $\mathbb{P}\mathcal{H}(1^{2g-2})$. By [CM12a, Proposition 3.1] we know $\mathcal{T}$ is disjoint with $\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})$, hence we have

$$\mathcal{T} \cdot (N + A) = 0.$$

Since A is ample, $\mathcal{T} \cdot A > 0$. Therefore, $\mathcal{T} \cdot N < 0$ and consequently N contains $\mathcal{T}$. However, the union of such $\mathcal{T}$ is Zariski dense in $\mathbb{P}\mathcal{H}$, see e.g. [Che10, Theorem 1.21], and thus cannot be contained in a divisor, leading to a contradiction. □

4. Extremal effective divisors on $\overline{\mathcal{M}}_{g,n}$

We say that an effective divisor class D in a projective variety X is extremal, if for any linear combination $D = D_1 + D_2$ with D_i pseudo-effective, D and D_i are proportional. In this case, we also say that D spans an extremal ray of the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$. Let us first present a method to test the extremality of an effective divisor. Recall that a divisor class is big if it lies in the interior of $\overline{\mathrm{Eff}}(X)$.

Lemma 4.1. *Suppose that D is an irreducible effective divisor and A a big divisor in X. Let S be a set of irreducible effective curves contained in D such that the union of these curves is Zariski dense in D. If for every curve C in S we have*

$$C \cdot (D + dA) \leq 0$$

for a fixed $d > 0$, then D is an extremal divisor.

Proof. Suppose that $D = D_1 + D_2$ with D_i pseudo-effective. If D_i and D are not proportional, we can assume that D_i lies in the boundary of $\overline{\mathrm{Eff}}(X)$ and moreover that $D_i - sD$ is not pseudo-effective for any $s > 0$, because otherwise we can replace D_1 and D_2 by the intersections of the linear span $\langle D_1, D_2 \rangle$ with the boundary of $\overline{\mathrm{Eff}}(X)$, possibly after rescaling.

By assumption, we have $C \cdot (D_1 + D_2) = C \cdot D < 0$. Therefore, without loss of generality we may assume that S has a subset S_1 whose elements C satisfy

$$C \cdot D_1 \leq \frac{1}{2} \cdot (C \cdot D)$$

and the union of C in S_1 is Zariski dense in D as well.

Consider the divisor class $F_n = nD_1 + A$ for n sufficiently large. Since D_1 is pseudo-effective and A is big, F_n lies in the interior of $\overline{\mathrm{Eff}}(X)$, and hence can be represented by an effective divisor. It is easy to check that for $k < \frac{n}{2} - \frac{1}{d}$, we have $C \cdot (F_n - kD) < 0$ for every C in S_1. Since the union of such curves C is Zariski dense in D, it implies that the multiplicity of D in the base locus of F_n is at least equal to $\frac{n}{2} - \frac{1}{d}$. Consequently the class

$$E_n = F_n - \Big(\frac{n}{2} - \frac{1}{d}\Big) D$$

is pseudo-effective. As n goes to infinity, the limit of the divisor classes $\{\frac{1}{n} E_n\}$ is equal to $D_1 - \frac{1}{2}D$, thus $D_1 - \frac{1}{2}D$ is also pseudo-effective. But this contradicts our assumption that $D_1 - sD$ is not pseudo-effective for any $s > 0$. □

In what follows we will apply Lemma 4.1 to Teichmüller curves contained in a stratum of abelian differentials. Since the union of Teichmüller curves contained in a stratum (component) $\mathcal{H}(\mu)$ is Zariski dense in $\mathcal{H}(\mu)$, if $\mathcal{H}(\mu)$ dominates an irreducible effective divisor D in $\overline{\mathcal{M}}_{g,n}$, then the union of the images of these Teichmüller curves is also Zariski dense in D.

In order to apply Lemma 4.1 to show the extremality of D, we need to understand the intersection of a Teichmüller curve with divisor classes on $\overline{\mathcal{M}}_{g,n}$. Luckily this has been worked out in [CM12a, Section 4]. For the reader's convenience, let us recall the relevant results.

Let C be (the closure of) a Teichmüller curve in the stratum $\mathcal{H}(\mu)$, where $\mu = (m_1, \ldots, m_n)$ is a partition of $2g - 2$. Let L be the sum of Lyapunov exponents of C and χ its orbifold Euler characteristic. Lift C to $\overline{\mathcal{M}}_{g,n}$ by choosing an order and marking the n zeros of its generating differential. Let Ω be the relative dualizing sheaf for the map $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$. Let ω_i be the first Chern class of $\pi^*\Omega$ associated to the map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,1}$ that forgets all but the ith marked point, see e.g. [Log03, Section 2]. Recall that

$$\kappa_\mu = \frac{1}{12} \sum_{i=1}^{n} \frac{m_i(m_i+2)}{m_i+1}.$$

By [CM12a, Proposition 4.8] we have

$$C \cdot \lambda = \frac{\chi}{2} \cdot L,$$

$$C \cdot \delta_0 = \frac{\chi}{2} \cdot (12L - 12\kappa_\mu),$$

$$C \cdot \omega_i = \frac{\chi}{2} \cdot \frac{1}{m_i+1}.$$

Moreover, we use δ_{other} to denote an arbitrary linear combination of boundary divisor classes of $\overline{\mathcal{M}}_{g,n}$ that does not contain δ_0. The purpose of doing this is because Teichmüller curves generated by abelian differentials do not intersect any boundary divisors except δ_0 [CM12a, Corollary 3.2]. Therefore, we can write a divisor class in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}$ as

$$D = a\lambda + \sum_{i=1}^{n} b_i \omega_i + c\delta_0 + \delta_{\mathrm{other}}.$$

By the above intersection numbers, we have

$$\frac{C \cdot D}{C \cdot \lambda} = a + \sum_{i=1}^{n} \frac{b_i}{(m_i+1)L} + c\Big(12 - \frac{12\kappa_\mu}{L}\Big).$$

Let $\underline{a} = (a_1, \ldots, a_n)$ be a sequence of positive integers such that $\sum_{i=1}^n a_i = g$. Consider the pointed Brill-Noether divisor $BN^1_{g,\underline{a}}$ in $\overline{\mathcal{M}}_{g,n}$ parameterizing $(X, p_1, \ldots, p_n)$ such that $h^0(X, \sum_{i=1}^n a_i p_i) \geq 2$. Its divisor class was first calculated in [Log03] as

$$[BN^1_{g,\underline{a}}] = -\lambda + \sum_{i=1}^{n} \frac{a_i(a_i+1)}{2}\omega_i - \delta_{\text{other}}.$$

We have a dominant map from the lift of $\mathcal{H}(\underline{a}, 1^{g-2})$ to $BN^1_{g,\underline{a}}$ by choosing an order and marking the first n zeros of an abelian differential in $\mathcal{H}(\underline{a}, 1^{g-2})$.

In order to apply Lemma 4.1, we need a big divisor class on $\overline{\mathcal{M}}_{g,n}$. Note that the restriction of λ to the interior of the moduli space $\mathcal{M}_g$ is ample. Therefore, the map associated to some multiple $m\lambda$ is a birational map of $\overline{\mathcal{M}}_g$, and we thus conclude that λ is big.

Lemma 4.2. *Let C be a Teichmüller curve in $\mathcal{H}(\underline{a}, 1^{g-2})$ mapping to $BN^1_{g,\underline{a}}$. Then $C \cdot BN^1_{g,\underline{a}} < 0$ if and only if $L > \frac{g}{2}$. Moreover, suppose that for every C contained in this stratum we have $L \geq \frac{g}{2} + \epsilon$ for a given $\epsilon > 0$, then there exists some $d > 0$, depending on ϵ and g only, such that*

$$C \cdot (BN^1_{g,\underline{a}} + d\lambda) \leq 0.$$

Proof. We have

$$\frac{C \cdot BN^1_{g,\underline{a}}}{C \cdot \lambda} = -1 + \sum_{i=1}^{n} \frac{a_i}{2L}.$$

The first part of the lemma follows from the assumption that $\sum_{i=1}^n a_i = g$. For the other part, take any $d \leq \frac{2\epsilon}{g+2\epsilon}$ and it is easy to check that it satisfies the desired inequality. □

For $n = 1$, $BN^1_{g,(g)}$ parameterizes a genus g curve with a marked Weierstrass point, hence we also use W to denote the divisor in this case. In what follows we list a number of extremal pointed Brill-Noether divisors on $\overline{\mathcal{M}}_{g,n}$ for small g. We also order the list according to the number of marked points n.

Theorem 4.3. *The following divisors are extremal.*

(i) *W in $\overline{\mathcal{M}}_{g,1}$ for $2 \leq g \leq 4$.*

(ii) *$BN^1_{2,(1,1)}$ in $\overline{\mathcal{M}}_{2,2}$, $BN^1_{3,(2,1)}$ in $\overline{\mathcal{M}}_{3,2}$, $BN^1_{4,(3,1)}$ and $BN^1_{4,(2,2)}$ in $\overline{\mathcal{M}}_{4,2}$.*

(iii) *$BN^1_{3,(1^3)}$ in $\overline{\mathcal{M}}_{3,3}$ and $BN^1_{4,(2,1,1)}$ in $\overline{\mathcal{M}}_{4,3}$.*

(iv) *$BN^1_{4,(1^4)}$ in $\overline{\mathcal{M}}_{4,4}$.*

(v) *$BN^1_{5,(1^5)}$ in $\overline{\mathcal{M}}_{5,5}$.*

Proof. Let C be a Teichmüller curve in $\mathcal{H}(g, 1^{g-2})$. For $g = 2$ and $g = 3$, by [CM12a, Corollary 4.3, Section 5.2] we know L is equal to $\frac{4}{3} > 1$ and $\frac{7}{4} > \frac{3}{2}$, respectively. Then combining Lemmas 4.2 and 4.1 we conclude that W is extremal in $\overline{\mathcal{M}}_{2,1}$ and $\overline{\mathcal{M}}_{3,1}$. For $g = 4$, despite that Teichmüller curves in $\mathcal{H}(4,1,1)$ have varying sums L of Lyapunov exponents, the limit of L is equal to the sum $L_{(4,1,1)}$ of Lyapunov exponents associated to the whole stratum [Che11,

Appendix A]. Based on the recursive algorithm in [EMZ03], we know $L_{(4,1,1)} = \frac{1137}{550} > 2$ [CM12a, Figure 3]. It implies that we can find infinitely many Teichmüller curves in $\mathcal{H}(4,1,1)$ such that the union of them is Zariski dense in that stratum and all of them have $L > 2+\epsilon$ for some fixed $\epsilon > 0$. Then the result follows by combining Lemmas 4.2 and 4.1. This thus completes the proof of (i).

The same argument applies to all the other cases without any change. So let us just list the non-varying values or the limits of L for the remaining cases. For $BN^1_{2,(1,1)}$ and $BN^1_{3,(2,1)}$, Teichmüller curves in $\mathcal{H}(1,1)$ and $\mathcal{H}(2,1,1)$ have $L = \frac{3}{2} > 1$ [CM12a, Corollary 4.3] and $L = \frac{11}{6} > \frac{3}{2}$ [CM12a, Section 5.4], respectively. For $BN^1_{4,(3,1)}$ and $BN^1_{4,(2,2)}$, the limits of L for Teichmüller curves in $\mathcal{H}(3,1^3)$ and in $\mathcal{H}(2,2,1,1)$ are equal to $\frac{66}{31} > 2$ and $\frac{5045}{2358} > 2$ [CM12a, Figure 3], respectively. For $BN^1_{3,(1^3)}$ and $BN^1_{4,(2,1,1)}$, the limits of L for Teichmüller curves in $\mathcal{H}(1^4)$ and in $\mathcal{H}(2,1^4)$ are equal to $\frac{53}{28} > \frac{3}{2}$ [CM12a, Figure 2] and $\frac{131}{60} > 2$ [CM12a, Figure 3], respectively. For $BN^1_{4,(1^4)}$ and $BN^1_{5,(1^5)}$, the limits of L for Teichmüller curves in $\mathcal{H}(1^6)$ and $\mathcal{H}(1^8)$ are equal to $\frac{839}{377} > 2$ [CM12a, Figure 3] and $\frac{235761}{93428} > \frac{5}{2}$ [CM12a, Figure 5], respectively. $\square$

Remark 4.4. The extremality of W was first showed by Rulla for $g = 2$ [Rul01] and by Jensen for $g = 3$ and $g = 5$ [Jenar, Jen12]. After the paper was written, the author learnt from Farkas that $BN^1_{g,(1^g)}$ is always extremal in $\overline{\mathcal{M}}_{g,g}$ [FVar]. These authors all took a different approach by using certain birational contraction of $\overline{\mathcal{M}}_{g,n}$ to conclude the extremality of the corresponding exceptional divisor. The question remains open to determine whether W as well as other pointed Brill-Noether divisors are extremal in higher genera.

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