COVERS OF THE PROJECTIVE LINE AND THE MODULI SPACE OF QUADRATIC DIFFERENTIALS

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ABSTRACT. Consider the Hurwitz space parameterizing covers of $\mathbb{P}^1$ branched at four points. We study its intersection with divisor classes on the moduli space of curves. As applications, we calculate the slope of Teichmüller curves parameterizing square-tiled cyclic covers. In addition, we come up with a relation among the slope of Teichmüller curves, the sum of Lyapunov exponents and the Siegel-Veech constant for the moduli space of quadratic differentials, which yields information for the effective cone of the moduli space of curves.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Let $\overline{\mathcal{M}}_g$ denote the Deligne-Mumford moduli space of stable genus g curves. The geometry of $\overline{\mathcal{M}}_g$ can be often revealed by studying one-dimensional families of genus g curves. A useful construction of such families arises from branched covers. Cornalba and Harris [CH] studied 1-dimensional families of hyperelliptic curves, which have very special numerical classes. Stankova [S] generalized the result to families of trigonal curves. Along a different direction, Harris and Morrison [HM1] studied one-dimensional families of degree $d \gg 0$, simply branched covers of $\mathbb{P}^1$, which provided lower bounds for the slope of effective divisors on $\overline{\mathcal{M}}_g$. In [C1], families of covers of elliptic curves were studied for the same purpose. In addition, one-dimensional families of covers of elliptic curves with a unique branch point, also called arithmetic Teichmüller curves, are among the central objects in the study of flat surfaces, polygon billiards and moduli spaces. In [C2], a relation between their slopes and the sum of Lyapunov exponents of the Hodge bundle parameterizing Abelian differentials was established.

In this paper, we consider covers of $\mathbb{P}^1$ branched at four points. Varying a branch point in $\mathbb{P}^1$, we obtain a one-dimensional Hurwitz space parameterizing such covers. Let us first summarize the main results.

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In Section 2, we give a criterion of distinguishing irreducible components of the Hurwitz space (Theorem 2.1). Then we analyze its intersection with divisor classes on $\overline{\mathcal{M}}_g$ and derive a slope formula (Theorem 2.2).

In Section 3, we apply the formula to the case of cyclic covers (Theorem 3.2). Interpreting the Hurwitz space as a Teichmüller curve, we calculate the sum of its Lyapunov exponents (Theorem 3.4). This sum has also been obtained by Forni, Matheus and Zorich [FMZ] using a different method based on the remarkable work of Eskin, Kontsevich and Zorich [EKZ1].

In Section 4, we exhibit a relation between the slope of Teichmüller curves, the Siegel-Veech constant and the sum of Lyapunov exponents for the moduli space of quadratic differentials (Theorem 4.7). This provides a new viewpoint for computing those quantities in dynamics by the intersection theory on moduli spaces. Aiming to understanding the effective cone of $\overline{\mathcal{M}}_g$, we discuss when $\overline{\mathcal{M}}_g$ is dominated by the moduli space of quadratic differentials (Lamma 4.2), as well as bounding the slope of effective divisors from below (Corollary 4.4). Understanding the effective cone of $\overline{\mathcal{M}}_g$ can yield a novel approach to the geometric Schottky problem about describing the locus of Jacobians among Abelian varieties (Corollary 4.8).

Throughout the paper, we work over the complex number field $\mathbb{C}$.

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2. Covers of $\mathbb{P}^1$ with four branch points

Let p_1, p_2, p_3, p_4 be four distinct points on $\mathbb{P}^1$ and c_1, c_2, c_3, c_4 four conjugacy classes of the permutation group S_d such that the ramification type over p_i corresponds to c_i. Fix the ramification profile

$$\mathbf{c} = (c_1, c_2, c_3, c_4).$$

Suppose c_i consists of k_i cycles, each of which is of length $a_{i,j}$ for $1 \leq j \leq k_i$, where $\sum_{j=1}^{k_i} a_{i,j} = d$ for $1 \leq i \leq 4$. The genus of degree d connected covers of $\mathbb{P}^1$ with the ramification profile $\mathbf{c}$ is determined by the Riemann-Hurwitz formula:

$$g = d + 1 - \frac{1}{2}\sum_{i=1}^{4} k_i.$$

Fix p_1, p_2, p_3 and vary p_4. We obtain a 1-dimensional Hurwitz space $\mathcal{H}_d(\mathbf{c})$ parameterizing such covers. When p_4 meets p_i, $1 \leq i \leq 3$, we get degenerate covers between nodal curves in the sense of admissible covers. Let $\overline{\mathcal{H}}_d(\mathbf{c})$ denote the compactification of $\mathcal{H}_d(\mathbf{c})$ parameterizing degree d, connected admissible covers with the ramification profile $\mathbf{c}$, cf. [HM2, 3.G]. The space $\overline{\mathcal{H}}_d(\mathbf{c})$ admits two natural morphisms:

$$\begin{array}{ccc} \overline{\mathcal{H}}_d(\mathbf{c}) & \xrightarrow{h} & \overline{\mathcal{M}}_g \\ \big\downarrow e & & \\ \overline{M}_{0,4} & & \end{array}$$

The morphism h sends a cover to the stabilization of its domain curve parameterized by $\overline{\mathcal{M}}_g$. The morphism e sends a cover to the target rational curve marked at the four branch points parameterized by $\overline{M}_{0,4} \cong \mathbb{P}^1$. Note that e restricted to $\mathcal{H}_d(\mathbf{c})$ is an unramified finite morphism of degree $N_d(\mathbf{c})$, where $N_d(\mathbf{c})$ is the Hurwitz number of non-isomorphic covers of $\mathbb{P}^1$ with four fixed branch points and the ramification profile $\mathbf{c}$.

Let $\pi_1(\mathbb{P}^1, p_1, p_2, p_3, p_4; b)$ be the fundamental group of $\mathbb{P}^1$ punctured at p_i's and b a base point. Let α_i be a closed oriented path circular around p_i such that $\alpha_1\alpha_2\alpha_3\alpha_4 = id$ in $\pi_1(\mathbb{P}^1, p_1, p_2, p_3, p_4; b)$, as shown in Figure 1.

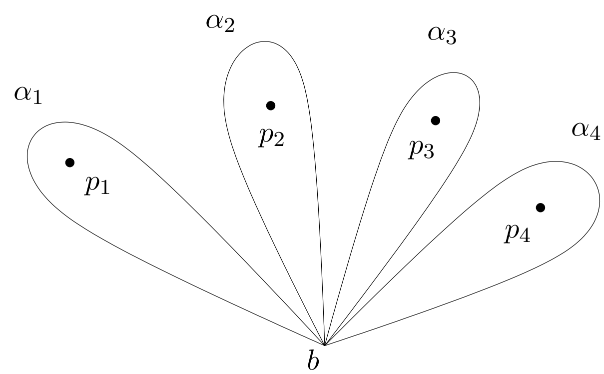

FIGURE 1.

Label the d sheets of a cover by $1, \ldots, d$. A cover parameterized by $\mathcal{H}_d(\mathbf{c})$ corresponds to an element in $\mathrm{Hom}(\pi_1(\mathbb{P}^1, p_1, p_2, p_3, p_4; b), S_d)$. Let γ_i denote the monodromy action associated to α_i in S_d. We call $\mathbf{r} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ the monodromy datum of the corresponding cover. Define the following set of equivalence classes

$$\mathrm{Cov}_d(\mathbf{c}) = \{\mathbf{r} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mid \gamma_i \in c_i,\ \gamma_1\gamma_2\gamma_3\gamma_4 = id,$$
$$\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle \text{ is a transitive subgroup of } S_d\}/\sim .$$

The equivalence relation $\sim$ is defined for two data $\mathbf{r} \sim \mathbf{r}'$ if there exists $\tau \in S_d$ such that

$$\tau(\gamma_1, \gamma_2, \gamma_3, \gamma_4)\tau^{-1} = (\gamma_1', \gamma_2', \gamma_3', \gamma_4').$$

Two covers corresponding to $\mathbf{r}$ and $\mathbf{r}'$ are isomorphic if and only if $\mathbf{r} \sim \mathbf{r}'$, where the conjugate action by τ amounts to relabeling the d sheets. The transitivity guarantees that the covers are connected. Hence, the set of non-isomorphic covers of $\mathbb{P}^1$ with four fixed branch points and the ramification profile $\mathbf{c}$ can be identified with $\mathrm{Cov}_d(\mathbf{c})$. Namely, a fiber of the finite morphism $e : \mathcal{H}_d(\mathbf{c}) \to M_{0,4} \cong \mathbb{P}^1 \backslash \{p_1, p_2, p_3\}$ is parameterized by $\mathrm{Cov}_d(\mathbf{c})$ and the degree of e equals $N_d(\mathbf{c}) = |\mathrm{Cov}_d(\mathbf{c})|$.

The Hurwitz space $\overline{\mathcal{H}}_d(\mathbf{c})$ may be reducible. Since e restricted to $\mathcal{H}_d(\mathbf{c})$ is unramified, the fundamental group $\pi_1(M_{0,4}; b')$ acts on $\mathrm{Cov}_d(\mathbf{c})$ and each orbit corresponds to an irreducible component of $\mathcal{H}_d(\mathbf{c})$. Let β_1 and β_2 denote two closed oriented paths around p_3 and p_2, p_3, respectively, as shown in Figure 2.

Let g_i be the action on $\mathrm{Cov}_d(\mathbf{c})$ induced by β_i for $i = 1, 2$.

Theorem 2.1. *Let $\mathbf{r} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be a representative of an equivalence class in $\mathrm{Cov}_d(\mathbf{c})$. We have*

$$g_1(\mathbf{r}) = (\gamma_1, \gamma_2, \gamma_4^{-1}\gamma_3\gamma_4, (\gamma_3\gamma_4)^{-1}\gamma_4(\gamma_3\gamma_4)),$$
$$g_2(\mathbf{r}) = (\gamma_1, \gamma_4^{-1}\gamma_2\gamma_4, \gamma_4^{-1}\gamma_3\gamma_4, (\gamma_2\gamma_3\gamma_4)^{-1}\gamma_4(\gamma_2\gamma_3\gamma_4)).$$

Two covers are parameterized in the same irreducible component of $\mathcal{H}_d(\mathbf{c})$ if and only if their monodromy data are in the same orbit under the actions generated by g_1, g_2.

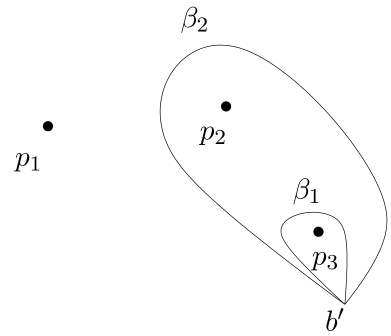

FIGURE 2.

Proof. Recall that α_i denotes a closed oriented path around p_i such that $\alpha_1\alpha_2\alpha_3\alpha_4 = id$ in $\pi_1(\mathbb{P}^1, p_1, p_2, p_3, p_4; b)$.

Vary p_4 along the path β_1 once. The resulting paths have the expression

$$(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4) = (\alpha_1, \alpha_2, \alpha_4^{-1}\alpha_3\alpha_4, (\alpha_3\alpha_4)^{-1}\alpha_4(\alpha_3\alpha_4))$$

by the original paths, hence the corresponding action g_1 on the monodromy datum $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ has the same expression, as shown in Figure 3.

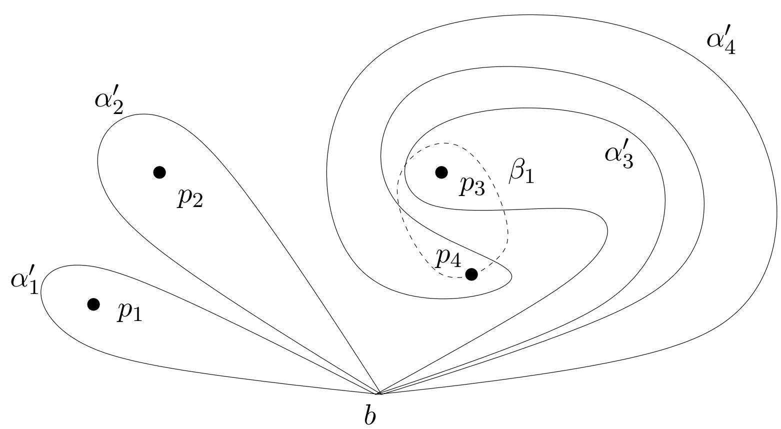

FIGURE 3.

Vary p_4 along the path β_2 once. The resulting paths have the expression

$$(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4) = (\alpha_1, \alpha_4^{-1}\alpha_2\alpha_4, \alpha_4^{-1}\alpha_3\alpha_4, (\alpha_2\alpha_3\alpha_4)^{-1}\alpha_4(\alpha_2\alpha_3\alpha_4))$$

by the original paths, hence the corresponding action g_2 on the monodromy datum $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ has the same expression, as shown in Figure 4.

Since $\mathcal{H}_d(\mathbf{c}) \to M_{0,4} \cong \mathbb{P}^1 \backslash \{p_1, p_2, p_3\}$ is unramified and $\pi_1(\mathbb{P}^1, p_1, p_2, p_3; b')$ is generated by β_1, β_2, the irreducible components of $\mathcal{H}_d(\mathbf{c})$ correspond to the orbits of $\mathrm{Cov}_d(\mathbf{c})$ under the actions generated by g_1, g_2. Note that these actions are well-defined respect to the equivalence relation $\sim$. □

Let $\mathcal{O} \subset \mathrm{Cov}_d(\mathbf{c})$ denote an orbit of the above actions. Let $Z_{\mathcal{O}}$ be the corresponding irreducible component of $\mathcal{H}_d(\mathbf{c})$ and $\overline{Z}_{\mathcal{O}}$ its closure in $\overline{\mathcal{H}}_d(\mathbf{c})$. The morphism h maps $\overline{Z}_{\mathcal{O}}$ to

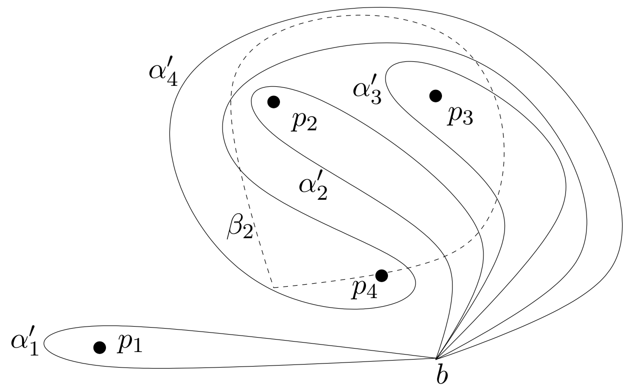

FIGURE 4.

an irreducible curve in $\overline{\mathcal{M}}_g$. Let λ denote the first Chern class of the Hodge bundle and δ be the total boundary class of $\overline{\mathcal{M}}_g$. Define the slope of $\overline{Z}_{\mathcal{O}}$ by

$$s(\overline{Z}_{\mathcal{O}}) = \frac{\deg\, h^*\delta|_{\overline{Z}_{\mathcal{O}}}}{\deg\, h^*\lambda|_{\overline{Z}_{\mathcal{O}}}}.$$

The slope is invariant under a finite base change.

In order to calculate $s(\overline{Z}_{\mathcal{O}})$, we need to analyze singular admissible covers that arise in $\overline{Z}_{\mathcal{O}}$. Suppose a smooth cover π corresponding to $\mathbf{r}$ degenerates to a singular cover π_0 when the moving branch point p_4 approaches p_3. The image of π_0 is a nodal union of two smooth rational curves $Q_{12} \cup_{p_0} Q_{34}$ with p_0, p_1, p_2 in Q_{12} and p_0, p_3, p_4 in Q_{34}. Locally there is a vanishing cycle α_0 that shrinks to the node p_0, as shown in Figure 5.

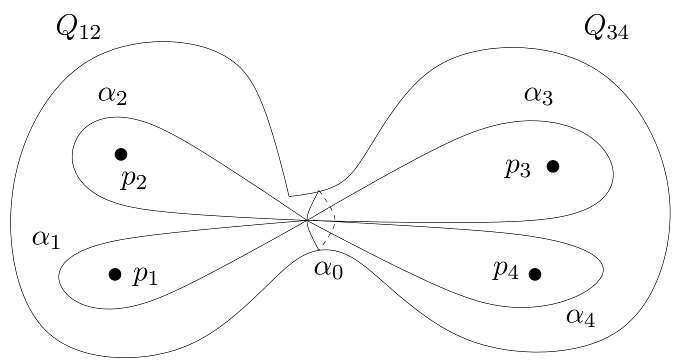

FIGURE 5.

Let γ_0 be the monodromy image of α_0 in S_d. We have $\gamma_0 = (\gamma_1\gamma_2)^{-1} = \gamma_3\gamma_4$. Suppose γ_0 consists of m cycles of length $a_1, \ldots, a_m$, respectively. Then the domain curve C_0 of π_0 has m nodes q_i for $1 \leq i \leq m$ and π_0 restricted to a neighborhood of q_i is given by $(x_i, y_i) \to (u, v) = (x_i^{a_i}, y_i^{a_i})$, where (x_i, y_i) and (u, v) are the local coordinates of the two branches of the nodes q_i and p_0, respectively. Let C_{12}, C_{34} be the two components of C_0 corresponding to the preimages of Q_{12}, Q_{34}, respectively. The restriction of π_0 to C_{12} is a degree d cover of Q_{12} branched at p_0, p_1, p_2 with the monodromy datum $(\gamma_0, \gamma_1, \gamma_2)$ as an element in $\mathrm{Hom}(\pi_1(Q_{12}, p_0, p_1, p_2; b), S_d)$. The connected components of C_{12} correspond to the orbits of $\{1,\ldots,d\}$ under the permutations generated by $\gamma_0,\gamma_1,\gamma_2$. The same analysis holds for π_0 restricted to C_{34}. In particular, the topological type of π_0 is uniquely determined by the monodromy datum $\mathbf{r}=(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ of the nearby smooth cover π.

Let C_0^{st} be the stabilization of C_0 by blowing down unstable rational components. Define $\delta(q_i)=1$ or 0, depending on whether or not the node q_i maps to a node of C_0^{st} via $C_0\to C_0^{\mathrm{st}}$. Assign $\mathbf{r}$ the following two weights:

$$\delta_3(\mathbf{r})=\sum_{i=1}^{m}\frac{1}{a_i}\delta(q_i) \text{ and } \delta_3'(\mathbf{r})=\sum_{i=1}^{m}\frac{1}{a_i}.$$

We similarly define $\delta_1(\mathbf{r}),\delta_1'(\mathbf{r})$ and $\delta_2(\mathbf{r}),\delta_2'(\mathbf{r})$ when p_4 approaches p_1 and p_2, respectively. Summing up the weights over all $\mathbf{r}$ in the orbit $\mathcal{O}$, we define

$$\delta_{\mathcal{O}}=\sum_{\mathbf{r}\in\mathcal{O}}\sum_{i=1}^{3}\delta_i(\mathbf{r}) \text{ and } \delta_{\mathcal{O}}'=\sum_{\mathbf{r}\in\mathcal{O}}\sum_{i=1}^{3}\delta_i'(\mathbf{r}).$$

Theorem 2.2. *The slope of $\overline{Z}_{\mathcal{O}}$ has the following expression:*

$$s(\overline{Z}_{\mathcal{O}})=\frac{12\delta_{\mathcal{O}}}{\delta_{\mathcal{O}}'+\Big(d-\sum_{i=1}^{4}\sum_{j=1}^{k_i}\frac{1}{a_{i,j}}\Big)|\mathcal{O}|},$$

where $|\mathcal{O}|$ denotes the cardinality of the orbit $\mathcal{O}$.

Proof. Recall how a nodal cover π_0 arises as the limit of a smooth cover π when the moving branch point p_4 approaches p_3. Suppose locally around a node of the domain curve C_0, π_0 is given by $(x,y)\to(u,v)=(x^a,y^a)$. If this node maps to a node of the stable curve C_0^{st}, it contributes $1/a$ to the degree of the boundary class δ restricted to $\overline{Z}_{\mathcal{O}}$. The weight $1/a$ comes from an orbifold viewpoint: for $s=xy$, $t=uv=x^ay^a=s^a$, locally a base change of degree a is needed to realize a smooth universal covering family. If this node maps to a smooth point of C_0^{st}, it does not contribute to the intersection with δ. Summing over all nodes of C_0, the limit π_0 of π contributes $\delta(\mathbf{r})$ to the restriction of δ. Summing over all π parameterized in the orbit $\mathcal{O}$ and also considering the case when p_4 approaches p_1 or p_2, we obtain by definition that $\deg h^*\delta|_{\overline{Z}_{\mathcal{O}}}=\delta_{\mathcal{O}}$.

Since the slope is invariant under a finite base change, assume there is a universal covering map (after the base change) as follows:

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow & & \downarrow \\ \overline{Z}_{\mathcal{O}} & \xrightarrow{e} & \overline{M}_{0,4} \end{array}$$

The surface S is a genus g fibration over $\overline{Z}_{\mathcal{O}}$ parameterizing the domain curves of the covers in $\overline{Z}_{\mathcal{O}}$. The surface T is isomorphic to the blow-up of $\mathbb{P}^1\times\mathbb{P}^1$ at the intersection points of three horizontal sections with the diagonal, where the sections and the diagonal parameterize the fixed branch points p_1,p_2,p_3 and the moving point p_4, respectively. Let Σ_i in T denote the proper transform of the three sections and the diagonal for $1\le i\le 4$. The horizontal morphisms e and f are finite of degree $|\mathcal{O}|$ and $d|\mathcal{O}|$, respectively. We have

$$f^*\Sigma_i = \sum_{j=1}^{k_i} a_{i,j}\Gamma_{i,j},$$

where $\Gamma_{i,j}$ are sections of S parameterizing points in the domain of the covers that map to p_i for $1 \leq i \leq 4$.

Let $\omega_{S/\overline{Z}_{\mathcal{O}}}$ and $\omega_{T/\overline{M}_{0,4}}$ be the relative dualizing sheaves of the two families S and T, respectively. By Riemann-Roch, we have

$$\omega_{S/\overline{Z}_{\mathcal{O}}} = f^*\omega_{T/\overline{M}_{0,4}} + \sum_{i=1}^{4}\sum_{j=1}^{k_i}(a_{i,j}-1)\Gamma_{i,j}.$$

Hence, we can compute the self-intersection of the divisor class

$$(\omega_{S/\overline{Z}_{\mathcal{O}}})^2 = (f^*\omega_{T/\overline{M}_{0,4}})^2 + 2\sum_{i=1}^{4}\sum_{j=1}^{k_i}(a_{i,j}-1)(f^*\omega_{T/\overline{M}_{0,4}}).\,\Gamma_{i,j} + \Big(\sum_{i=1}^{4}\sum_{j=1}^{k_i}(a_{i,j}-1)\Gamma_{i,j}\Big)^2.$$

Moreover, we have

$$(f^*\omega_{T/\overline{M}_{0,4}})^2 = d|\mathcal{O}|(\omega_{T/\overline{M}_{0,4}})^2 = -3d|\mathcal{O}|,$$

$$(f^*\omega_{T/\overline{M}_{0,4}}).\,\Gamma_{i,j} = \omega_{T/\overline{M}_{0,4}}.\,(f_*\Gamma_{i,j}) = |\mathcal{O}|(\omega_{T/\overline{M}_{0,4}}.\,\Sigma_i) = -|\mathcal{O}|(\Sigma_i)^2 = |\mathcal{O}|,$$

$$\Gamma_{i,j}.\,\Gamma_{i',j'} = 0 \text{ for } (i,j)\neq(i',j') \text{ and } (\Gamma_{i,j})^2 = \frac{|\mathcal{O}|}{a_{i,j}}(\Sigma_i)^2 = -\frac{|\mathcal{O}|}{a_{i,j}}.$$

Using the condition $\sum_{j=1}^{k_i} a_{i,j} = d$ for $1 \leq i \leq 4$, a routine calculation shows that

$$(\omega_{S/\overline{Z}_{\mathcal{O}}})^2 = \Big(d - \sum_{i=1}^{4}\sum_{j=1}^{k_i}\frac{1}{a_{i,j}}\Big)|\mathcal{O}|.$$

For the family S over $\overline{Z}_{\mathcal{O}}$, by Mumford's relation we have $12\lambda = \delta' + \kappa$ restricted to $\overline{Z}_{\mathcal{O}}$, where $\kappa = (\omega_{S/\overline{Z}_{\mathcal{O}}})^2$ and δ' enumerates the nodes of the fibers in S, which equals $\delta'_{\mathcal{O}}$ by definition. The weight $1/a_{i,j}$ in the definition of δ' is due to the need of a base change of degree $a_{i,j}$ to realize a smooth universal covering around such a node. Note that $\delta'_{\mathcal{O}}$ may be different from $\delta_{\mathcal{O}}$. If a node of the domain curve C_0 of a degenerate cover π_0 maps to a smooth point of the stabilization C_0^{st}, it does not contribute to the intersection with the boundary class δ. Nevertheless, on the surface S this node remains in the fiber C_0 and it is counted in δ'.

Overall, we obtain that

$$\deg\,\delta|_{\overline{Z}_{\mathcal{O}}} = \delta_{\mathcal{O}},$$

$$\deg\,\lambda|_{\overline{Z}_{\mathcal{O}}} = \frac{\delta'_{\mathcal{O}} + \Big(d - \sum_{i=1}^{4}\sum_{j=1}^{k_i}\frac{1}{a_{i,j}}\Big)|\mathcal{O}|}{12}.$$

The desired expression for the slope of $\overline{Z}_{\mathcal{O}}$ follows immediately. □

To illustrate how to apply Theorems 2.1 and 2.2, let us consider the following example.

Example 2.3. Consider the case when the length of the c_i's are all equal to d, where d is odd and $d \geq 3$. An orbit $\mathcal{O}_1$ of $\mathrm{Cov}_d(\mathbf{c})$ consists of a unique $\mathbf{r} = (\gamma, \gamma^{-1}, \gamma, \gamma^{-1})$ (up to equivalence) under the actions generated by g_1, g_2 in Theorem 2.1, where γ is any cycle of length d in S_d. Hence, we have the cardinality $|\mathcal{O}_1| = 1$. The genus g of the covers equals $d - 1$.

When p_4 approaches p_3, since $\gamma_3\gamma_4 = id$ consists of d cycles of length 1, the degenerate covering curve has d nodes. Its two components C_{12} and C_{34} are both rational and they meet at d nodes whose local branches map to p_0 as $(x, y) \to (x, y)$. There is no unstable rational component, hence each node contributes 1 and $\delta_3(\mathbf{r}) = \delta'_3(\mathbf{r}) = d$. The same conclusion holds when p_4 approaches p_1, so $\delta_1(\mathbf{r}) = \delta'_1(\mathbf{r}) = d$. When p_4 approaches p_2, since d is odd, $\gamma_2\gamma_4 = \gamma^2$ consists of a single cycle of length d. The degenerate covering curve has a single node whose local branches map to p_0 as $(x, y) \to (x^d, y^d)$. Its two components C_{12} and C_{34} both have genus $(d - 1)/2$ and there is no unstable rational component. Hence, the unique node contributes $1/d$ and $\delta_2(\mathbf{r}) = \delta'_2(\mathbf{r}) = 1/d$.

Since $|\mathcal{O}_1| = 1$, we have

$$\delta_{\mathcal{O}_1} = \delta'_{\mathcal{O}_1} = d + d + \frac{1}{d} = \frac{2d^2 + 1}{d}.$$

By Theorem 2.2, we obtain that

$$s(\overline{Z}_{\mathcal{O}_1}) = \frac{8d^2 + 4}{d^2 - 1} = 8 + \frac{12}{g^2 + 2g}.$$

Now assume that d is coprime with 6. Another orbit $\mathcal{O}_2$ of $\mathrm{Cov}_d(\mathbf{c})$ consists of a unique monodromy datum $\mathbf{r} = (\gamma, \gamma, \gamma, \gamma^{-3})$ (up to equivalence) under the actions generated by g_1, g_2 in Theorem 2.1, where γ is any cycle of length d in S_d. Hence, we have the cardinality $|\mathcal{O}_2| = 1$. The genus g of the covers equals $d - 1$. By a similar analysis as above, we have $\delta_1(\mathbf{r}) = \delta_2(\mathbf{r}) = \delta_3(\mathbf{r}) = 1/d$. Since $|\mathcal{O}_2| = 1$, we have $\delta_{\mathcal{O}_2} = \delta'_{\mathcal{O}_2} = 3/d$. By Theorem 2.2, we obtain that

$$s(\overline{Z}_{\mathcal{O}_2}) = \frac{36}{d^2 - 1} = \frac{36}{g^2 + 2g}.$$

Note that $s(\overline{Z}_{\mathcal{O}_1})$ converges to 8 while $s(\overline{Z}_{\mathcal{O}_2})$ converges to 0 for $g \gg 0$.

The covers in Example 2.3 are special cases of cyclic covers of $\mathbb{P}^1$. In Section 3, we will come up with a general formula to calculate the slope of Hurwitz spaces of cyclic covers.

Let us consider an example of non-cyclic covers.

Example 2.4. Let $d = 4$ and $\mathbf{r} = ((1234), (1432), (123), (132))$. The notation $(a_1 \cdots a_k)$ stands for a permutation in S_d that sends a_i to a_{i+1}. The orbit $\mathcal{O}$ generated by $\mathbf{r}$ under the action in Theorem 2.1 consists of six elements (up to equivalence):

$$\mathbf{r_1} = ((1234), (1432), (123), (132)); \ \mathbf{r_2} = ((1234), (1432), (132), (123));$$

$$\mathbf{r_3} = ((1234), (1324), (123), (243)); \ \mathbf{r_4} = ((1234), (1324), (134), (123));$$

$$\mathbf{r_5} = ((1234), (1324), (243), (134)); \ \mathbf{r_6} = ((1234), (1234), (123), (124)).$$

By Riemann-Hurwitz, the genus of these covers equals 2. Hence, we get an irreducible family $\overline{Z}_{\mathcal{O}}$ of degree 4, genus 2 covers of $\mathbb{P}^1$ with two ramification points of order 3 and two ramification points of order 2. Since the cardinality $|\mathcal{O}|$ equals 6, $\overline{Z}_{\mathcal{O}} \to \overline{M}_{0,4}$ is a finite morphism of degree 6.

When p_4 meets p_3, the degenerate admissible cover arising nearby $\mathbf{r_1}$ is shown in Figure 6. The covering curve consists of a $\mathbb{P}^1$ meeting another $\mathbb{P}^1$ at three nodes and meeting a third $\mathbb{P}^1$

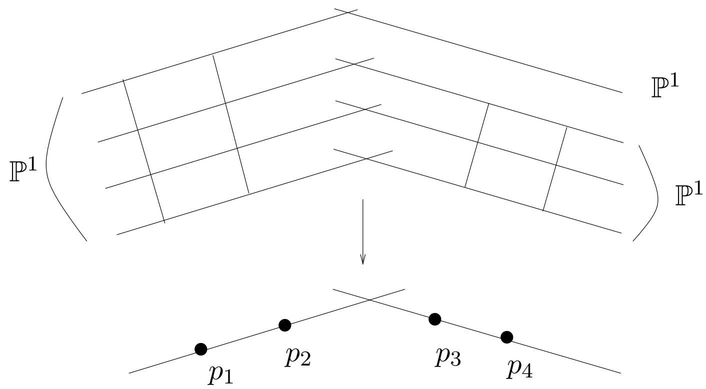

Figure 6.

at one node. Its stabilization consists of the first two $\mathbb{P}^1$ components meeting at three nodes. Around any node of the cover, the map is a local isomorphism, so each node has weight 1. The four nodes in total contribute 4 to δ' by definition. The node on top of the figure belongs to the semistable $\mathbb{P}^1$ component that maps to a smooth point of the stabilization, so it does not contribute to δ by definition. The other three nodes in total contribute 3 to δ. In the same way, one can analyze the degenerate admissible cover nearby $\mathbf{r_i}$ when p_4 meets p_j and count the contributions to δ' and δ, for $i = 1, \ldots, 6$ and $j = 1, 2, 3$. We skip the enumeration and just write down the result: $\delta'_{\mathcal{O}} = 25$ and $\delta_{\mathcal{O}} = 23$. By Theorem 2.2, we obtain that

$$s(\overline{Z}_{\mathcal{O}}) = \frac{12 \cdot 23}{25 + 6\Big(4 - \frac{1}{4} - \frac{1}{4} - \big(1 + \frac{1}{3}\big) - \big(1 + \frac{1}{3}\big)\Big)} = \frac{46}{5}.$$

3. Square-tiled cyclic covers

In this section, we fix the monodromy datum $\mathbf{r} = (\gamma^{a_1}, \gamma^{a_2}, \gamma^{a_3}, \gamma^{a_4})$, where γ is a cycle of length d in S_d, $\sum_{i=1}^4 a_i \equiv 0 \pmod d$, $\gcd(a_1, a_2, a_3, a_4, d) = 1$ and $1 \leq a_i \leq d - 1$. Under the actions g_1, g_2 in Theorem 2.1, it is easy to see that all the monodromy data in the orbit $\mathcal{O}$ generated by $\mathbf{r}$ are equivalent up to the S_d conjugate actions. Hence, the morphism $\overline{Z}_{\mathcal{O}} \to \overline{M}_{0,4}$ is one to one.

We can concretely write down such a cover corresponding to $\mathbf{r} = (\gamma^{a_1}, \gamma^{a_2}, \gamma^{a_3}, \gamma^{a_4})$ as

$$y^d = (x - z_1)^{a_1}(x - z_2)^{a_2}(x - z_3)^{a_3}(x - z_4)^{a_4},$$

where z_1, z_2, z_3, z_4 are four distinct points in $\mathbb{P}^1$. The covering map is induced by $(x, y) \to x$. Given $\sum_{i=1}^4 a_i \equiv 0 \pmod d$ and $\gcd(a_1, a_2, a_3, a_4, d) = 1$, it is a cyclic cover with Galois group $\mathbb{Z}/d$ branched at four points z_1, z_2, z_3, z_4. Since $\gcd(a_1, a_2, a_3, a_4, d) = 1$, the group $\langle \gamma^{a_1}, \gamma^{a_2}, \gamma^{a_3}, \gamma^{a_4} \rangle$ is the same as $\langle \gamma \rangle$, which acts transitively on $\{1, \ldots, d\}$. Hence, the covering curve is connected.

Now let z_4 be the moving point in $\mathbb{P}^1$. In order to apply Theorem 2.2, we need to analyze the degenerate covers when z_4 meets z_1, z_2 or z_3. Set up some notation as follows:

$$d_i = \gcd(a_i, d), \ 1 \leq i \leq 4,$$

$$s_{ij} = \gcd(a_i + a_j, d),\ 1 \leq i < j \leq 4,$$

$$t_{ij} = \gcd(a_i, a_j, d),\ 1 \leq i < j \leq 4.$$

Note that by the assumption $\sum_{i=1}^{4} a_i \equiv 0 \pmod d$, we have $s_{ij} = s_{kl}$ for any $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Consider the degenerate cover when z_4 meets z_3. The other two cases follow by changing the indices accordingly. Let $\pi_0 : C_0 \to Q_{12} \cup_{p_0} Q_{34}$ be the degenerate admissible cover when z_4 meets z_3, where $Q_{12} \cup_{p_0} Q_{34}$ is the nodal union of two $\mathbb{P}^1$'s at the node p_0. As we discussed in Section 2, C_0 consists of two components C_{12} and C_{34} such that π_0 restricted to C_{12} and C_{34} are degree d covers of Q_{12} and Q_{34}, respectively. Furthermore, $C_{12} \to Q_{12}$ is branched at z_1, z_2, p_0 with the monodromy datum $(\gamma^{a_1}, \gamma^{a_2}, \gamma^{-(a_1+a_2)})$ and $C_{34} \to Q_{34}$ is branched at z_3, z_4, p_0 with the monodromy datum $(\gamma^{a_3}, \gamma^{a_4}, \gamma^{-(a_3+a_4)})$. In particular, $\gamma^{-(a_1+a_2)}$ consists of s_{12} cycles of length d/s_{12}. Therefore, C_{12} and C_{34} intersect at s_{12} nodes. Locally around each node, π_0 is given by $(x, y) \to (u, v) = (x^{d/s_{12}}, y^{d/s_{12}})$. By definition, we have

$$\delta_3'(\mathbf{r}) = s_{12} \cdot \frac{1}{d/s_{12}} = \frac{s_{12}^2}{d}.$$

For the value of $\delta_3(\mathbf{r})$, it depends on whether or not a node belongs to a component of C_0 that can be contracted to a smooth point of the stabilization C_0^{st}. Denote by R a rational tail of C_0 if R is a smooth rational component that meets $\overline{C_0 \backslash R}$ at one point.

Lemma 3.1. *The covering curve C_0 does not have any rational tail.*

Proof. Note that the group $\langle \gamma^{a_1}, \gamma^{a_2}, \gamma^{-(a_1+a_2)} \rangle$ equals $\langle \gamma^{t_{12}} \rangle$. Its action on $\{1, \ldots, d\}$ has t_{12} orbits and each orbit contains d/t_{12} elements. It implies that C_{12} has t_{12} connected components and π_0 restricted to each component is a degree d/t_{12} cover of Q_{12} with the monodromy datum $(\eta^{a_1/t_{12}}, \eta^{a_2/t_{12}}, \eta^{-(a_1+a_2)/t_{12}})$ over the three branch points z_1, z_2, p_0, where η is a single cycle of length d/t_{12}.

By $\gcd(a_1/t_{12}, a_2/t_{12}, d/t_{12}) = 1$, the monodromy datum $(\eta^{a_1/t_{12}}, \eta^{a_2/t_{12}}, \eta^{-(a_1+a_2)/t_{12}})$ yields a connected cover R of degree d/t_{12} to $Q_{12} \cong \mathbb{P}^1$. If R is a rational tail of C_0, then R meets C_{34} at a unique node. It implies that $\eta^{-(a_1+a_2)/t_{12}}$ consists of a single cycle of length d/t_{12}, namely, $(a_1 + a_2)/t_{12}$ and d/t_{12} are coprime. Moreover, $\gcd(a_i/t_{12}, d/t_{12}) = d_i/t_{12}$ for $i = 1, 2$, hence $\eta^{a_i/t_{12}}$ consists of d_i/t_{12} cycles of length d/d_i. By the Riemann-Hurwitz formula, the genus of R satisfies

$$2g_R - 2 + 2d/t_{12} = (d/t_{12} - 1) + (d/t_{12} - d_1/t_{12}) + (d/t_{12} - d_2/t_{12}),$$

$$2g_R = \frac{d - d_1 - d_2}{t_{12}} + 1.$$

Since $g_R = 0$, we get $d < d_1 + d_2$. But $d_i = \gcd(a_i, d) \leq d/2$ for $i = 1, 2$, a contradiction. □

Since C_0 does not have any rational tail, its nodes map to the nodes of C_0^{st}. By definition, we have

$$\delta_3(\mathbf{r}) = \delta_3'(\mathbf{r}) = \frac{s_{12}^2}{d}.$$

Theorem 3.2. *The Hurwitz component $\overline{Z}_{\mathcal{O}}$ of cyclic covers of $\mathbb{P}^1$ with four branch points has slope*

$$s(\overline{Z}_{\mathcal{O}}) = \frac{12(s_{12}^2 + s_{23}^2 + s_{13}^2)}{s_{12}^2 + s_{23}^2 + s_{13}^2 + d^2 - \sum_{i=1}^{4} d_i^2},$$

where $s_{ij} = \gcd(a_i + a_j, d)$ and $d_i = \gcd(a_i, d)$.

Proof. We have shown that $\delta_3(\mathbf{r}) = \delta_3'(\mathbf{r}) = s_{12}^2/d$. Similarly, we have $\delta_1(\mathbf{r}) = \delta_1'(\mathbf{r}) = s_{23}^2/d$ and $\delta_2(\mathbf{r}) = \delta_2'(\mathbf{r}) = s_{13}^2/d$. Since the orbit $\mathcal{O}$ generated by $\mathbf{r}$ under the actions in Theorem 2.1 contains a unique equivalence class up to the S_d conjugate action, by definition we have $|\mathcal{O}| = 1$ and

$$\delta_{\mathcal{O}} = \delta_{\mathcal{O}}' = \frac{s_{12}^2 + s_{23}^2 + s_{13}^2}{d}.$$

Moreover, γ^{a_i} consists of d_i cycles of length d/d_i for $1 \leq i \leq 4$. By Theorem 2.2 we thus obtain the desired formula for $s(\overline{Z}_{\mathcal{O}})$. □

Let us revisit the two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of cyclic covers in Example 2.3. For $\mathcal{O}_1$, it corresponds to the case when $a_1 = a_3 = 1$, $a_2 = a_4 = d-1$ and d is odd. Then we have $d_i = 1$ for $1 \leq i \leq 4$, $s_{12} = s_{23} = d$ and $s_{13} = 1$. By Theorem 3.2, we have

$$s(\overline{Z}_{\mathcal{O}_1}) = \frac{8d^2+4}{d^2-1}.$$

For $\mathcal{O}_2$, it corresponds to the case when $a_1 = a_2 = a_3 = 1$, $a_4 = d-3$ and d is coprime with 6. Then we have $d_i = 1$ for $1 \leq i \leq 4$ and $s_{ij} = 1$ for $1 \leq i < j \leq 3$. By Theorem 3.2, we have

$$s(\overline{Z}_{\mathcal{O}_2}) = \frac{36}{d^2-1}.$$

These results coincide with our previous analysis in Example 2.3.

Remark 3.3. In [C1], the author considered covers of an elliptic curve with a unique branch point. For fixed g, there exist such covers of arbitrarily large degree and the limit of slopes of corresponding Hurwitz spaces can provide a lower bound for the slope of effective divisors on $\overline{\mathcal{M}}_g$. Nevertheless, the degree is bounded for genus g cyclic covers of $\mathbb{P}^1$ with four branch points. In fact, by the Riemann-Hurwitz formula we know

$$g = d + 1 - \frac{1}{2}\sum_{i=1}^{4} \gcd(a_i, d)$$

for a cyclic cover given by $y^d = (x-z_1)^{a_1}(x-z_2)^{a_2}(x-z_3)^{a_3}(x-z_4)^{a_4}$. Since $1 \leq a_i \leq d-1$ for $1 \leq i \leq 4$ and $\gcd(a_1, a_2, a_3, a_4, d) = 1$, we have

$$\sum_{i=1}^{4} \gcd(a_i, d) \leq \frac{d}{2} + \frac{d}{2} + \frac{d}{2} + \frac{d}{3} = \frac{11}{6}d.$$

Hence, $g \geq d + 1 - \frac{11}{12}d$ and $d \leq 12(g-1)$.

For an effective divisor D in $\overline{\mathcal{M}}_g$ with divisor class $a\lambda - b\delta$, define its slope

$$s(D) = \frac{a}{b},$$

where $a, b > 0$. If an irreducible curve C in $\overline{\mathcal{M}}_g$ has slope $s(C) > s(D)$, then it implies $C.D < 0$ and C is contained in D. By this argument, our calculation of $s(\overline{Z}_{\mathcal{O}})$ can induce various applications to the geometry of covering curves in the Hurwitz space $\overline{Z}_{\mathcal{O}}$. For instance, the divisor Θ on $\overline{\mathcal{M}}_g$ parameterizing curves with an even theta-characteristic η such that $h^0(\eta) > 0$ has slope $8 + 1/2^{g-3}$ [F]. The Hurwitz space $\overline{Z}_{\mathcal{O}_1}$ in Example 2.3 has slope

$$s(\overline{Z}_{\mathcal{O}_1}) = \frac{8d^2+4}{d^2-1} = 8 + \frac{12}{g^2+2g}$$

for odd d and even $g = d-1$. When $g \geq 6$, we have $s(\overline{Z}_{\mathcal{O}_1}) > s(\Theta)$, hence a curve in $\overline{Z}_{\mathcal{O}_1}$ admits an even theta-characteristic η with $h^0(\eta) > 0$.

Consider a cyclic cover $\pi\colon C \to \mathbb{P}^1$ defined by

$$y^d = (x-z_1)^{a_1}(x-z_2)^{a_2}(x-z_3)^{a_3}(x-z_4)^{a_4}.$$

The quadratic differential

$$q_0 = \frac{(dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

defines a flat metric on $\mathbb{P}^1$. For a suitable choice of z_1, z_2, z_3, z_4, the sphere can be realized as a pillow case by gluing two copies of a unit square along their boundary. Consequently C can be tiled by $2d$ unit squares, i.e. it is a square-tiled (Riemann) surface, cf. [FMZ] for a detailed description.

Regard the pair $(C, \pi^*(dz))$ as a point in the moduli space of quadratic differentials $\mathcal{Q}$. There is an $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{Q}$ by acting on the real and imaginary parts of quadratic differentials. Note that the orbit generated by $(C, \pi^*(dz))$ is the component $Z_{\mathcal{O}}$ of the 1-dimensional Hurwitz space containing the cover π (up to a finite base change, depending on whether the branch points are ordered), since the $\mathrm{SL}(2,\mathbb{R})$ action on $(C, \pi^*(dz))$ amounts to varying the cross-ratio of z_1, z_2, z_3, z_4. Thus $\overline{Z}_{\mathcal{O}}$ projects to $\overline{\mathcal{M}}_g$ as a closed algebraic curve, which is (the closure of) a Teichmüller curve by definition. In general, it is well-known that the $\mathrm{SL}(2,\mathbb{R})$-orbit of a square-tiled surface yields a so-called arithmetic Teichmüller curve.

Let L denote the sum of the nonnegative Lyapunov exponents of the Hodge bundle over a Teichmüller curve with respect to the Teichmüller geodesic flow. Roughly speaking, the Lyapunov exponents measure the growth rate of the length of a vector in the bundle under parallel transport along the flow, cf. [K, §5] for an introduction to Lyapunov exponents and also Section 4 for more details. For a Teichmüller curve $\overline{Z}_{\mathcal{O}}$ parameterizing cyclic covers of $\mathbb{P}^1$ with four branch points, our analysis for the slope of $\overline{Z}_{\mathcal{O}}$ can deduce the value of L.

Theorem 3.4. *The sum of Lyapunov exponents of $\overline{Z}_{\mathcal{O}}$ is equal to*

$$L = \frac{d}{6} - \frac{1}{6d}\sum_{i=1}^{4}\gcd(d,a_i)^2 + \frac{1}{6d}\Big(\gcd(d,a_1+a_2)^2 + \gcd(d,a_1+a_3)^2 + \gcd(d,a_2+a_3)^2\Big).$$

Proof. By the formula [K, §7] and [BM, Theorem 8.2] for the sum of Lyapunov exponents, we have

$$L = \frac{2(\deg \lambda|_{\overline{Z}_{\mathcal{O}}})}{2g(\overline{Z}_{\mathcal{O}}) - 2 + s},$$

where $g(\overline{Z}_{\mathcal{O}})$ is the genus of $\overline{Z}_{\mathcal{O}}$ and s is the number of singular covers. The Hurwitz component $\overline{Z}_{\mathcal{O}}$ is isomorphic to $\mathbb{P}^1$ and it contains three singular covers, as the moving branch point z_4 approaches one of the fixed branch points z_1, z_2, z_3. Therefore, we have

$$L = 2(\deg \lambda|_{\overline{Z}_{\mathcal{O}}}).$$

The degree of λ has been calculated in the proof of Theorem 2.2 in a more general setting, and worked out explicitly in the proof of Theorem 3.2 for cyclic covers. Using these data, we thus obtain the desired formula for L. □

Remark 3.5. The sum L was also calculated in [FMZ] via a different method. Individual Lyapunov exponents of $\overline{Z}_{\mathcal{O}}$ were further calculated in [EKZ2].

4. Moduli space of quadratic differentials

We first introduce the moduli space of quadratic differentials, cf. [EO2] and [L] for more details.

Let $\mu = (2m_1, \ldots, 2m_k)$ and $\nu = (2n_1 - 1, \ldots, 2n_l - 1)$ be two partitions, where $m_i, n_j \geq 1$ and l is even. Let $\mathcal{Q}(\mu, \nu)$ denote the moduli space of quadratic differentials parameterizing pairs (C, ψ), where ψ is a quadratic differential of a smooth curve C, the divisor

$$(\psi)_0 = \sum_{i=1}^{k} 2m_i p_i + \sum_{j=1}^{l} (2n_j - 1) q_j$$

and ψ is not a global square of an Abelian differential. The genus of C satisfies

$$4g - 4 = \sum_{i=1}^{k} 2m_i + \sum_{j=1}^{l} (2n_j - 1).$$

One can associate (C, ψ) a canonical double cover $\pi : \tilde{C} \to C$ such that $\pi^*\psi = \omega^2$, where ω is an Abelian differential of $\tilde{C}$, cf. [L, Construction 1]. The double cover π is branched exactly at the zeros $q_1, \ldots, q_l$ of odd order. Let $\pi^{-1}(p_i) = \{p'_i, p''_i\}$ and $\pi^{-1}(q_j) = q'_j$. One checks that $(\omega)_0 = \sum m_i(p'_i + p''_i) + \sum 2n_j q'_j$. Namely, $(\tilde{C}, \omega)$ is parameterized by the moduli space of Abelian differentials whose zeros are of type $(m_i, m_i, 2n_j)$. The genus of $\tilde{C}$ satisfies

$$2\tilde{g} - 2 = 2(2g - 2) + l.$$

Let $\sigma : \tilde{C} \to \tilde{C}$ be the involution such that $C \cong \tilde{C}/\sigma$. Since ψ is not a global square, we have $\sigma^*\omega = -\omega$. The relative homology group $H_1(\tilde{C}, p'_i, p''_i, q'_j; \mathbb{Z})$ splits into a direct sum of two eigenspaces $H_+ \oplus H_-$ under the involution σ, where $H_+ = \pi^* H_1(C, p_i, q_j; \mathbb{Z})$. For any $\gamma \in H_+$, we have $\int_\gamma \omega = 0$. Then H_- has dimension equal to

$$n = (2\tilde{g} + 2k + l - 1) - (2g + k + l - 1) = 2g - 2 + k + l.$$

Let $\gamma_1, \ldots, \gamma_n$ be a basis of H_-. The period map

$$\Phi : (C, \psi) \to \Big(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_n} \omega \Big) \in \mathbb{C}^n$$

yields a local coordinate system for $\mathcal{Q}(\mu, \nu)$, cf. [V] and [K, §3].

Let $\mathbb{T} = \mathbb{C}/\mathbb{Z}^2$ be the standard torus. Consider the double cover $\rho : \mathbb{T} \to \mathbb{P}^1 \cong \mathbb{T}/\pm$. In this way, $\mathbb{P}^1$ can be regarded as a pillow case with four branch points at

$$z_1 = 0, \; z_2 = \frac{1}{2}, \; z_3 = \frac{\sqrt{-1}}{2}, \; z_4 = \frac{1 + \sqrt{-1}}{2}.$$

The quadratic differential $q_0 = (dz)^2/\prod_{i=1}^4(z - z_i)$ on $\mathbb{P}^1$ has a simple pole at each z_i. Its pullback $\rho^* q_0$ is a global square $(dz)^2$ (up to a scalar).

For a branched cover f, if $f^{-1}(q) = k_1 p_1 + \cdots + k_r p_r$ for distinct points $p_1, \cdots, p_r$, we say that the preimages of q are of type $(k_1, \ldots, k_r)$. Define the covering set

$$\mathrm{Cov}(\mu, \nu) = \{f : C \to \mathbb{P}^1\}$$

parameterizing connected covers of even degree such that f is branched at z_1 whose preimages are of type $(2n_1 + 1, \ldots, 2n_l + 1, 2, \ldots, 2)$, branched at z_i for $2 \leq i \leq 4$ whose preimages are of type $(2, \ldots, 2)$ and branched at k other fixed points whose preimages are of type $(m_i + 1, 1, \ldots, 1)$ for $1 \leq i \leq k$. Here we do not fix the degree of the branched covers. For any $f \in \mathrm{Cov}(\mu, \nu)$, one checks that $(C, f^* q_0) \in \mathcal{Q}(\mu, \nu)$. Moreover, by the construction of canonical double covers [EKZ1, Lemma 2.1], there exists a finite cover $\tilde{f} : \tilde{C} \to \mathbb{T}$ that completes the following commutative diagram:

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\pi} & C \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{T} & \xrightarrow{\rho} & \mathbb{P}^1 \end{array}$$

We take an explicit basis $\gamma_1, \ldots, \gamma_n$ of H_- as follows. Let $\gamma_1, \ldots, \gamma_{2g-2+l}$ be a basis of $H_1(\tilde{C}; \mathbb{Z})/\pi^* H_1(C; \mathbb{Z})$. Let $\gamma_{2g-2+l+i} = \overline{p_i' q_1'} - \sigma(\overline{p_i' q_1'}) = \overline{p_i' q_1'} - \overline{p_i'' q_1'}$ for $1 \leq i \leq k$, where $\overline{pq}$ denotes a path connecting p and q, as shown in Figure 7.

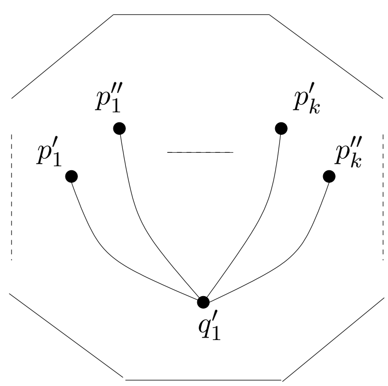

FIGURE 7.

The following result implies that covers in $\mathrm{Cov}(\mu, \nu)$ correspond to lattice points in $\mathcal{Q}(\mu, \nu)$ under the period map coordinates. It was implicitly used in [EO2, §1.2.4] to evaluating the volume of $\mathcal{Q}(\mu, \nu)$, whose analogue for Abelian differentials was proved in [EO1, Lemma 3.1].

Lemma 4.1. *For $(C, \psi) \in \mathcal{Q}(\mu, \nu)$, consider its coordinates $\Phi(C, \psi) = (\phi_1, \ldots, \phi_n) \in \mathbb{C}^n$, where $\phi_{i_1} \neq \phi_{i_2}, z_j \mod \mathbb{Z}^2, \pm$ for any $i_1, i_2 > 2g - 2 + l$ and $1 \leq j \leq 4$. Then we have $\phi_i \in \mathbb{Z}^2$ for $1 \leq i \leq 2g - 2 + l$ if and only if the following holds:*
(1) there exists a branched cover $f : C \to \mathbb{P}^1 = \mathbb{T}/\pm$ and $\psi = f^* q_0$;
(2) $f(q_j) = z_1 = 0$ for $1 \leq j \leq l$ and $f(p_i) = \phi_{2g-2+l+i} \mod \mathbb{Z}^2, \pm$ for $1 \leq i \leq k$;
(3) locally around p_i and q_j, f is given by $z \to z^{m_i+1}$ and $z \to z^{2n_j+1}$, respectively;

(4) f is branched at z_1 whose preimages are of type $(2n_1+1,\dots,2n_l+1,2,\dots,2)$, branched at z_j whose preimages are of type $(2,\dots,2)$ for $2\le j\le 4$ and branched at $\phi_{2g-2+l+i}$ whose preimages are of type $(m_i+1,1,\dots,1)$ for $1\le i\le k$.

Note that branched covers satisfying the above conditions are parameterized by $\mathrm{Cov}(\mu,\nu)$.

Proof. If such a map f exists, by the above diagram, we know $\pi^*\psi=\omega^2$ and $\omega=\tilde{f}^*dz$. Then we have

$$\int_{\gamma_i}\omega=\int_{\gamma_i}\tilde{f}^*dz=\int_{\tilde{f}_*\gamma_i}dz\in\mathbb{Z}^2$$

for $1\le i\le 2g-2+l$, since $\tilde{f}_*\gamma_i$ is closed in $\mathbb{T}$.

For the other direction, define

$$f(z)=\tilde{f}(z')=\int_{z'}^{\sigma(z')}\omega,$$

where $z\in C$ and $z'\in\pi^{-1}(z)$. Since $\phi_i\in\mathbb{Z}^2$ for $1\le i\le 2g-2+l$, modulo $\mathbb{Z}^2$, $f(z)$ does not depend on the integration path. Modulo $\pm$, $f(z)$ is independent of the choices of $z'\in\pi^{-1}(z)$. Therefore, we obtain an induced map $f:C\to\mathbb{P}^1\cong\mathbb{T}/\pm$ which satisfies all the desired conditions. The assumption that $\phi_{i_1}\ne\phi_{i_2},z_j \mod \mathbb{Z}^2,\pm$ for any $i_1,i_2>2g-2+l$ and $1\le j\le 4$ guarantees that $f(p_1),\dots,f(p_k)$ are k distinct points away from z_j. □

Up to a scalar, $\mathcal{Q}(\mu,\nu)/\mathbb{C}^*$ has dimension equal to $2g-3+k+l$ and it naturally maps to $\mathcal{M}_g$. When dim $\mathcal{Q}(\mu,\nu)/\mathbb{C}^*\ge$ dim $\mathcal{M}_g$, we prove that $\mathcal{Q}(\mu,\nu)/\mathbb{C}^*\to\mathcal{M}_g$ is dominant.

Lemma 4.2. *For $k+l\ge g$, a general genus g curve admits a quadratic differential parameterized by $\mathcal{Q}(\mu,\nu)$.*

Proof. There is a stratification among all $\mathcal{Q}(\mu,\nu)$'s by refining the partitions, hence it suffices to verify the case when $k+l=g$. We apply the De Jonquières' Formula [ACGH, VIII §5]. Let $a_1,\dots,a_m$ be distinct integers such that a_i appears h_i times in the partition (μ,ν) of $4g-4$. We have

$$\sum_{i=1}^m h_i=k+l=g,$$
$$\sum_{i=1}^m h_ia_i=4g-4.$$

Define

$$R(t)=1+\sum_{i=1}^m a_i^2t_i.$$

On a general genus g curve, the virtual number of quadratic differentials (up to a scalar) that have h_i zeros of order a_i is given by

$$[R(t)^g]_{t_1^{h_1}\cdots t_m^{h_m}},$$

the coefficient of $t_1^{h_1}\cdots t_m^{h_m}$ in $R(t)^g$. Note that $R(t)$ has positive coefficients, hence the virtual number is also positive, which implies the existence of such quadratic differentials (possibly infinitely many if the curve is special). □

Corollary 4.3. *For $k+l\ge g$, the union of genus g curves that admit covers in* $\mathrm{Cov}(\mu,\nu)$ *is a Zariski dense subset of $\mathcal{M}_g$.*

Proof. By Lemma 4.1 covers in $\mathrm{Cov}(\mu,\nu)$ correspond to lattice points under the period map coordinates, hence their union is a Zariski dense subset of $\mathcal{Q}(\mu,\nu)$. Now the conclusion follows from Lemma 4.2. $\square$

Let $\mu=\emptyset$ and $\nu=(2n_1-1,\ldots,2n_l-1)$ a partition of $4g-4$. Let c_1 be the conjugacy class $(2n_1+1,\ldots,2n_l+1,2,\ldots,2)$ and c_2,c_3,c_4 the same conjugacy class $(2,\ldots,2)$ of S_d, where d is even and each entry of c_i corresponds to the length of a cycle contained in c_i. Let $\mathbf{c}_\nu=(c_1,c_2,c_3,c_4)$ denote the ramification profile. By the notation in Section 2, the set $\mathrm{Cov}_d(\mathbf{c}_\nu)$ parameterizes degree d, genus g connected covers of $\mathbb{P}^1$ with four fixed branch points and the ramification profile $\mathbf{c}_\nu$. Let $\mathcal{O}\subset\mathrm{Cov}_d(\mathbf{c}_\nu)$ denote an orbit of the action in Theorem 2.1 and $Z_{\mathcal{O}}$ the corresponding irreducible component of the Hurwitz space $\mathcal{H}_d(\mathbf{c}_\nu)$.

Corollary 4.4. *For $l\geq g$, the union over all d of irreducible components of $\mathcal{H}_d(\mathbf{c}_\nu)$ maps to a Zariski dense subset of $\mathcal{M}_g$. Let s_ν be the limit (if it exists) of their slopes as d approaches infinity. For an effective divisor D on $\overline{\mathcal{M}}_g$, it has slope $s(D)\geq s_\nu$.*

Proof. For $l\geq g$, the density result follows from Corollary 4.3. If D is an effective divisor on $\overline{\mathcal{M}}_g$, there exist infinitely many Hurwitz components $Z_{\mathcal{O}}$ not entirely contained in D. We have the intersection number $D.\overline{Z}_{\mathcal{O}}\geq 0$, hence $s(D)\geq s(\overline{Z}_{\mathcal{O}})$ by the definition of slopes. Taking the limit as d approaches infinity, we obtain that $s(D)\geq s_\nu$. The existence of s_ν is verified in Theorem 4.7 using a relation between slopes and Siegel-Veech constants. $\square$

In order to evaluate s_ν, we first simplify the slope formula for components of $\mathcal{H}_d(\mathbf{c}_\nu)$.

Lemma 4.5. *Let $\pi_0\colon C_0\to Q_{12}\cup_{z_0}Q_{34}$ be a degenerate cover parameterized by $\overline{\mathcal{H}}_d(\mathbf{c}_\nu)$, where Q_{12} and Q_{34} are two $\mathbb{P}^1$'s containing the branch points z_1,z_2 and z_3,z_4, respectively, and they are joined at a node z_0. Then C_0 does not have a rational tail.*

Proof. Suppose C' is a rational tail of C_0. Let $z'=C'\cap\overline{C_0\backslash C'}$ and d' the degree of π_0 restricted to C'. Then z' is the unique preimage of the node z_0 in C'.

If C' maps to Q_{34}, it is branched at z_1,z_2 whose preimages are of type $(2,\ldots,2)$ and branched at z_0 whose preimage is of type (d'). By the Riemann-Hurwitz formula, we have

$$2d'-2=(d'-1)+\frac{d'}{2}+\frac{d'}{2},$$

which is impossible.

If C' maps to Q_{12}, it is branched at z_2, z_0 and z_1 whose preimages are of type $(2,\ldots,2)$, (d') and $(2n_{a_1}+1,\ldots,2n_{a_i}+1,2,\ldots,2)$, respectively. By Riemann-Hurwitz, we have

$$2d'-2=(d'-1)+\frac{d'}{2}+\sum_{j=1}^{i}2n_{a_j}+\frac{d'-\sum_{j=1}^{i}(2n_{a_j}+1)}{2}.$$

After simplifying the expression, we have

$$\sum_{j=1}^{i}(2n_{a_j}-1)=-2.$$

Since $n_i\geq 1$ in the partition $\nu=(2n_1-1,\ldots,2n_l-1)$, the last equality is impossible. $\square$

Corollary 4.6. *Let $d_i = 2n_i - 1$ for the partition $\nu = (2n_1 - 1, \ldots, 2n_l - 1)$. Then the slope of a component $\overline{Z}_{\mathcal{O}}$ of $\overline{\mathcal{H}}_d(\mathbf{c}_\nu)$ equals*

$$s(\overline{Z}_{\mathcal{O}}) = \frac{12}{1 + \frac{1}{4}\Big(\sum_{i=1}^{l} \frac{d_i(d_i+4)}{d_i+2}\Big)\frac{|\mathcal{O}|}{\delta_{\mathcal{O}}}}.$$

Proof. By Lemma 4.5, a node of a degenerate cover in $\overline{\mathcal{H}}_d(\mathbf{c}_\nu)$ maps to a node of its stabilization. By definition we have $\delta_{\mathcal{O}} = \delta'_{\mathcal{O}}$ for any component $\overline{Z}_{\mathcal{O}}$ of $\overline{\mathcal{H}}_d(\mathbf{c}_\nu)$. Plugging the ramification profile $\mathbf{c}_\nu$ in Theorem 2.2, we obtain the desired expression for $s(\overline{Z}_{\mathcal{O}})$. □

For the canonical double cover $\tilde{C} \to C$, the cohomology $H^1(\tilde{C};\mathbb{R})$ splits into the direct sum $H^1_+(\tilde{C};\mathbb{R}) \oplus H^1_-(\tilde{C};\mathbb{R})$ of the invariant and anti-invariant subspaces under the action σ^*, where σ is the involution of the double cover. We know that $H^1_+(\tilde{C};\mathbb{R}) \cong \sigma^* H^1(C;\mathbb{R})$ has dimension $2g$. Let H^1_+ be the invariant subbundle of the Hodge bundle on the moduli space $\mathcal{Q}(\nu)$ whose fiber over (C,ψ) is naturally identified with $H^1_+(\tilde{C};\mathbb{R})$. Let $\lambda_1^+ \geq \ldots \geq \lambda_g^+$ be the nonnegative Lyapunov exponents of H^1_+ with respect to the Teichmüller geodesic flow on $\mathcal{Q}(\nu)$. Denote their sum by

$$L_\nu = \lambda_1^+ + \ldots + \lambda_g^+.$$

The reader may refer to [EKZ1, §2] for a detailed introduction to the sum of Lyapunov exponents and its properties.

For the partition $\nu = (d_1, \ldots, d_l)$, where $d_i = 2n_i - 1$, define

$$\kappa_\nu = \frac{1}{24}\sum_{i=1}^{l} \frac{d_i(d_i+4)}{d_i+2}.$$

Let c_ν be the area Siegel-Veech constant of the stratum $\mathcal{Q}(\nu)$ satisfying $c_\nu = \frac{\pi^2}{3} c_{\text{area}}(\mathcal{Q}(\nu))$ in the context of [EKZ1, §1.5]. It measures the average number of horizontal cylinders with weight "height/length" on a random flat surface parameterized by $\mathcal{Q}(\nu)$. By [EKZ1, Theorem 2], we know that

$$L_\nu = \kappa_\nu + c_\nu.$$

Let $N_{d,\nu} = |\mathrm{Cov}_d(\mathbf{c}_\nu)|$ denote the cardinality of the covering set $\mathrm{Cov}_d(\mathbf{c}_\nu)$. Then $N_{d,\nu}$ is equal to the degree of the finite morphism $\overline{\mathcal{H}}_d(\mathbf{c}_\nu) \to \overline{M}_{0,4}$. Let $\delta_{d,\nu}$ be the sum of $\delta_{\mathcal{O}}$ ranging over all the orbits $\mathcal{O}$ of $\mathrm{Cov}_d(\mathbf{c}_\nu)$. Recall that the covers in $\mathrm{Cov}(\nu)$ correspond to lattice points of $\mathcal{Q}(\nu)$. During the degeneration of a smooth cover, a horizontal cylinder of height 1 and length k on its square-tiled surface model shrinks to a node of weight $1/k$. By the same argument as in [C2, Appendix A], we have

$$c_\nu = \lim_{d\to\infty} \frac{\delta_{d,\nu}}{6N_{d,\nu}}.$$

The coefficient 6 on the bottom arises by the following reason. First, there are three directions of degeneration for a cover in $\mathrm{Cov}(\nu)$. To get $\delta_{d,\nu}$, we enumerate all the weighted nodes appearing in the degeneration. But the Siegel-Veech constant only counts those arising in the horizontal direction. Hence, we need to correct the relation by a factor 1/3. In addition, the orbifold $\mathbb{P}^1$ as a pillow case is tiled by a white square and a black one. If a node of a degenerate cover is formed by shrinking a horizontal cylinder of height 1 and length k in our setting, it gives rise to a horizontal cylinder consisting of k pairs of squares with height 1 and total length $2k$ in the context of [EKZ1, Theorem 4] and [FMZ]. Hence we need another factor $1/2$.

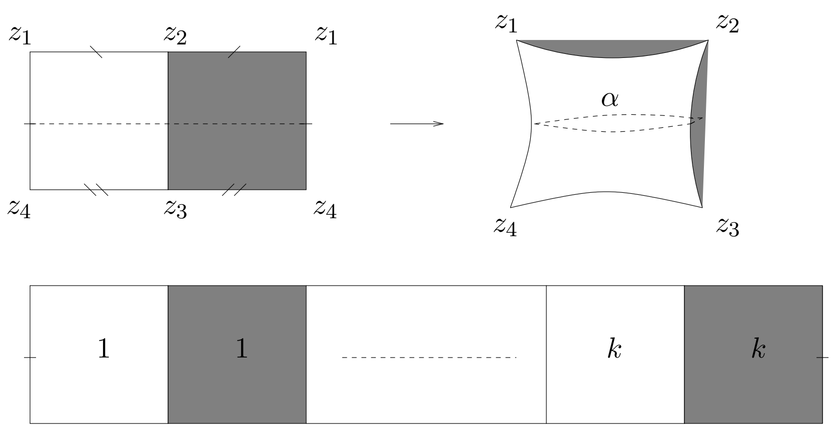

FIGURE 8.

Figure 8 shows $\mathbb{P}^1$ as a pillow case, along with a horizontal cylinder consisting of k pairs of white and black squares on a covering square-tiled surface. The horizontal vanishing cycle α shrinks to a node that separates z_1, z_2 on one rational component and z_3, z_4 on the other in the degeneration process. The horizontal cylinder gives rise to a node of weight $1/k$ for enumerating δ. The reader may refer to [FMZ] for a detailed illustration on square-tiled surfaces of this type.

Combining the above results, we derive a relation among the limit of slopes, the Siegel-Veech constant and the sum of Lyapunov exponents as follows.

Theorem 4.7. *For a partition $\nu = (d_1, \ldots, d_l)$ of $4g-4$ into odd parts, let s_ν be the limit of $s(\overline{\mathcal{H}}_d(\mathbf{c}_\nu))$ as d approaches infinity. Then we have*

$$s_\nu = \frac{12c_\nu}{L_\nu}.$$

Proof. Applying Corollary 4.6 to $\overline{\mathcal{H}}_d(\mathbf{c}_\nu)$, we have

$$s(\overline{\mathcal{H}}_d(\mathbf{c}_\nu)) = \frac{12}{1 + \kappa_\nu \cdot \dfrac{6N_{d,\nu}}{\delta_{d,\nu}}}.$$

Since $c_\nu = \lim\limits_{d\to\infty} \frac{\delta_{d,\nu}}{6N_{d,\nu}}$ and $L_\nu = \kappa_\nu + c_\nu$, we obtain that

$$s_\nu = \lim_{d\to\infty} s(\overline{\mathcal{H}}_d(\mathbf{c}_\nu)) = \frac{12}{1 + \dfrac{\kappa_\nu}{c_\nu}} = \frac{12c_\nu}{L_\nu}.$$

□

For any (non-hyperelliptic) $\mathrm{SL}(2,\mathbb{R})$ invariant submanifolds $\tilde{\mathcal{M}}$ in the moduli space of Abelian differentials, Eskin and Zorich obtained strong numerical evidence, which predicts that their Siegel-Veech constants $c(\tilde{\mathcal{M}})$ approach 2 as g tends to infinity. Let $\mathcal{M}$ be an

SL$(2,\mathbb{R})$ submanifold in the moduli space of quadratic differentials. Via the canonical double cover construction, we can lift $\mathcal{M}$ to $\tilde{\mathcal{M}}$ in the corresponding moduli space of Abelian differentials. Their Siegel-Veech constants satisfy the relation [EKZ1, Lemma 1.1]:

$$c(\tilde{\mathcal{M}}) = 2c(\mathcal{M}).$$

Therefore, we expect that

$$\lim_{g\to\infty} c_\nu = 1.$$

Combining this with Corollary 4.4 and Theorem 4.7, a lower bound for slopes of effective divisors on $\overline{\mathcal{M}}_g$ would be arbitrarily close to

$$s = \frac{12}{1+\kappa_\nu} = \frac{288}{24 + \sum_{i=1}^{l} \frac{d_i(d_i+4)}{d_i+2}}$$

for $g \gg 0$, where $(d_1,\ldots,d_l)$ is a partition of $4g-4$ into odd parts and $l \geq g$. The partition $(1,\ldots,1,3g-3)$ for $l = g$ maximizes the bound as

$$s \sim \frac{432}{7g}$$

for $g \gg 0$.

Understanding the effective cone of $\overline{\mathcal{M}}_g$ is related to the (geometric) Schottky problem about describing the locus of Jacobians in the moduli space $\mathcal{A}_g$ of principally polarized Abelian varieties. By sending a curve to its Jacobian polarized by the theta divisor, $\mathcal{M}_g$ can be embedded into $\mathcal{A}_g$ via the Torelli map:

$$\tau : \mathcal{M}_g \hookrightarrow \mathcal{A}_g.$$

Note that τ can be extended as a rational map $\tau : \overline{\mathcal{M}}_g \dashrightarrow \mathcal{A}_g^*$ which is a morphism in codimension one, where $\mathcal{A}_g^*$ is a partial compactification of $\mathcal{A}_g$ [G, Definition 4.2]. The rational Picard group of $\mathcal{A}_g^*$ is generated by the first Chern class of the Hodge bundle λ and the total boundary δ satisfying $\tau^*(\lambda) = \lambda$ and $\tau^*(\delta) = \delta$. For a Siegel modular form of class $a\lambda - b\delta$ on $\mathcal{A}_g^*$, define its slope similarly by a/b. If it has slope smaller than the lower bound of slopes of effective divisors on $\overline{\mathcal{M}}_g$, then it automatically vanishes on the image of $\mathcal{M}_g$ in $\mathcal{A}_g$. Otherwise it would cut out an effective divisor on $\overline{\mathcal{M}}_g$ whose slope is smaller than the lower bound, a contradiction. Therefore, a good understanding of the cone of effective divisors on $\overline{\mathcal{M}}_g$ would provide a novel approach to the Schottky problem. Combining the above discussion, we obtain a corollary as follows.

Corollary 4.8. *Assuming that $\lim_{g\to\infty} c_\nu = 1$, a Siegel modular form of slope smaller than $\sim 432/7g$ is vanishing on the locus of Jacobians in $\mathcal{A}_g$ for $g \gg 0$.*

It remains to be a very interesting question if a lower bound of type $O(1/g)$ could be asymptotically sharp for the Schottky problem.

Another advantage of Theorem 4.7 is that it is usually easier to compute slopes in the first place, and then to derive information about the Siegel-Veech constants and Lyapunov exponents based on their relation. For instance, the sum of Lyapunov exponents is non-varying for all Teichmüller curves in some low genus strata of Abelian differentials [CM1], hence equal to that of the stratum. The idea is to first show that slopes of those Teichmüller curves are non-varying, by exhibiting a geometric divisor that does not intersect Teichmüller curves in that stratum. We plan to study this question for Teichmüller curves generated by quadratic differentials in a forthcoming paper [CM2].

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