SQUARE-TILED SURFACES AND RIGID CURVES ON MODULI SPACES

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Abstract. We study the algebro-geometric aspects of Teichmüller curves parameterizing square-tiled surfaces with two applications.

(a) There exist infinitely many rigid curves on the moduli space of hyperelliptic curves. They span the same extremal ray of the cone of moving curves. Their union is a Zariski dense subset. Hence they yield infinitely many rigid curves with the same properties on the moduli space of stable n-pointed rational curves for even n.

(b) The limit of slopes of Teichmüller curves and the sum of Lyapunov exponents for the Teichmüller geodesic flow determine each other, which yields information about the cone of effective divisors on the moduli space of curves.

Contents

1. Introduction

Let $\mu = (m_1, \ldots, m_k)$ be a partition of $2g-2$ for $g \geq 2$. The moduli space $\mathcal{H}(\mu)$ of Abelian differentials parameterizes pairs (C, ω), where C is a smooth complex curve of genus g and ω is a holomorphic 1-form whose divisor $(\omega) = m_1p_1 + \cdots + m_kp_k$ for distinct points $p_1, \ldots, p_k$ on C. The space $\mathcal{H}(\mu)$ is a complex orbifold of dimension $2g-1+k$ and the period map yields its local coordinates [K, §3]. It may have up to three connected components [KZ, §2.3], corresponding to hyperelliptic, odd or even spin structures.

Consider a degree d connected cover $\pi : C \to E$ from a genus g curve C to the standard torus E with a unique branch point q, such that $\pi^{-1}(q) = (m_1+1)p_1 + \cdots + (m_k+1)p_k + p_{k+1} + \cdots + p_l$. Then C admits a holomorphic 1-form $\omega = \pi^{-1}(dz)$ whose divisor $(\omega) = m_1p_1 + \cdots + m_kp_k$. It is known [EO, Lemma 3.1] that such a pair (C, ω) has integer coordinates under the period map. Varying the complex structure of E, we obtain a Teichmüller curve $\mathcal{T}_{d,\mu}$ in $\mathcal{H}(\mu)$ generated by (C, ω), which is invariant under the natural $\mathrm{SL}(2, \mathbb{R})$ action on $\mathcal{H}(\mu)$. One can regard $\mathcal{T}_{d,\mu}$ as the 1-dimensional Hurwitz space parameterizing degree d, genus g connected covers of elliptic curves with a unique branch point q and the ramification profile μ. Use $\overline{\mathcal{T}}_{d,\mu}$ to denote the compactification of $\mathcal{T}_{d,\mu}$ in the sense of admissible covers [HM2, 3.G]. The boundary points of $\overline{\mathcal{T}}_{d,\mu}$ parameterize admissible covers of rational nodal curves. We call them cusps of $\overline{\mathcal{T}}_{d,\mu}$. Note that $\mathcal{T}_{d,\mu}$ may be reducible. There is a monodromy criterion [C, Theorem 1.18] to distinguish its irreducible components, which correspond to the orbits of the $\mathrm{SL}(2, \mathbb{Z})$ action. Let $n_{d,\mu}$ be the number of irreducible components of $\mathcal{T}_{d,\mu}$ and label these components as $\mathcal{T}_{d,\mu,i}$ for $1 \leq i \leq n_{d,\mu}$. If $\mathcal{T}_{d,\mu,i}$ is contained in the hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(\mu)$, we denote it by $\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i}$.

Our motivation is to use $\mathcal{T}_{d,\mu}$ to study the birational geometry of moduli spaces of stable pointed rational curves and stable curves of genus g. Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli stack of stable n-pointed curves of genus g. The space $\overline{\mathcal{M}}_{g,n}$ has a natural S_n action by reordering the marked points. When $g = 0$, let $f : \overline{M}_{0,n} \to \widetilde{M}_{0,n}$ be the finite quotient morphism to the space $\widetilde{M}_{0,n}$ of stable rational curves with n unordered marked points. There are two natural morphisms as follows:

$$\begin{tikzcd} \mathcal{T}_{d,\mu} \arrow[r, "h"] \arrow[d, "e"] & \overline{\mathcal{M}}_g \\ \overline{\mathcal{M}}_{1,1} & \end{tikzcd}$$

The map h sends a branched cover to the stable limit of its domain curve. The map e sends a cover to its target elliptic curve marked at the unique branch point. Moreover, e is finite of degree $N_{d,\mu}$, where $N_{d,\mu}$ is the number of non-isomorphic such covers of a fixed elliptic curve. Let $N_{d,\mu,i}$ denote the degree of e restricted to the component $\mathcal{T}_{d,\mu,i}$ for $1 \leq i \leq n_{d,\mu}$.

Let $\overline{H}_g \subset \overline{\mathcal{M}}_g$ denote the closure of locus of hyperelliptic curves. One can regard $\overline{H}_g$ as the Hurwitz space parameterizing genus g admissible double covers of rational curves. Such a cover uniquely corresponds to a stable $(2g+2)$-pointed rational curve, by marking the branch points of the cover. Thus $\overline{H}_g$ can be further identified with $\widetilde{M}_{0,2g+2}$. The moduli spaces $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$ both have hyperelliptic components. For Teichmüller curves $\mathcal{T}^{\text{hyp}}_{d,\mu,i}$ in these hyperelliptic components, the map h sends $\overline{\mathcal{T}}^{\text{hyp}}_{d,\mu,i}$ to $\overline{H}_g \cong \widetilde{M}_{0,2g+2}$.

Theorem 1.1 (Density)**.** *For either $\mu = (2g-2)$ or $\mu = (g-1, g-1)$, the union over all d and i of $h(\overline{\mathcal{T}}^{\text{hyp}}_{d,\mu,i})$ is a Zariski dense subset of $\widetilde{M}_{0,2g+2}$ and its preimage in $\overline{M}_{0,2g+2}$ is also Zariski dense.*

For a projective variety X, let $\overline{\text{NE}}_1(X)$ denote its Mori cone of effective curves. By a vital curve, we mean an irreducible component of the 1-dimensional locus in $\overline{M}_{0,n}$ (resp. $\widetilde{M}_{0,n}$) parameterizing pointed rational curves with at least $n-3$ components. Fulton conjectured that $\overline{\text{NE}}_1(\overline{M}_{0,n})$ (resp. $\overline{\text{NE}}_1(\widetilde{M}_{0,n})$) is generated by vital curves. Fulton's conjecture has been verified for $n \leq 7$ for $\overline{M}_{0,n}$ [KM] and $n \leq 24$ for $\widetilde{M}_{0,n}$ [Gi].

We call a map $f : C \to X$ from an irreducible curve C to a variety X rigid if there does not exist a non-isotrivial family f_t of maps to X such that $f = f_0$. If there is no confusion about the map, we also call C rigid on X. Suppose the class of an effective curve R generates an extremal ray of $\overline{\text{NE}}_1(\overline{M}_{0,n})$. Keel and McKernan [KM] proved that if $R \cap M_{0,n} \neq \emptyset$, then any finite morphism $f : C \to R$ must be rigid, cf. [CT, §8] for a precise statement. Hence, it is natural to study rigid curves on $\overline{M}_{0,n}$ intersecting its interior, as candidate counterexamples to Fulton's conjecture. Few rigid curves are known on $\overline{M}_{0,n}$. Castravet and Tevelev studied exceptional loci on $\overline{M}_{0,n}$ using hypergraph curves and found a rigid curve on $\overline{M}_{0,12}$ [CT, Theorem 7.8]. Tevelev also informed the author that Kollár came up with a series of potentially rigid curves (unpublished), whose construction relies on rigid line configurations in $\mathbb{P}^2$.

Theorem 1.2 (Rigidity)**.** *For $g \geq 2$, $\mu = (2g-2)$ or $\mu = (g-1, g-1)$, $\overline{\mathcal{T}}^{\text{hyp}}_{d,\mu,i}$ is rigid on $\widetilde{M}_{0,2g+2}$. If $g \geq 3$, infinitely many of the pullbacks of $\overline{\mathcal{T}}^{\text{hyp}}_{d,\mu,i}$ via $f : \overline{M}_{0,2g+2} \to \widetilde{M}_{0,2g+2}$ are rigid on $\overline{M}_{0,2g+2}$.*

Corollary 1.3. *For even $n \geq 6$ (resp. ≥ 8), there exist infinitely many rigid curves on $\widetilde{M}_{0,n}$ (resp. $\overline{M}_{0,n}$) and the union of their images is a Zariski dense subset.*

For $I \subset \{1, \ldots, n\}$ satisfying $2 \leq |I| \leq [n/2]$, let Δ_I denote the boundary component of $\overline{M}_{0,n}$ whose general point parameterizes a nodal union of two rational curves marked by indices in I and $\overline{I}$, respectively. Define Δ_2 to be the union of Δ_I over all I of cardinality 2. For $2 \leq k \leq [n/2]$, let $\widetilde{\Delta}_k$ be the boundary component of $\widetilde{M}_{0,n}$ whose general point parameterizes a nodal union of two rational curves with k marked points in one component and $n-k$ marked points in the other.

Let $\phi : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ be the morphism forgetting the last marked point. For a rigid curve R on $\overline{M}_{0,n}$, $\phi^{-1}(R)$ is the universal curve over R, hence it is a ruled surface with n sections. These sections are rigid curves on $\overline{M}_{0,n+1}$ contained in the boundary.

Corollary 1.4. *For even $n \geq 8$, there exist infinitely many rigid curves on $\overline{M}_{0,n+1}$. The union of their images is a Zariski dense subset contained in the boundary Δ_2.*

The rational Picard group of $\overline{H}_g$ is generated by boundary components $\Xi_0, \ldots, \Xi_{[(g-1)/2]}$ and $\Theta_1, \ldots, \Theta_{[g/2]}$ [HM2, 6.C]. A general point of Ξ_i parameterizes a double cover of a nodal union $\mathbb{P}^1 \cup \mathbb{P}^1$ branched at $2i+2$ points in one component and $2g-2i$ in the other. A general point of Θ_i parameterizes a double cover of $\mathbb{P}^1 \cup \mathbb{P}^1$ branched at $2i+1$ points in one component and $2g-2i+1$ in the other. The natural isomorphism $\overline{H}_g \cong \widetilde{M}_{0,2g+2}$ induces the identification $\Xi_i = \widetilde{\Delta}_{2i+2}$ and $\Theta_i = \widetilde{\Delta}_{2i+1}$.

Theorem 1.5 (Extremality)**.** *For either $\mu = (2g-2)$ or $\mu = (g-1, g-1)$, the image of $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}$ in $\widetilde{M}_{0,2g+2}$ does not intersect the boundary component $\widetilde{\Delta}_k$ for $k > 2$. The numerical class of $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}$ spans the extremal ray of the cone of moving curves on $\widetilde{M}_{0,2g+2}$ that is dual to the face $\langle \widetilde{\Delta}_3, \ldots, \widetilde{\Delta}_{g+1} \rangle$ of the cone of effective divisors.*

Unfortunately (or fortunately, depending on the reader's perspective), this extremal ray can be represented by nonnegative linear combinations of vital curves. Hence these Teichmüller curves are not counterexamples of Fulton's conjecture. Nevertheless, they are very different from the rigid curves constructed in [CT, §7], cf. Remark 2.4.

Hassett [H] studied moduli spaces of weighted pointed stable curves. The moduli space $\overline{M}_{0,n}$ can be regarded as parameterizing stable pointed rational curves with weight 1 on each marked point. Let $\mathcal{A}(i) = \{1/i, \ldots, 1/i\}$ be the symmetric weight that assigns $1/i$ to each marked point for $2 \leq i \leq [(n-1)/2]$. The morphism $\rho_i : \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}(i)}$ contracts all boundary divisors Δ_I satisfying $2 < |I| \leq i$. For a Teichmüller curve on $\overline{M}_{0,n}$, its image remains rigid on $\overline{M}_{0,\mathcal{A}(i)}$.

Corollary 1.6. *The infinitely many Teichmüller curves on $\overline{M}_{0,n}$ descend to infinitely many rigid curves on $\overline{M}_{0,\mathcal{A}(i)}$ for even $n \geq 8$ and $2 \leq i \leq [(n-1)/2]$. The union of their images is a Zariski dense subset in $\overline{M}_{0,\mathcal{A}(i)}$.*

For a curve C mapped to $\overline{\mathcal{M}}_g$ by a morphism h, define its slope

$$s(C) = \frac{\deg\, h^*\delta}{\deg\, h^*\lambda},$$

where δ is the total boundary class of $\overline{\mathcal{M}}_g$ and λ is the first Chern class of the Hodge bundle.

Corollary 1.7. *For either $\mu = (2g-2)$ or $\mu = (g-1, g-1)$, the slope of $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}$ is equal to $8 + 4/g$ for all d and i.*

The slope $s(\overline{\mathcal{T}}_{d,\mu})$ is determined by the quotient of two summations $M_{d,\mu}$ and $N_{d,\mu}$ [C, Theorem 1.15]. Understanding the asymptotic behavior of $s(\overline{\mathcal{T}}_{d,\mu})$ is crucial in a number of applications, e.g. it can provide information for the cone of effective divisors of $\overline{\mathcal{M}}_g$. Since $\mathcal{H}(\mu)$ may have up to three connected components due to hyperelliptic, odd or even spin structures, we use $\mathcal{T}^{\mathrm{hyp}}_{d,\mu}$, $\mathcal{T}^{\mathrm{odd}}_{d,\mu}$ or $\mathcal{T}^{\mathrm{even}}_{d,\mu}$ to denote the parts of $\mathcal{T}_{d,\mu}$ contained in each component, respectively.

For a stratum $\mathcal{H}(\mu)$ (or its connected components), define the limit (if it exists) of slopes of $\overline{\mathcal{T}}_{d,\mu}$ as

$$s_\mu = \lim_{d \to \infty} s(\overline{\mathcal{T}}_{d,\mu}).$$

Let $\lambda_1 \geq \cdots \geq \lambda_g \geq 0$ denote the nonnegative Lyapunov exponents associated to the Hodge bundle with respect to the Teichmüller geodesic flow $\mathrm{diag}(e^t, e^{-t})$ on $\mathcal{H}(\mu)$. Roughly speaking, these numbers measure the growth rate of the length of a vector in the bundle under parallel transport along the flow, cf. [K, §5] for an introduction to Lyapunov exponents. Let L_μ be the sum

$$L_\mu = \lambda_1 + \cdots + \lambda_g.$$

Use c_μ to denote the (area) Siegel-Veech constant of $\mathcal{H}(\mu)$, which satisfies the relation

$$c_\mu = \frac{\pi^2}{3} \cdot c_{\text{area}}(\mathcal{H}(\mu))$$

in the context of [EKZ, §1.5]. Further define

$$\kappa_\mu = \frac{1}{12}\Big(\sum_{i=1}^{k} \frac{m_i(m_i+2)}{m_i+1}\Big),$$

which is determined by $\mu = (m_1, \ldots, m_k)$. If $\mathcal{H}(\mu)$ has more than one component, the above quantities s, L and c can be defined in the same way for each component. We distinguish them by adding subscripts hyp, odd or even, respectively. By [EKZ, Theorem 1] we have the relation

$$L_\mu = \kappa_\mu + c_\mu. \tag{1}$$

The key observation here is that the quotient $M_{d,\mu}/N_{d,\mu}$ approaches the Siegel-Veech constant c_μ for large d, hence we obtain a relation among the limit of slopes, the sum of Lyapunov exponents and the Siegel-Veech constant as follows.

Theorem 1.8 (Slope). *For a connected stratum $\mathcal{H}(\mu)$, we have*

$$s_\mu = \frac{12c_\mu}{L_\mu} = 12 - \frac{12\kappa_\mu}{L_\mu}.$$

If $\mathcal{H}(\mu)$ has more than one connected component, the same formula holds for each component.

In Section 3, we provide two explanations of Theorem 1.8. The first one is an explicit proof but depends on the equality (1). The other is a conceptual explanation based on a formula of Kontsevich [K, §7], cf. Remark 3.1.

For the hyperelliptic strata, we can calculate L and c explicitly.

Corollary 1.9. *For the hyperelliptic component $\mathcal{H}^{\text{hyp}}(2g-2)$, we have*

$$c^{\text{hyp}}_{(2g-2)} = \frac{g(2g+1)}{3(2g-1)},\ L^{\text{hyp}}_{(2g-2)} = \frac{g^2}{2g-1}.$$

For the hyperelliptic component $\mathcal{H}^{\text{hyp}}(g-1, g-1)$, we have

$$c^{\text{hyp}}_{(g-1,g-1)} = \frac{(g+1)(2g+1)}{6g},\ L^{\text{hyp}}_{(g-1,g-1)} = \frac{g+1}{2}.$$

The above data match with the results of [EKZ, Corollary 1].

The slopes of $\overline{T}_{d,\mu}$ can be applied to study the cone of effective divisors on $\overline{\mathcal{M}}_g$. For an effective divisor $D = a\lambda - b\delta$ on $\overline{\mathcal{M}}_g$ for $a, b > 0$, define its slope

$$s(D) = \frac{a}{b}.$$

We do not know any effective divisor on $\overline{\mathcal{M}}_g$ whose slope is smaller than or equal to 6. On the other hand, any known lower bound of slopes of effective divisors has asymptotics $O(1/g)$ as g approaches infinity. Note that an effective divisor D cannot contain all Teichmüller curves, if their union is Zariski dense in $\overline{\mathcal{M}}_g$. Hence, the limit of slopes of these Teichmüller curves provides a lower bound for the slope of effective divisors. Given an speculation by Eskin and Zorich on the asymptotics of Siegel-Veech constants, the slope growth of these Teichmüller curves turns out to be $576/5g$, which (heuristically) coincides with that of Harris and Morrison's moving curves [HM1, Remark 3.23]. This coincidence is amusing, since curves used in [HM1] are moving on $\overline{\mathcal{M}}_g$ while the Teichmüller curves are rigid.

Conjecture 1.10. *There exist effective divisors on $\overline{\mathcal{M}}_g$ whose slopes are arbitrarily close to $576/5g$ as g approaches infinity.*

See the end of Section 3 for more details on this slope problem.

This paper is organized as follows. In Section 2, we study $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu}$ for $\mu = (2g-2)$ and $\mu = (g-1, g-1)$. In Section 3, we interpret the branched covers of tori as square-tiled surfaces and study the relation between slopes of $\overline{\mathcal{T}}_{d,\mu}$ and the sum of Lyapunov exponents. In Section 4, we analyze a few examples of square-tiled surfaces with a unique zero. In Appendix A, we sketch a proof for the relation between the limit of slopes of Teichmüller curves and the Siegel-Veech constant of the corresponding stratum of Abelian differentials. In Appendix B, we list the limit of slopes of $\overline{\mathcal{T}}_{d,\mu}$ in each stratum for small g. Throughout the paper, we work over the complex number field $\mathbb{C}$. A divisor means a $\mathbb{Q}$-Cartier divisor. We use cusp to denote an intersection point of a Teichmüller curve with the boundary of the moduli space. When we consider a torus covering as a square-tiled surface, we emphasize it as a Riemann surface.

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2. Density, rigidity and extremality

In this section we will prove Theorems 1.1, 1.2 and 1.5.

Proof of Theorem 1.1. The complex dimension of $\mathcal{H}(2g-2)$ equals $2g$. Take a standard basis $\gamma_1, \ldots, \gamma_{2g}$ of $H_1(C; \mathbb{Z})$ where C is a Riemann surface of genus g. The period map $\Phi : (C, \omega) \to \mathbb{C}^{2g}$ given by

$$\Phi(C, \omega) = \Big(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{2g}} \omega \Big)$$

provides a local coordinate chart for $\mathcal{H}(2g-2)$ [K, §3].

Take the standard torus E by gluing the parallel sides of $[0,1]^2$. Consider the local coordinates $\Phi(C,\omega) = (\phi_1, \ldots, \phi_{2g}) \in \mathbb{C}^{2g} \cong \mathbb{R}^{4g}$. By [EO, Lemma 3.1], we have $\phi_i \in \mathbb{Z}^2$ for $i = 1, \ldots, 2g$ if and only if the following holds:
(1) there exists a morphism $f : C \to E$;
(2) $\omega = f^{-1}(dz)$;
(3) f has a unique ramification point at p and $(\omega) = (2g-2)p$;
(4) the degree of f is equal to $\frac{\sqrt{-1}}{2}\int_C \omega \wedge \overline{\omega}$.

This establishes a correspondence between integer points of $\mathcal{H}(2g-2)$ and genus g covers $C \to E$ with a unique ramification point. It follows that such covers form a Zariski dense subset in $\mathcal{H}(2g-2)$. Since $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ is a connected component of $\mathcal{H}(2g-2)$, the union over all d and i of hyperelliptic components $\mathcal{T}^{\mathrm{hyp}}_{d,(2g-2),i}$ of Teichmüller curves $\mathcal{T}_{d,(2g-2)}$ forms a Zariski dense subset in $\mathcal{H}^{\mathrm{hyp}}(2g-2)$. Since $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ admits a dominant map to $\overline{H}_g$, the image of the union is Zariski dense in $\overline{H}_g$. Using the isomorphism $\overline{H}_g \cong \widetilde{M}_{0,2g+2}$ and the finite morphism $\overline{M}_{0,2g+2} \to \widetilde{M}_{0,2g+2}$, the preimage of this union is also Zariski dense in $\overline{M}_{0,2g+2}$.

The dimension of $\mathcal{H}(g-1,g-1)$ equals $2g+1$. Take a path γ_{2g+1} connecting the two zeros p_1 and p_2 of ω. The period map $\Phi : (C,\omega) \to \mathbb{C}^{2g+1}$ is given by

$$\Phi(C,\omega) = \Big(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{2g+1}} \omega \Big).$$

The rest of the argument is the same as the previous case. □

A Teichmüller curve on $\mathcal{M}_g$ is an algebraic geodesic with respect to the Kobayashi (equivalently, Teichmüller) metric. More precisely, pull back the Hodge bundle from $\mathcal{M}_g$ to the Teichmüller space $\mathcal{T}_g$ and consider it as a real manifold. There is an $\mathrm{SL}(2,\mathbb{R})$ action on the Hodge bundle, induced by the natural $\mathrm{SL}(2,\mathbb{R})$ action on the real and imaginary parts of a holomorphic 1-form. The fibers of the Hodge bundle are stabilized by $\mathrm{SO}(2,\mathbb{R})$. The induced map $\mathbb{H} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \to \mathcal{T}_g$ is a holomorphic isometry. In some rare occasions, the image of the composite map $\mathbb{H} \to \mathcal{T}_g \to \mathcal{M}_g$ is an algebraic subcurve D of $\mathcal{M}_g$. We say that $f : C \to D \subset \mathcal{M}_g$ is a Teichmüller curve if f lifts to such a local isometry from $\mathbb{H}$.

The 1-dimensional Hurwitz spaces $\mathcal{T}_{d,\mu,i}$ parameterizing torus coverings with a unique branch point (also called square-tiled surfaces or origamis, as we will see in Section 3) are invariant under the $\mathrm{SL}(2,\mathbb{R})$ action, since the action amounts to varying the defining lattice of the target elliptic curve. Then $h : \mathcal{T}_{d,\mu,i} \to \overline{\mathcal{M}}_g$ is a Teichmüller curve (called arithmetic Teichmüller curve). Its rigidity follows from the rigidity of general Teichmüller curves.

Proof of Theorem 1.2. McMullen [McM] and Möller [Möl] both proved that a Teichmüller curve is rigid on $\overline{\mathcal{M}}_g$. In fact, they proved that the rigidity is inherited by any finite covering of the image of a Teichmüller curve unramified away from cusps, where a cusp corresponds to a singular covering in the Hurwitz space in our setting. A component $\mathcal{T}_{d,\mu,i}$ of $\mathcal{T}_{d,\mu}$ is an arithmetic Teichmüller curve, hence it is rigid on $\overline{\mathcal{M}}_g$. If $\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i}$ maps into $\overline{H}_g \subset \overline{\mathcal{M}}_g$, then $\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i}$ is also rigid on $\overline{H}_g \cong \widetilde{M}_{0,2g+2}$.

Consider the finite morphism $f : \overline{M}_{0,2g+2} \to \widetilde{M}_{0,2g+2}$. Let $U \subset H_g$ denote the branch locus of f restricted to the interior of the moduli space. For $g \geq 3$, we claim that infinitely many of $h(\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i})$ do not intersect U. Then $f^{-1}(h(\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i})) \to h(\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i})$ is unramified, hence the pullback of $\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i}$ via f is also rigid on $\overline{M}_{0,2g+2}$.

By [KM, Lemma 3.3], the codimension of U in H_g is at least two for $2g+2 \geq 7$, i.e. $g \geq 3$. Let $U(\mu)$ denote the locus of pairs (C,ω) in $\mathcal{H}^{\mathrm{hyp}}(\mu)$ for either $\mu = (2g-2)$ or $\mu = (g-1,g-1)$, where C is parameterized in U. If U intersects $h(\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i})$, then $U(\mu)$ has to intersect the $\mathrm{SL}(2,\mathbb{R})$ orbit $\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i}$ in $\mathcal{H}^{\mathrm{hyp}}(\mu)$ generated by points whose period map coordinates are integers. By a dimension count, the $\mathrm{SL}(2,\mathbb{R})$ orbit of $U(\mu)$ is a proper subspace of $\mathcal{H}^{\mathrm{hyp}}(\mu)$, which must miss infinitely many points that have integer coordinates. Hence the $\mathrm{SL}(2,\mathbb{R})$ orbits of these points do not intersect $U(\mu)$. Consequently the images of the Teichmüller curves $\mathcal{T}^{\mathrm{hyp}}_{d,\mu,i}$ generated by these points do not intersect U in H_g. □

Now Corollary 1.3 follows as a consequence of Theorems 1.1 and 1.2.

Let R denote a rigid curve on $\overline{M}_{0,n}$. Let $\phi : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ be the morphism forgetting the last marked point and stabilizing the curve, if necessary. Note that $S = \phi^{-1}(R)$ is a ruled surface over R with n sections Γ_i for $1 \leq i \leq n$. Each section can be regarded as a curve in $\overline{M}_{0,n+1}$ lying in $\Delta_{\{i,n+1\}} \subset \Delta_2$.

Lemma 2.1. *The sections $\Gamma_1, \ldots, \Gamma_n$ are rigid curves on $\overline{M}_{0,n+1}$.*

Proof. Since $\phi(\Gamma_i) = R$, if Γ_i deforms in a surface S' in $\overline{M}_{0,n+1}$, the image $\phi(S')$ must be R. Otherwise R would deform in $\phi(S')$, which contradicts its rigidity. Therefore, we conclude that $S' = S$. But S can be constructed from successive blow-ups of $R \times \mathbb{P}^1$ and each blow-up decreases the self-intersection of a section passing through the blow-up center. Hence, $\Gamma_i^2 < 0$ and Γ_i does not deform in S. □

Proof of Corollary 1.4. Consider the following diagram:

$$\begin{tikzcd} \overline{M}_{0,n+1} \arrow[d, "\phi_i"] & \\ \overline{M}_{0,n} \arrow[r, "f"] & \widetilde{M}_{0,n} \end{tikzcd}$$

The morphism ϕ_i forgets the i-th marked point and reorders the other n marked points. In other words, ϕ_i is determined by a set bijection $\psi : \{1, \ldots, n\} \to \{1, \ldots, \hat{i}, \ldots, n+1\}$.

Let R denote a Teichmüller curve on $\widetilde{M}_{0,n}$ such that $f^{-1}(R) \to R$ is unramified away from the cusps of R, i.e. the image of R does not intersect the branch locus of f in the interior of $\widetilde{M}_{0,n}$. Then R and $f^{-1}(R)$ are rigid on $\widetilde{M}_{0,n}$ and $\overline{M}_{0,n}$, respectively. Let $\Gamma_{i,\psi(j)}$ be the j-th section of the universal curve with respect to ϕ_i over $f^{-1}(R)$ for $1 \leq j \leq n$. By Lamma 2.1, $\Gamma_{i,\psi(j)}$ is rigid on $\overline{M}_{0,n+1}$ and contained in Δ_I, where $I = \{i, \psi(j)\}$. Consider all possible R, ψ, i and j. We thus obtain infinitely many rigid curves $\Gamma_{i,\psi(j)}$ lying in Δ_2. Corollary 1.4 follows as a consequence of Theorems 1.1 and 1.2. □

Let us analyze the cusps of $\overline{\mathcal{T}}_{d,\mu}$ parameterizing singular admissible covers of a rational nodal curve for $\mu = (2g-2)$ and $\mu = (g-1, g-1)$. This can help us understand the intersection of $h(\overline{\mathcal{T}}_{d,\mu})$ with the boundary components of moduli spaces. See [HM2, 3.G] for an introduction to admissible covers.

Proposition 2.2. *If $\pi : C \to E_0$ parameterized by $\overline{\mathcal{T}}_{d,(2g-2)}$ is an admissible cover of a rational nodal curve E_0, the stable limit of C is an irreducible nodal curve. For $\pi : C \to E_0$ parameterized by $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,(g-1,g-1)}$, the stable limit of C is either irreducible or consists of two smooth rational curves joint at $g+1$ nodes.*

Proof. Consider the case $\mu = (2g-2)$ first. Let C_0 be the irreducible component of C that contains the unique ramification point p. If C_1 is another irreducible component of C, note that π restricted to C_1 is not ramified away from the nodes of C. We claim that C_1 is a smooth rational curve and it intersects $\overline{C \backslash C_1}$ at two points.

Suppose the restriction of π to C_1 is a map of degree d_1 to E_0. Let q be the node of E_0 and (U, V) the two local branches of q in E_0 with local coordinates (u, v). If C_1 is singular, let $r_1, \ldots, r_m$ denote its nodes. Note that $\pi(r_i) = q$ and locally around r_i, the map π is given by $(x, y) \to (u = x^{a_i}, v = y^{a_i})$. Let s be an intersection point in $\{C_1 \cap \overline{C \backslash C_1}\}$ and $\pi(s) = q$. If a local neighborhood of s on C_1 maps to U (resp. V) given by $u = x^b$ (resp. $v = x^b$), we say that s is of type (u, b) (resp. (v, b)). Starting from s of type (u, b), by the definition of admissible covers, s also belongs to another component C_1' of C, where s is of type (v, b) as a smooth point in C_1'. Then there exists s' in C_1' such that $\pi(s') = q$. As a smooth point in C_1', s' is of type (u, b). To pair with s', there ought to be some s'' as a smooth point in a component C_1'' such that $\pi(s'') = q$ and s'' in C_1'' is of type (v, b). Since there are finitely many components, the process has to stop at some stage. Suppose it ends with a point, say t in the starting component C_1 such that t is of type (v, b) and t pairs with s of type (u, b) that maps to q. We conclude that the set $\{C_1 \cap \overline{C \backslash C_1}\}$ contains even number of points that decompose into pairs $(s_1, t_1), \ldots, (s_k, t_k)$. Locally around (s_j, t_j) in C_1, the map π is given by $(x, y) \to (u = x^{b_j}, v = y^{b_j})$.

Let C_1^ν be the normalization of C_1. Use r_i' and r_i'' to denote the preimages of r_i in C_1^ν. Let q' and q'' be the two points in $\mathbb{P}^1$ glued together as the node q of E_0. The map π induces a degree d_1 branched cover $\pi^\nu : C_1^\nu \to \mathbb{P}^1$ with ramification points $r_1', r_1'', \ldots, r_m', r_m''$ and $s_1, t_1, \ldots, s_k, t_k$. The ramification order of r_i' and r_i'' equals $a_i - 1$. The ramification order of s_j and t_j equals $b_j - 1$. By Riemann-Roch, we have

$$2g(C_1^\nu) - 2 + 2d_1 = 2\sum_{i=1}^{m}(a_i - 1) + 2\sum_{j=1}^{k}(b_j - 1),$$

$$g(C_1^\nu) - 1 + d_1 = \sum_{i=1}^{m} a_i + \sum_{j=1}^{k} b_j - m - k.$$

Since π^ν maps $r'_1, \dots, r'_m$ and $s_1, \dots, s_k$ to q', we have $\sum_{i=1}^m a_i + \sum_{j=1}^k b_j \leq d_1$. Then we get $0 \leq g(C_1^\nu) \leq 1 - m - k$, which implies $m + k \leq 1$. But $2k = |C_1 \cap \overline{C \backslash C_1}|$ is positive, otherwise C would be disconnected. So the only possibility is $g(C_1^\nu) = m = 0$ and $k = 1$. It says that C_1 is a smooth rational curve and $|C_1 \cap \overline{C \backslash C_1}| = 2$.

Using the same argument, the intersection points $\{C_0 \cap \overline{C \backslash C_0}\}$ decompose in pairs (s_i, t_i) such that s_i and t_i are connected by a chain of smooth rational curves, each of which has the same property as C_1. An example of such a curve C is shown in Figure 1. Blowing down all the smooth rational components, the

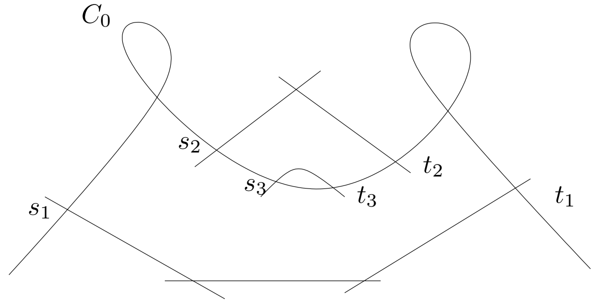

Figure 1.

stable limit of C is an irreducible nodal curve by gluing the points $\{C_0 \cap \overline{C \backslash C_0}\}$ in pairs.

For $\mu = (g-1, g-1)$, by the definition of $\mathcal{H}^{\mathrm{hyp}}(g-1, g-1)$ [KZ, Remark 3], the two zeros p_1 and p_2 of the holomorphic 1-form ω are switched by the hyperelliptic involution ι. If p_1 and p_2 belong to two different irreducible components C_1 and C_2 of the stable limit of C, then C admits a double cover of a rational curve that maps C_1 and C_2 to the same $\mathbb{P}^1$ component of the rational curve, since the target rational curve is fixed by ι and p_1 and p_2 are switched by ι. It implies that C_1 and C_2 are both isomorphic to $\mathbb{P}^1$ and they are joint by $g+1$ rational bridges $B_1, \dots, B_{g+1}$. Blowing down the semistable components B_i, the stable limit of C consists of the two smooth rational curves C_1 and C_2 joint at $g+1$ nodes.

If p_1 and p_2 are contained in the same component C_0 of C, the rest of the argument is the same as in the preceding case. □

Remark 2.3. Proposition 2.2 may fail for singular admissible covers in non-hyperelliptic components of $\mathcal{T}_{d,(g-1,g-1)}$. For instance, let E be an elliptic curve that admits a triple cover of $\mathbb{P}^1$ with three ramification points p, q and r of ramification order 2. Glue E with another copy E' at $p = q'$ and $q = p'$. We obtain a reducible curve C with two nodes. The arithmetic genus of C is equal to 3. Then C admits a degree 6 cover of a rational nodal curve E_0 such that p and q map to the node of E_0. The two ramification points r and r' are smooth on C with ramification order 2, as shown in Figure 2. This cover $C \to E_0$ is a limit of smooth

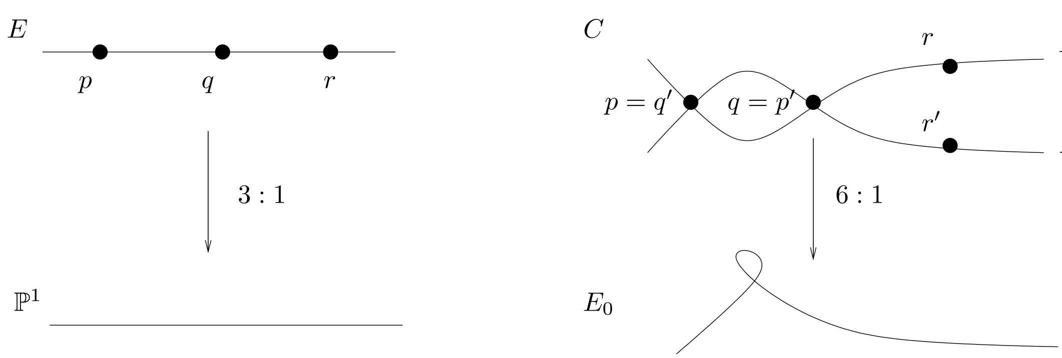

Figure 2.

covers in $\overline{\mathcal{T}}_{6,(2,2)}$. Although C possesses irreducible components of positive genus, C itself is not contained in the boundary component Δ_i of $\overline{M}_g$ for $i > 0$, since removing any node of C does not disconnect the whole curve. This holds in general for singular admissible covers in any $\overline{\mathcal{T}}_{d,\mu}$ [C, Proposition 3.1].

Now we prove the extremality of $\overline{\mathcal{T}}^{\text{hyp}}_{d,\mu,i}$ for $\mu = (2g-2)$ and $\mu = (g-1, g-1)$.

Proof of Theorem 1.5. We want to show that $h(\overline{\mathcal{T}}^{\text{hyp}}_{d,\mu,i})$ does not intersect the boundary component $\widetilde{\Delta}_k$ of $\widetilde{M}_{0,2g+2}$ for $k > 2$. Let $\phi : C \to B$ be an admissible double cover of a stable $(2g+2)$-pointed rational curve B such that the stable limit of C is the same as that of an admissible cover of a rational nodal curve E_0 parameterized by $\overline{\mathcal{T}}^{\text{hyp}}_{d,\mu,i}$. The map ϕ is branched at the $2g+2$ marked points of B.

We first take care of the exceptional case in Proposition 2.2, when the stable limit of C consists of two copies of $\mathbb{P}^1$ joint at $g+1$ nodes. For the corresponding admissible double cover, the target curve consists of an unmarked $\mathbb{P}^1$ with $g+1$ rational tails. Each tail is marked at the 2 branch points of the cover. As a point in $\widetilde{M}_{0,2g+2}$, this target curve does not lie in $\widetilde{\Delta}_k$ for $k > 2$. By Proposition 2.2, from now on we assume that the stable limit of C is an irreducible nodal curve. Then C consists of a smooth component C_0 with m pairs of points (s_i, t_i) on C_0, each linked by a chain of smooth rational curves, as shown in Figure 3.

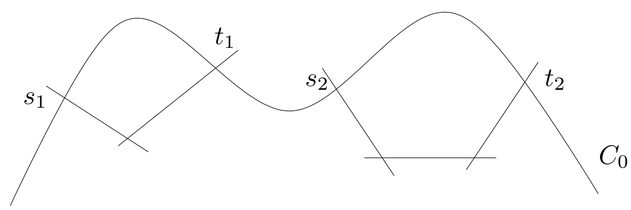

FIGURE 3.

Let B_0 be the image $\phi(C_0)$ in B. Call an irreducible component B_1 of $B \backslash B_0$ a tail if removing it does not disconnect B. We claim that a tail must intersect B_0. Suppose B_1 is a tail that does not meet B_0. Let C_1 denote the preimage $\phi^{-1}(B_1)$. By the stability of B, the tail B_1 contains two marked points and ϕ restricted to C_1 is a double cover of B_1 branched at these two points. Hence C_1 is an irreducible component of a chain of rational curves in C. Let r be the node $B_1 \cap \overline{B \backslash B_1}$ and B_r the unique irreducible component of B that intersects B_1 at r. Let p and q be the preimages of r under ϕ. Note that p and q are not contained in C_0. By the description of C, there exist two different components C_p and C_q of C that intersect C_1 at p and q, respectively. Since r is not a branch point, ϕ maps both C_p and C_q isomorphically to B_r. Then B_r does not contain any marked point, which contradicts the stability of B. Figure 4 illustrates the idea of this argument.

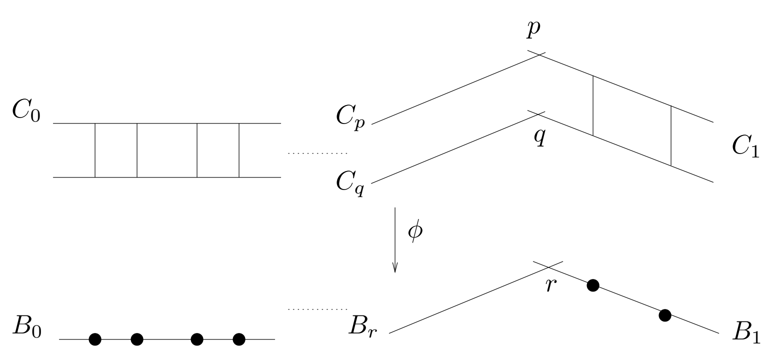

FIGURE 4.

Therefore, B consists of a main component B_0, plus a set of rational tails each of which contains exactly 2 marked points. The preimage of a tail under ϕ is an irreducible rational component of C which intersects C_0 at a pair of points s_i and t_i. An example of such admissible covers ϕ is shown in Figure 5.

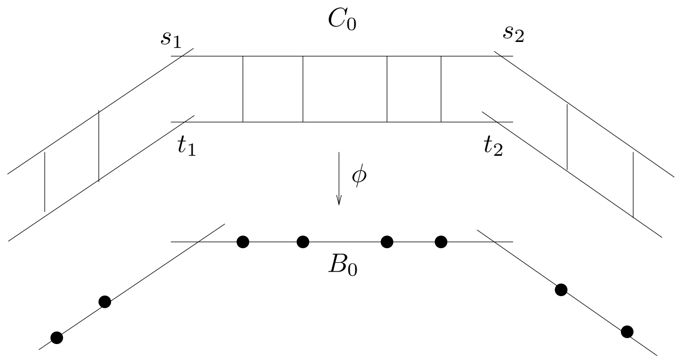

FIGURE 5.

Note that removing any node of B, the resulting two connected components have 2 and $2g$ marked points, respectively. Hence B as a $(2g+2)$-pointed rational curve does not lie in the boundary component $\widetilde{\Delta}_k$ of $\widetilde{M}_{0,2g+2}$ for $k > 2$. Then the intersection $h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}).\,\widetilde{\Delta}_k$ is zero for $k > 2$. Keel and McKernan [KM, Theorem 1.3] showed that the effective cone of $\widetilde{M}_{0,2g+2}$ is generated by $\widetilde{\Delta}_2, \ldots, \widetilde{\Delta}_{g+1}$. The pseudo-effective cone of a projective variety is dual to its cone of moving curves with respect to the natural intersection paring [BDPP]. Therefore, the numerical class of $h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i})$ spans the extremal ray of the cone of moving curves on $\widetilde{M}_{0,2g+2}$ that is dual to the face $\langle \widetilde{\Delta}_3, \ldots, \widetilde{\Delta}_{g+1} \rangle$ of the cone of effective divisors. $\square$

Remark 2.4. Take a $(2g+2)$-pointed smooth rational curve and vary a marked point. By semistable reduction, we get a curve in $\overline{M}_{0,2g+2}$ whose image C in $\widetilde{M}_{0,2g+2}$ does not intersect $\widetilde{\Delta}_k$ for $k > 2$. Hence C and $h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i})$ span the same extremal ray of the cone of moving curves on $\widetilde{M}_{0,2g+2}$, which is surprising because $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}$ is rigid while the deformations of C cover an open subset of $\widetilde{M}_{0,2g+2}$. Nevertheless, this ray can be expressed as a nonnegative linear combination of vital curves. So these rigid curves are not counterexamples to Fulton's conjecture. Note that our Teichmüller curves are different from the rigid curves constructed in [CT, §7] as the exceptional loci of birational contractions of $\overline{M}_{0,n}$, because any contraction that blows down $h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i})$, and hence C, must contract $\widetilde{M}_{0,2g+2}$ to a lower dimensional target and cannot be birational.

The rigidity of $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}$ on the moduli space of weighted pointed rational curves and the value of their slopes for $\mu = (2g-2)$ and $\mu = (g-1, g-1)$ follow as direct consequences of Theorem 1.5.

Proof of Corollary 1.6. Let $n = 2g+2$. By Theorem 1.5, we know that $h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i})$ does not intersect the boundary component $\widetilde{\Delta}_j$ of $\widetilde{M}_{0,n}$ for $j > 2$ and even $n \geq 6$. The preimage $f^{-1}(h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}))$ in $\overline{M}_{0,n}$ does not intersect the boundary component Δ_J for $|J| > 2$. Since the morphism $\rho_i : \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}(i)}$ only contracts the boundary components D_J for $2 < |J| \leq i$, the restriction of ρ_i to a local neighborhood of $f^{-1}(h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}))$ is an isomorphism. Hence, if $f^{-1}(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i})$ is rigid on $\overline{M}_{0,n}$, its projection to $\overline{M}_{0,\mathcal{A}(i)}$ is also rigid. Corollary 1.6 now follows by combining Theorems 1.1 and 1.2. $\square$

Proof of Corollary 1.7. The inclusion $\iota : \overline{H}_g \hookrightarrow \overline{\mathcal{M}}_g$ induces a pull-back map

$$\iota^* : \mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q} \to \mathrm{Pic}(\overline{H}_g) \otimes \mathbb{Q},$$

such that $\iota^*(\Delta_0) = 2\sum \widetilde{\Delta}_{2i}$ and $\iota^*(\Delta_i) = \widetilde{\Delta}_{2i+1}/2$ for $i > 0$ [HM2, 6.C]. Moreover, we have

$$\iota^*(\lambda) = \sum_{i=0}^{[(g-1)/2]} \frac{(i+1)(g-i)}{4g+2}\widetilde{\Delta}_{2i+2} + \sum_{i=1}^{[g/2]} \frac{i(g-i)}{4g+2}\widetilde{\Delta}_{2i+1}.$$

For both $\mu = (2g-2)$ and $\mu = (g-1, g-1)$, $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}$ maps to $\overline{H}_g$ and its image does not intersect $\widetilde{\Delta}_k$ for $k > 2$. Then it has slope

$$\frac{h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}).\,\delta}{h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}).\,\lambda} = 2\cdot\frac{h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}).\,\widetilde{\Delta}_2}{h(\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}).\,\iota^*(\lambda)} = 2\cdot\frac{4g+2}{g} = 8+\frac{4}{g}.$$

$\square$

Remark 2.5. There is a general formula [C, Theorem 1.15] to compute the slope of such 1-dimensional Hurwitz spaces of covers. But the combinatorics involved in the formula is so complicated that the author was only able to calculate the slope for $g = 2$. Here the result implies that for a hyperelliptic component $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu,i}$ for $\mu = (2g-2)$ and $\mu = (g-1, g-1)$, we know its slope equal to $8 + 4/g$ without doing any explicit calculation! Nevertheless, see Example 4.5 for a reducible Hurwitz space whose hyperelliptic components have slope $8 + 4/g$, while the others have a different slope.

3. Slopes, Lyapunov exponents and Siegel-Veech constants

In this section, we will prove Theorem 1.8.

Let a, b and c denote a standard basis of $\pi_1(E, q)$ for a torus E punctured at q, satisfying the relation $b^{-1}a^{-1}ba = c$ as shown in Figure 6.

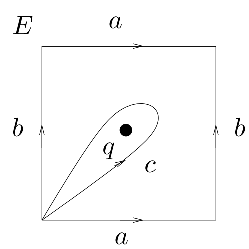

Figure 6.

Consider a degree d connected cover π of E with a unique branch point q and

$$\pi^{-1}(q) = (m_1+1)p_1 + \cdots + (m_k+1)p_k + p_{k+1} + \cdots + p_l.$$

Such a cover corresponds to an element in $\mathrm{Hom}(\pi_1(E, q), S_d)$, where S_d is the permutation group on d letters, such that the images α, β and γ of a, b and c satisfy

$$\beta^{-1}\alpha^{-1}\beta\alpha = \gamma \in (m_1+1)\cdots(m_k+1)(1)\cdots(1),$$

where $(m_1+1)\cdots(m_k+1)(1)\cdots(1)$ is the conjugacy class in S_d with the parenthesized cycle lengths. The cover is connected if and only if the subgroup generated by α and β acts transitively on the d letters.

Define an equivalence relation $\sim$ between two pairs (α, β) and (α', β') if there exists a permutation τ such that $\tau(\alpha, \beta)\tau^{-1} = (\alpha', \beta')$. Two covers are called isomorphic if and only if there is a commutative diagram as follows:

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr] & & C' \arrow[dl] \\ & E & \end{tikzcd}$$

where ϕ is an isomorphism between C and C'. For two isomorphic covers, their monodromy images (α, β) and (α', β') are equivalent to each other with τ the permutation induced by ϕ. Hence, non-isomorphic degree d, genus g connected covers of a fixed E can be parameterized by the following set of equivalence classes:

$$\mathrm{Cov}_{d,\mu} = \{(\alpha, \beta) \in S_d \times S_d \mid \beta^{-1}\alpha^{-1}\beta\alpha \in (m_1+1)\cdots(m_k+1)(1)\cdots(1),\ \langle \alpha, \beta \rangle \text{ is transitive}\}/\sim .$$

Varying the j-invariant of the elliptic curve E, we obtain the 1-dimensional Hurwitz space $\overline{\mathcal{T}}_{d,\mu}$ parameterizing such covers. The fiber of the finite map $e : \overline{\mathcal{T}}_{d,\mu} \to \overline{\mathcal{M}}_{1,1}$ over a fixed elliptic curve can be identified with the equivalence classes of pairs in $\mathrm{Cov}_{d,\mu}$. The degree $N_{d,\mu}$ of e counts the number of non-isomorphic covers, hence we have

$$N_{d,\mu} = |\mathrm{Cov}_{d,\mu}|.$$

The Teichmüller curves $\mathcal{T}_{d,\mu}$ are contained in $\mathcal{H}(\mu)$ as closed $\mathrm{SL}(2, \mathbb{R})$ orbits. If $\mathcal{H}(\mu)$ has more than one connected component, the covering set will have a corresponding decomposition. In this case, we add subscripts hyp, odd or even to distinguish them.

For each equivalence class of monodromy pairs (α, β) in $\mathrm{Cov}_{d,\mu}$, suppose α has a_i cycles of length i so that $\sum_{i=1}^{d} ia_i = d$. Associate to (α, β) the following weight

$$\sum_{i=1}^{d} \frac{a_i}{i}.$$

This weight is well-defined up to equivalence, since permutations in the same conjugacy class have the same number of cycles of length i. Let $M_{d,\mu}$ denote the summation of such weights, ranging over the $N_{d,\mu}$ equivalence classes of monodromy pairs in $\mathrm{Cov}_{d,\mu}$. By [C, Theorem 1.15], the slope of $\overline{\mathcal{T}}_{d,\mu}$ is determined by the quotient of $M_{d,\mu}$ and $N_{d,\mu}$ as follows:

$$s(\overline{\mathcal{T}}_{d,\mu}) = \frac{12M_{d,\mu}}{M_{d,\mu} + \kappa_\mu N_{d,\mu}}, \tag{2}$$

where $\kappa_\mu = \frac{1}{12}\left(\sum_{i=1}^{k} \frac{m_i(m_i+2)}{m_i+1}\right)$ is a constant determined by μ. The same formula applies to $\overline{\mathcal{T}}^{\mathrm{hyp}}_{d,\mu}$, $\overline{\mathcal{T}}^{\mathrm{odd}}_{d,\mu}$ and $\overline{\mathcal{T}}^{\mathrm{even}}_{d,\mu}$ in case $\mathcal{H}(\mu)$ has more than one connected component. Accordingly the summations M and N are taken over the corresponding subsets in the decomposition of $\mathrm{Cov}_{d,\mu}$.

For the reader's convenience, we briefly explain the idea behind the slope formula (2). When an elliptic curve E degenerates to a rational nodal curve, locally there is a vanishing cycle $a \in H_1(E; \mathbb{Z})$ that shrinks to the node. Let α be the monodromy image of a in S_d for a smooth cover in $\overline{\mathcal{T}}_{d,\mu}$. If α has a cycle of length i, around the resulting node of the singular cover, the map is locally given by $(x, y) \to (x^i, y^i)$. Moreover, one has to make a degree i base change to realize a universal covering map over $\overline{\mathcal{T}}_{d,\mu} \to \overline{\mathcal{M}}_{1,1}$ around this node. From an orbifold point of view, such a node contributes $1/i$ to the intersection $\overline{\mathcal{T}}_{d,\mu}.\,\delta$, hence the weighted sum $M_{d,\mu}$ counts the total intersection $\overline{\mathcal{T}}_{d,\mu}.\,\delta$. For $\overline{\mathcal{T}}_{d,\mu}.\,\lambda$, the proof of Theorem 1.15 in [C, §3] gives the details of this standard calculation on the universal covering map.

To connect the slope of $\overline{\mathcal{T}}_{d,\mu}$ to the Siegel-Veech constant and the sum of Lyapunov exponents, we need to interpret covers of elliptic curves with a unique branch point as square-tiled surfaces (or origamis). The domain Riemann surface of such a cover has a plane polygon model, which is called a square-tiled surface, cf. [GJ] for an introduction and relevant references. There is a correspondence between square-tiled surfaces and the monodromy pairs of the covers. Let $\pi : C \to E$ be a degree d cover of the standard torus E with a unique branch point at the vertices of E. Take d unit squares and mark them by $1, \ldots, d$. For the i-th square, mark its upper and lower horizontal edges by b_i and b'_i, respectively. Similarly, mark its right and left vertical edges by a_i and a'_i, respectively. Let (α, β) be the monodromy pair corresponding to π. One can realize C as a flat surface tiled by d unit squares by identifying a_i with $a'_{\alpha(i)}$ and identifying b_i with $b'_{\beta(i)}$ via parallel transport. See Section 4 for many concrete examples of square-tiled surfaces.

A holomorphic 1-form defines a flat structure on a Riemann surface C so we can talk about its geodesics along a fixed direction. A saddle connection on C is a geodesic connecting two zeros of the 1-form. Geodesics with the same direction fill in a maximal cylindrical area, which is bounded by saddle connections. The key observation is that the summation $M_{d,\mu}$ counts the weighted number of maximal horizontal cylinders of height 1, where the weight is given by $1/l$ for a cylinder of length l.

Proof of Theorem 1.8. For a smooth cover corresponding to the monodromy pair (α, β) in $\mathrm{Cov}_{d,\mu}$, we glue d unit squares by the monodromy pair (α, β) to construct a square-tiled surface. Suppose α corresponds to the monodromy image of the horizontal edge of the torus. If α has a cycle of length i, say $(a_1 \cdots a_i)$, we line up i unit squares horizontally into a rectangle of length i and height 1 with two vertical edges glued together. On the resulting square-tiled surface, this rectangle corresponds to a cylinder filled in by maximal horizontal geodesics of length i, as shown in Figure 7. Therefore, the sum $M_{d,\mu}$ equals the total number of

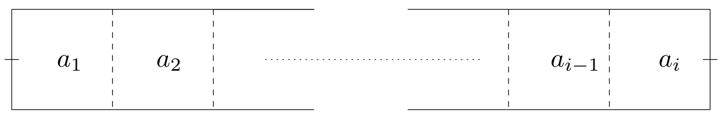

FIGURE 7.

maximal horizontal cylinders of height 1, weighted by the inverse of their lengths, in all square-tiled surfaces parameterized by $\mathrm{Cov}_{d,\mu}$. Eskin explained to the author why the Siegel-Veech constant c_μ of a stratum $\mathcal{H}(\mu)$ (or one of its connected components) equals the following limit:

$$c_\mu = \lim_{d\to\infty} \frac{M_{d,\mu}}{N_{d,\mu}}.$$

For the convenience of the reader, we sketch a proof in Appendix A. Because our square-tiled surfaces only have cylinders of height 1, our identifications are compatible with those [EKZ, §2.5] where a cylinder of arbitrary height h and length l is assigned weight equal to h/l. Combining the equality (1) $L_\mu = \kappa_\mu + c_\mu$ with the slope formula (2), we have

$$s_\mu = \lim_{d\to\infty} s(\mathcal{T}_{d,\mu}) = \frac{12 c_\mu}{L_\mu}.$$

□

Remark 3.1. We sketch another explanation of Theorem 1.8, which leads to the equality (1). Using the period map, the tangent space of $\mathcal{H}(\mu)$ can be identified with the relative cohomology group $H^1(C, p_1, \ldots, p_k; \mathbb{C})$. The $\mathrm{SL}(2,\mathbb{R})$ action induces an invariant splitting $H^1(C, p_1, \ldots, p_k; \mathbb{C}) = L_1 \oplus L_2$, where L_1 is the direction of the $\mathrm{SL}(2,\mathbb{R})$ orbit and L_2 is the orthogonal complement with respect to the Hodge inner product. Then L_2 supports an invariant differential form β in $H^{2n-2}(\mathcal{H}_1(\mu); \mathbb{R})$, where $\mathcal{H}_1(\mu)$ is the quotient $\mathcal{H}(\mu)/\mathbb{C}^*$ by scaling the 1-forms and $n = 2g - 2 + k$ is the complex dimension of $\mathcal{H}_1(\mu)$. Let γ be the first Chern class of the holomorphic $\mathbb{C}^*$ bundle $\mathcal{H}(\mu) \to \mathcal{H}_1(\mu)$. In [K, §7], there is a formula to compute the sum of Lyapunov exponents as follows:

$$L_\mu = \frac{\int_{\mathcal{H}_1(\mu)} \beta \wedge \lambda}{\int_{\mathcal{H}_1(\mu)} \beta \wedge \gamma}. \tag{3}$$

The Poincare dual of β can be regarded as the limit of $\mathcal{T}_{d,\mu}$ as d approaches infinity. The bottom integral encodes the volume of $\mathcal{H}_1(\mu)$, which can be calculated using the limit of $N_{d,\mu}$ [EO, Proposition 1.6]. Therefore, the formula (3) can be interpreted as

$$L_\mu = \frac{\lim\limits_{d\to\infty} \overline{\mathcal{T}}_{d,\mu}.\,\lambda}{\lim\limits_{d\to\infty} N_{d,\mu}} = \frac{\lim\limits_{d\to\infty} \overline{\mathcal{T}}_{d,\mu}.\,\lambda/(\frac{1}{12}\overline{\mathcal{T}}_{d,\mu}.\,\delta)}{\lim\limits_{d\to\infty} N_{d,\mu}/M_{d,\mu}} = \frac{12 \lim\limits_{d\to\infty} 1/s(\mathcal{T}_{d,\mu})}{1/c_\mu} = \frac{12 c_\mu}{s_\mu}.$$

Moreover, taking the limit of the slope formula (2), we have $s_\mu = 12 c_\mu/(\kappa_\mu + c_\mu)$. Hence, we obtain that $L_\mu = \kappa_\mu + c_\mu$. The coefficient 12 appears from the need to make a base change of order 12 to realize a universal covering map, say by a general pencil of plane cubics, after which the intersection with δ is counted by $12 M_{d,\mu}$. The reader may also refer to [BM, Theorem 8.2] for a discussion relating Lyapunov exponents of Teichmüller curves to local systems.

We can calculate c and L explicitly for the hyperelliptic components $\mathcal{H}^{\rm hyp}(2g-2)$ and $\mathcal{H}^{\rm hyp}(g-1,g-1)$.

Proof of Corollary 1.9. By Corollary 1.7, the Teichmüller curves $\overline{\mathcal{T}}^{\rm hyp}_{d,\mu}$ for $\mu=(2g-2)$ and $\mu=(g-1,g-1)$ have slopes $s=8+4/g$, independent of d.

For $\mu=(2g-2)$, we have

$$\kappa=\frac{(g-1)g}{3(2g-1)}.$$

By Theorem 1.8, we get

$$L=\frac{12\kappa}{12-s}=\frac{g^2}{2g-1},\ \ c=L-\kappa=\frac{g(2g+1)}{3(2g-1)}.$$

For $\mu=(g-1,g-1)$, we have

$$\kappa=\frac{(g-1)(g+1)}{6g}.$$

By Theorem 1.8, we get

$$L=\frac{12\kappa}{12-s}=\frac{g+1}{2},\ \ c=L-\kappa=\frac{(g+1)(2g+1)}{6g}.$$

□

These numbers are also calculated in [EKZ, Corollary 1] using the determinant of the Laplacian.

The Teichmüller curves $\overline{\mathcal{T}}_{d,\mu}$ were studied in [C] for the purpose of bounding slopes of effective divisors on $\overline{\mathcal{M}}_g$. For an effective divisor $D=a\lambda-b\delta$, define its slope

$$s(D)=\frac{a}{b}.$$

If an effective curve C on $\overline{\mathcal{M}}_g$ is not contained in D, then $D.C\geq 0$, hence $s(D)\geq s(C)$. For a partition $\mu=(m_1,\ldots,m_k)$ of $2g-2$ with $k\geq g-1$, by [C, Theorem 1.21] the map $\mathcal{H}(\mu)\to\overline{\mathcal{M}}_g$ is dominant and the images of $\overline{\mathcal{T}}_{d,\mu}$ cannot be all contained in D, since the union of square-tiled surfaces is a Zariski dense subset in $\mathcal{H}(\mu)$. Hence, the limit of slopes of $\overline{\mathcal{T}}_{d,\mu}$ as d approaches infinity can provide a lower bound for slopes of effective divisors on $\overline{\mathcal{M}}_g$.

There are two difficulties in this method of bounding slopes. First, the Teichmüller curves $\overline{\mathcal{T}}_{d,\mu}$ may be reducible. It is logically possible that a divisor D contains some components of $\overline{\mathcal{T}}_{d,\mu}$ but not all, for infinitely many d. In principle we also need to compute the limit of slopes of irreducible components $\overline{\mathcal{T}}_{d,\mu,i}$ of $\overline{\mathcal{T}}_{d,\mu}$ as d approaches infinity. Second, we do not have an effective way to evaluate s, c and L for large g. Nevertheless, it is conjectured that for "generic" $\overline{\mathcal{T}}_{d,\mu,i}$, the quotient $M_{d,\mu,i}/N_{d,\mu,i}$ goes to c_μ, i.e. the limit of $s(\overline{\mathcal{T}}_{d,\mu,i})$ should equal s_μ, where "generic" means that $\overline{\mathcal{T}}_{d,\mu,i}$ is not contained in any proper $\mathrm{SL}(2,\mathbb{R})$ orbit closure in $\mathcal{H}(\mu)$. Moreover, Eskin and Zorich obtained strong numerical evidence, which predicts that

$$\lim_{g\to\infty} c_\mu=2$$

for non-hyperelliptic strata $\mathcal{H}(\mu)$. It is necessary to rule out hyperelliptic strata, since by Corollary 1.9 we know that $c^{\rm hyp}_\mu$ grows asymptotically as $g/3$ for $\mu=(2g-2)$ and $\mu=(g-1,g-1)$.

Given the above speculations, by Theorem 1.8, an heuristic lower bound for slopes of effective divisors can be arbitrarily close to

$$s=\frac{12}{1+\frac{1}{2}\kappa}=\frac{288}{2g+22+\displaystyle\sum_{i=1}^{k}\frac{m_i}{m_i+1}}$$

for large g, where $(m_1,\ldots,m_k)$ is a partition of $2g-2$ and $k\geq g-1$. The partition $(1,\ldots,1,g)$ maximizes this bound and we have $s\sim 576/5g$ as g approaches infinity for this partition.

An interesting feature is that the asymptotic bound $576/5g$ also appeared in [HM1, Remark 3.23] based on an heuristic analysis of monodromy data of 1-dimensional families obtained by varying a branch point of a simply branched cover of a rational curve. Using Hodge integrals, Pandharipande [P] established a rigorous lower bound $60/(g+4)$. To the author's best knowledge, on the one hand, there is no known effective divisor on $\overline{\mathcal{M}}_g$ with slope ≤ 6. On the other hand, there is no known lower bound better than $O(1/g)$ for slopes of effective divisors as g approaches infinity. The range $[O(1/g), 6]$ has been mysterious for a long time. The reader may look at [Mor, Chapter 2] for a nice survey on this problem and its recent development. In any case, we make the following conjecture.

Conjecture 3.2. *There exist effective divisors on $\overline{\mathcal{M}}_g$ whose slopes are arbitrarily close to $576/5g$ as g approaches infinity.*

The author has not been able to find further evidence for this conjecture except the following heuristic argument. Assume that $576/5g$ is the limit of slopes of $\overline{\mathcal{T}}_{d,\mu}$ for $\mu = (1,\ldots,1,g)$ as d and g approach infinity. Let $Z_{k,g}$ denote the moving curves used in [HM1]. Assume that the slopes of $Z_{k,g}$ have growth $576/5g$ as expected for large g. Let R be a curve class in $N_1(\overline{\mathcal{M}}_g)\otimes\mathbb{Q}$ such that $R.\,\delta_i = 0$ for $1 \leq i \leq [g/2]$ and $s(R) = (R.\,\delta_0)/(R.\,\lambda) = 576/5g$. Since $\overline{\mathcal{T}}_{d,\mu}$ does not intersect δ_i for $i > 0$ [C, Proposition 3.1] and $Z_{k,g}.\,\delta$ is dominated by $Z_{k,g}.\,\delta_0$ for large g, the rays spanned by the limit of $\overline{\mathcal{T}}_{d,\mu}$ as d approaches infinity and by $Z_{k,g}$ should be close to R for large g. Moreover, $Z_{k,g}$ is moving in $\overline{\mathcal{M}}_g$ while $\overline{\mathcal{T}}_{d,\mu}$ is rigid. It is natural to expect that their limits are close to an extremal ray R' of the cone of moving curves for large g, which is dual to the pseudo-effective cone of divisors by [BDPP]. Given such an expectation, the dual face of R' in the pseudo-effective cone would be spanned by $\delta_1,\ldots,\delta_{[g/2]}$ along with a pseudo-effective divisor D such that $D.\,R' = 0$. Since R' is close to R, the slope of D would be close to $s(R) = 576/5g$.

Remark 3.3. An extremal ray of the cone of moving curves on a projective variety may be generated by both a moving curve and a rigid curve. Remark 2.4 gives one example but another is provided by rational curves R on a quintic threefold X with normal bundle $N \cong \mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Since the Picard number of X equals one, R is numerically equivalent to a moving curve class but since $h^0(f^*N) = 0$ for any finite morphism f to R, no multiple of R deforms in X. Such R are called super rigid.

Combining Theorem 1.8 and the tables of Lyapunov exponents calculated in [EKZ, Appendix A], we list, in Appendix B, the limit s_μ of slopes of $\overline{\mathcal{T}}_{d,\mu}$ in each stratum $\mathcal{H}(\mu)$ for small g.

There is a similar slope problem for the partial compactification $\mathcal{A}_g^*$ of the moduli space of g-dimensional principally polarized abelian varieties. See Definition 4.2 of [Gr], which is a good general introduction to this topic. By work of Manni [Gr, Theorem 5.19], the lower bound for slopes of effective divisors on $\mathcal{A}_g^*$ is known to approach zero as g goes to infinity. In fact, there exists an effective divisor on $\mathcal{A}_g^*$ of slope at most

$$\frac{(2\pi)^2}{\big(2(g!)\zeta(2g)\big)^{1/g}}.$$

Since $\lim_{g\to\infty}\zeta(2g) = 1$ and $(g!)^{1/g}\sim g/e$, it is easy to check that the slope of this divisor is smaller than $576/5g$ for large g.

Corollary 3.4. *Assuming that $\lim_{g\to\infty} c_\mu = 2$ for $\mu = (1,\ldots,1,g)$, a divisor on $\mathcal{A}_g^*$ defined by a modular form of slope smaller than $576/5g$ must contain the image of $\mathcal{M}_g$ via the Torelli embedding $\mathcal{M}_g \hookrightarrow \mathcal{A}_g$ for sufficiently large g.*

Proof. We have seen that slopes of effective divisors on $\overline{\mathcal{M}}_g$ are bounded by $576/5g$ from below for $g \gg 0$, assuming that $\lim_{g\to\infty} c_\mu = 2$ for $\mu = (1,\ldots,1,g)$. If an effective divisor D on $\mathcal{A}_g^*$ does not contain $\mathcal{M}_g$, its restriction to $\mathcal{M}_g$ induces an effective divisor on $\overline{\mathcal{M}}_g$ with the same slope. Hence, the slope of D cannot be smaller than $576/5g$ for $g \gg 0$. ☐

4. Square-tiled surfaces with a unique zero

In this section we describe in detail some examples of square-tiled surfaces that have a unique zero.

Consider the partition $\mu = (2g-2)$. The covering set is reduced to

$$\mathrm{Cov}_{d,(2g-2)} = \{(\alpha, \beta) \in S_d \times S_d \mid \beta^{-1}\alpha^{-1}\beta\alpha \in (2g-1)(1)\cdots(1),\ \langle \alpha, \beta \rangle \text{ is transitive}\}/\sim .$$

By the description in Section 3, the fiber of the finite map $e : \mathcal{T}_{d,(2g-2)} \to \mathcal{M}_{1,1}$ can be identified with the equivalence classes of pairs in $\mathrm{Cov}_{d,(2g-2)}$. The degree $N_{d,(2g-2)}$ of e counts the number of non-isomorphic covers, hence we have

$$N_{d,(2g-2)} = |\mathrm{Cov}_{d,(2g-2)}|.$$

Example 4.1. Let $\alpha = (1234)(5)$ and $\beta = (15)(2)(3)(4)$ be two permutations on five letters. We have $\beta^{-1}\alpha^{-1}\beta\alpha = (154)(2)(3)$, so this monodromy pair yields a degree 5, genus 2 cover of a torus with a unique ramification point, whose corresponding square-tiled surface model is shown in Figure 8. It is an octagon

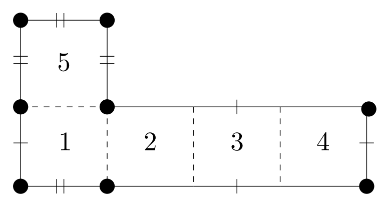

FIGURE 8.

whose edges with the same label are glued in pairs. The eight vertices are glued together as the unique zero. It is clear that along the horizontal direction, the permutation is $(1234)(5)$ and along the vertical direction, the permutation is $(15)(2)(3)(4)$.

We have seen that there are infinitely many genus g covers of a torus with a unique ramification point in the hyperelliptic locus. Veech [V1] showed that a hyperelliptic curve can be obtained by gluing the opposite sides of a centrally symmetric $2n$-gon. We want to study which monodromy pairs in $\mathrm{Cov}_{d,g}$ admit square-tiled surfaces that are hyperelliptic. Note that a genus g curve C is hyperelliptic if and only if C has an involution with $2g+2$ fixed point. A difficulty is that the hyperelliptic involution can never come from an automorphism of the covers parameterized by $\mathcal{T}_{d,(2g-2)}$.

Proposition 4.2. *A cover $\pi : C \to E$ parameterized by a point of $\mathcal{T}_{d,(2g-2)}$ has no automorphism of order two.*

Proof. Let (α, β) be a monodromy pair in $\mathrm{Cov}_{d,(2g-2)}$ associated to π. An order two automorphism of π corresponds to some non-trivial permutation τ such that $\tau(\alpha, \beta)\tau^{-1} = (\alpha, \beta)$ and $\tau^2 = id$. Let $\{1, \ldots, d\}$ be the d letters and assume that $\beta^{-1}\alpha^{-1}\beta\alpha = \gamma = (12\cdots(2g-1))(2g)\cdots(d)$. We get $\tau\gamma\tau^{-1} = \gamma$, hence $\tau(12\cdots(2g-1))\tau^{-1} = (12\cdots(2g-1))$. Then $\tau(i) = i + m \pmod{2g-1}$ for some m and $i = 1, \ldots, 2g-1$. But $\tau^2 = id$ implies $2m = 0 \pmod{2g-1}$, a contradiction. $\square$

Nevertheless, the elliptic curve E has involution ι and we can ask which monodromy pairs (α, β) yield covers with a compatible involution $\imath$:

$$\begin{tikzcd} C \arrow[r, "\imath"] \arrow[d, "\pi"'] & C \arrow[d, "\pi"] \\ E \arrow[r, "\iota"] & E \end{tikzcd}$$

Proposition 4.3. *A cover $\pi : C \to E$ admits an involution $\imath$ as in the above diagram if and only if there exists a non-trivial permutation τ such that $\tau(\alpha, \beta)\tau^{-1} = (\alpha^{-1}, \beta^{-1})$ and $\tau^2 = id$. For such τ, let n, n_a and n_b denote the numbers of letters in $\{1, \ldots, d\}$ fixed by τ, $\tau\alpha$ and $\tau\beta$, respectively. Let n_{ba} denote the number of letters fixed by both $\tau\beta\alpha$ and $\beta^{-1}\alpha^{-1}\beta\alpha$. Then the involution $\imath$ has $n + n_a + n_b + n_{ba} + 1$ fixed points. In particular, C is hyperelliptic if $n + n_a + n_b + n_{ba} = 2g + 1$.*

Proof. The composite map $\pi \circ \iota : C \to E$ has a unique ramification point, hence it corresponds to a cover π' parameterized by $\mathcal{T}_{d,(2g-2)}$. Since ι sends the basis (a,b) of the fundamental group $\pi_1(E,q)$ to (a^{-1},b^{-1}), the monodromy pair associated to π' is given by (α^{-1},β^{-1}). Note that π and π' are isomorphic if and only if there exists $\tau \in S_d$ such that $\tau(\alpha,\beta)\tau^{-1} = (\alpha^{-1},\beta^{-1})$. Moreover, $\tau^2 = id$ because the order of the induced automorphism is 2.

Take the standard torus E by gluing the parallel edges of $[0,1]^2$. We can realize C as a square-tiled surface by gluing d unit squares with respect to the monodromy pair (α,β). Note that ι has four 2-torsion points, one at the center of the square, one at the centers of the edges in each direction and one at the vertices. The fixed points of $\imath$ on the square-tiled surface C can only occur at these 2-torsion points. Mark them on the i-th unit square as shown in Figure 9.

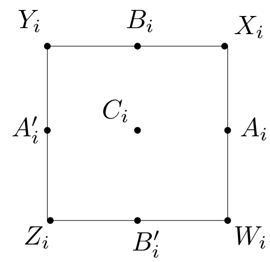

FIGURE 9.

The action $\imath$ sends the i-th square to the $\tau(i)$-th by parallel transport and then sends each point to its conjugate symmetric to the center of the square. We have $\imath(C_i) = C_{\tau_i}$, $\imath(A_i) = A'_{\tau(i)}$, $\imath(B_i) = B'_{\tau(i)}$, $\imath(X_i) = Z_{\tau(i)}$ and $\imath(Y_i) = W_{\tau(i)}$. Then C_i is fixed by $\imath$ if and only if i is fixed by τ. Recall that C is obtained by identifying the edges a_i with $a'_{\alpha(i)}$ and identifying b_i with $b'_{\beta(i)}$ by parallel transport. In this process, A_i is glued to $A'_{\alpha(i)}$ and B_i is glued to $B'_{\beta(i)}$. Hence, A_i is fixed by $\imath$ if and only if $\tau(i) = \alpha(i)$, namely, i is fixed by $\tau\alpha$ since $\tau = \tau^{-1}$. Similarly, B_i is fixed by $\imath$ if and only if i is fixed by $\tau\beta$. The vertices X_i, $Y_{\alpha(i)}$, $Z_{\beta\alpha(i)}$, $W_{\alpha^{-1}\beta\alpha(i)}$, $X_{\beta^{-1}\alpha^{-1}\beta\alpha(i)}, \ldots$ are identified as the same point on C. If $\beta^{-1}\alpha^{-1}\beta\alpha(i) = i$, we get an unramified point over the unique branch point. Since $\imath(X_i) = Z_{\tau(i)}$ and $\imath(Y_{\alpha(i)}) = W_{\tau\alpha(i)}$, this point is fixed by $\imath$ if and only if $\tau(i) = \beta\alpha(i)$ or $\alpha^{-1}\beta\alpha(i) = \tau\alpha(i)$. Namely, i is fixed by $\tau\beta\alpha$. If $\beta^{-1}\alpha^{-1}\beta\alpha(i) \neq i$, we get the unique ramification point on C, which is obviously fixed by $\imath$.

In sum, $\imath$ has n fixed points at the center of the d squares, n_a fixed points at the center of the vertical edges, n_b fixed points at the center of horizontal edges and $n_{ba}+1$ fixed points at the vertices. The total number of fixed points is equal to $n+n_a+n_b+n_{ba}+1$. If this equals $2g+2$, by Riemann-Hurwitz, $C \to C/\imath$ is a double cover of $\mathbb{P}^1$, hence C is hyperelliptic. □

Example 4.4. In Example 4.1, we have the monodromy pair $\alpha = (1234)(5)$ and $\beta = (15)(2)(3)(4)$. Take $\tau = (24)(1)(3)(5)$ and one can check that $\tau(\alpha,\beta)\tau^{-1} = (\alpha^{-1},\beta^{-1})$. By Proposition 4.3, C admits an involution $\imath$ compatible with the elliptic involution. The permutation $\tau = (24)(1)(3)(5)$ fixes the three letters 1, 3 and 5, $\tau\alpha = (14)(23)(5)$ fixes 5 and $\tau\beta = (15)(24)(3)$ fixes 3. Note that $\tau\beta\alpha = (145)(23)$ has no fixed letter. Overall, we get $n = 3$, $n_a = 1$, $n_b = 1$ and $n_{ba} = 0$, hence $\imath$ has six fixed points. This coincides with the fact that C is a genus two hyperelliptic curve with six Weierstrass points.

On the square-tiled surface model of C, we can visualize $\imath$ and its fixed points, as shown in Figure 10. Among the six Weierstrass points, three are at the centers of the squares, one is on a horizonal edge, one is on a vertical edge, and the last one is the unique zero of the 1-form by pulling back dz from the standard torus.

Example 4.5. Consider $d = 5$, $g = 3$ and the partition $\mu = (4)$. Let $\{1,2,3,4,5\}$ be the letters labeled on the five sheets of a cover parameterized by $\mathcal{T}_{5,(4)}$. Consider all possible $\alpha,\beta \in S_5$ such that $\beta^{-1}\alpha^{-1}\beta\alpha$ consists of a single cycle of length 5. By [C, Theorem 1.18], there is a group of actions generated by $h_\alpha : (\alpha,\beta) \to (\alpha,\alpha\beta)$ and $h_\beta : (\alpha,\beta) \to (\beta\alpha,\beta)$ on $\mathrm{Cov}_{d,g}$. Each orbit of the actions corresponds to an

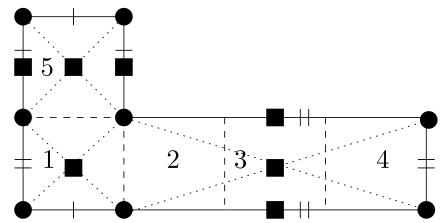

FIGURE 10.

irreducible component of $\mathcal{T}_{d,(2g-2)}$. A routine examination shows that $\mathrm{Cov}_{5,(4)}$ parameterizes 40 equivalence classes that fall into 4 orbits. Hence $\mathcal{T}_{5,(4)}$ has 4 irreducible components, as shown in Table 1.

Using the slope formula [C, Theorem 1.15], we find that

$$s(\mathcal{T}_{5,(4),1}) = s(\mathcal{T}_{5,(4),4}) = 28/3,$$

$$s(\mathcal{T}_{5,(4),2}) = s(\mathcal{T}_{5,(4),3}) = 9.$$

Since the hyperelliptic divisor $\overline{H}_3$ on $\overline{\mathcal{M}}_3$ has slope 9, it has negative intersection with $\mathcal{T}_{5,(4),1}$ and $\mathcal{T}_{5,(4),4}$, and has zero intersection with $\mathcal{T}_{5,(4),2}$ and $\mathcal{T}_{5,(4),3}$. This implies that covers parameterized by $\mathcal{T}_{5,(4),1}$ and $\mathcal{T}_{5,(4),4}$ are hyperelliptic. Hence $\mathcal{T}_{5,(4),1}$ and $\mathcal{T}_{5,(4),4}$ are irreducible, rigid curves on $\widetilde{M}_{0,8}$ with slope 28/3 as predicted in Corollary 1.7. The other two components $\mathcal{T}_{5,(4),2}$ and $\mathcal{T}_{5,(4),3}$ do not intersect $\overline{H}_3$. They map into the divisor of $\overline{\mathcal{M}}_3$ that parameterizes plane quartics with a hyperflex line.

We can also analyze the components from the viewpoint of square-tiled surfaces. Take a cover $\pi : C \to E$ corresponding to the case $\alpha = (12354)$ and $\beta = (12)(34)(5)$ in $\mathcal{T}_{5,(4),1}$. If $\tau = (12)(34)(5)$, then $\tau(\alpha, \beta)\tau^{-1} = (\alpha^{-1}, \beta^{-1})$ and we have $n = 1$, $n_a = 1$, $n_b = 5$ and $n_{ba} = 0$. Hence $n + n_a + n_b + n_{ba} = 7$, so by Proposition 4.3, C is hyperelliptic. The hyperelliptic involution of C and its eight Weierstrass points can be seen as in Figure 11.

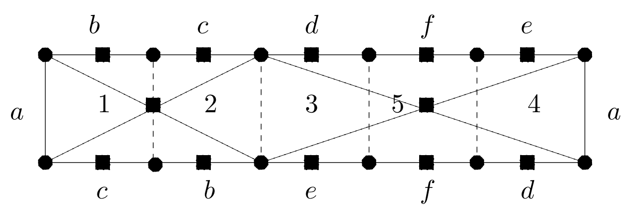

FIGURE 11.

Similarly for the case $\alpha = (1354)(2)$ and $\beta = (12)(34)(5)$ in $\mathcal{T}_{5,(4),4}$, the square-tiled surface has a hyperelliptic involution and its eight Weierstrass points are shown in Figure 12.

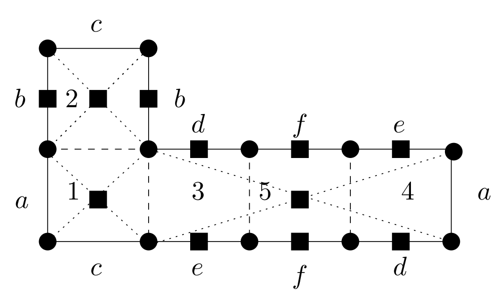

FIGURE 12.

Components of $\mathcal{T}_{5,(4)}$	α	β
$\mathcal{T}_{5,(4),1}$	(12)(35)(4)	(12345)
	(15)(23)(4)	(12)(34)(5)
	(12354)	(12)(34)(5)
$\mathcal{T}_{5,(4),2}$	(12)(34)(5)	(12345)
	(124)(3)(5)	(12345)
	(142)(3)(5)	(12345)
	(12453)	(12345)
	(13254)	(12345)
	(14)(25)(3)	(123)(4)(5)
	(12435)	(123)(4)(5)
	(13425)	(123)(4)(5)
	(135)(2)(4)	(12)(34)(5)
	(12345)	(12)(34)(5)
$\mathcal{T}_{5,(4),3}$	(1243)(5)	(12345)
	(1342)(5)	(12345)
	(15)(23)(4)	(1234)(5)
	(15)(34)(2)	(1234)(5)
	(1325)(4)	(1234)(5)
	(1352)(4)	(1234)(5)
	(1523)(4)	(1234)(5)
	(1253)(4)	(1234)(5)
	(12435)	(1234)(5)
	(14235)	(1234)(5)
	(1345)(2)	(12)(34)(5)
	(1534)(2)	(12)(34)(5)
$\mathcal{T}_{5,(4),4}$	(15)(24)(3)	(1234)(5)
	(135)(2)(4)	(1234)(5)
	(125)(34)	(1234)(5)
	(152)(34)	(1234)(5)
	(14)(23)(5)	(123)(45)
	(124)(3)(5)	(123)(45)
	(134)(2)(5)	(123)(45)
	(145)(23)	(123)(45)
	(1245)(3)	(123)(45)
	(1345)(2)	(123)(45)
	(1425)(3)	(123)(4)(5)
	(124)(35)	(123)(4)(5)
	(142)(35)	(123)(4)(5)
	(143)(25)	(12)(34)(5)
	(1354)(2)	(12)(34)(5)

TABLE 1.

Consider the case $\alpha = (12435)$ and $\beta = (123)(4)(5)$ in $\mathcal{T}_{5,(4),2}$. If τ satisfies $\tau(\alpha, \beta)\tau^{-1} = (\alpha^{-1}, \beta^{-1})$ and $\tau^2 = id$, then τ must equal (12)(45)(3). Note that $n = 1$, $n_a = 1$, $n_b = 1$ and $n_{ba} = 0$. By Proposition 4.3, the involution induced by τ is not hyperelliptic, but a double cover of an elliptic curve. So a covering curve in $\mathcal{T}_{5,(4),2}$ is not only a degree 5 cover of an elliptic curve with a unique ramification point, but also a double cover of another elliptic curve, as shown in Figure 13.

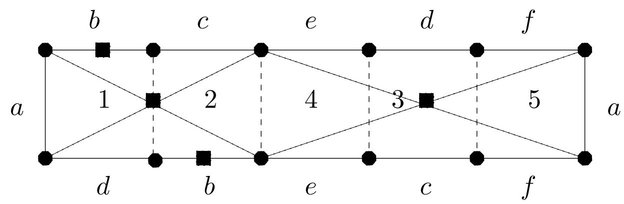

FIGURE 13.

For (α, β) parameterized by $\mathcal{T}_{5,(4),3}$, one can check that there does not exist such τ as in Proposition 4.3. So those covers do not admit an involution compatible with the elliptic involution. Consequently their square-tiled surface models do not have symmetry of order two.

5. Appendix A: Siegel-Veech constants

In this paper, we denote by c_μ the (area) Siegel-Veech constant of the stratum $\mathcal{H}(\mu)$, which equals $\frac{\pi^2}{3}c_{\mathrm{area}}$ in the context of [EKZ, §1.5]. In the proof of Theorem 1.8, we use an equality

$$c_\mu = \lim_{d\to\infty} \frac{M_{d,\mu}}{N_{d,\mu}}. \tag{4}$$

Here we sketch a proof of (4), which was explained to the author by Eskin. The reader may refer to [EKZ] and [EMZ] for broader discussions on Siegel-Veech constants.

We first fix some notation. Let $S_0 = (C, \omega)$ be a square-tiled surface that comes from a genus g standard torus covering $\pi : C \to E$ with a unique branch point and the ramification profile μ, where $\omega = \pi^{-1}(dz)$ and μ is a partition of $2g-2$. As a point in the stratum $\mathcal{H}(\mu)$ of Abelian differentials, S_0 has integer coordinates under the period map. The $\mathrm{SL}(2,\mathbb{R})$ action on the real and imaginary parts of ω induces an action on $\mathcal{H}(\mu)$. Let $\mathcal{H}_1(\mu)$ be the subset of surfaces of area one, which is $\mathrm{SL}(2,\mathbb{R})$ invariant in $\mathcal{H}(\mu)$. Let $\Gamma(S_0)$ be the Veech group of S_0, its stabilizer in $\mathrm{SL}(2,\mathbb{R})$. Use $\mathcal{O}(S_0)$ to denote the $\mathrm{SL}(2,\mathbb{Z})$ orbit of S_0. For simplicity, let $G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and η be the Haar measure on G normalized so that $\eta(G/\Gamma) = 1$.

Let $\phi : \mathcal{H}_1(\mu) \to \mathbb{R}$ be any L^1 function. Expressing the fundamental domain of $G/\Gamma(S_0)$ as a union of the fundamental domains of G/Γ yields the following standard equality

$$\int_{G/\Gamma(S_0)} \phi(gS_0)d\eta(g) = \sum_{S\in\mathcal{O}(S_0)} \int_{G/\Gamma} \phi(gS)d\eta(g). \tag{5}$$

Since ω induces a flat structure on S, we can talk about geodesics and cylinders on S with a fixed direction. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of compact support and define $\hat{f} : \mathcal{H}_1(\mu) \to \mathbb{R}$ by

$$\hat{f}(S) = \sum_{C\in\mathrm{Cyl}(S)} f(\vec{C})\mathrm{Area}(C),$$

where $\mathrm{Cyl}(S)$ is the set of cylinders on the flat surface S, $\vec{C} \in \mathbb{R}^2$ is the associated vector of $C \in \mathrm{Cyl}(S)$ and $\mathrm{Area}(C)$ is the area of the cylinder.

Let ν be the Lebesgue measure on $\mathcal{H}_1(\mu)$, cf. [EO, Definition 1.3]. We know that $\mathcal{H}_1(\mu)$ has finite volume under ν. By the Siegel-Veech formula [V2, (0.6)], the Siegel-Veech constants $c_{\mathrm{area}}(S_0)$ and c_μ have the property that for f as above,

$$\frac{1}{\eta(G/\Gamma(S_0))} \int_{G/\Gamma(S_0)} \hat{f}(gS)d\eta(g) = c_{\mathrm{area}}(S_0) \int_{\mathbb{R}^2} f, \tag{6}$$

and

$$\frac{1}{\nu(\mathcal{H}_1(\mu))} \int_{\mathcal{H}_1(\mu)} \hat{f}(S)d\nu(S) = c_\mu \int_{\mathbb{R}^2} f. \tag{7}$$

By [EKZ, Theorem 4], we know

$$c_{\text{area}}(S_0) = \frac{1}{|\mathcal{O}(S_0)|} \sum_{S \in \mathcal{O}(S_0)} w(S), \tag{8}$$

where $w(S)$ is the sum of height(C)/length(C) over all horizontal cylinders C on S.

In Section 3, we defined a set of equivalence classes $\mathrm{Cov}_{d,\mu}$ parameterizing degree d connected covers of a fixed torus with a unique branch point of ramification profile μ, as well as two summations

$$N_{d,\mu} = |\mathrm{Cov}_{d,\mu}|,$$

$$M_{d,\mu} = \sum_{S \in \mathrm{Cov}_{d,\mu}} w(S).$$

The action of the group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ on $\mathrm{Cov}_{d,\mu}$ is the same as the monodromy action [C, Theorem 1.18]. Each of its orbits corresponds to an irreducible component $\mathcal{T}_{d,\mu,i}$ of the Hurwitz space $\mathcal{T}_{d,\mu}$. Let Δ be a subset of $\mathrm{Cov}_{d,\mu}$ consisting of a single element from each orbit. Note that in our setting, we have $\eta(G/\Gamma(S_0)) = |\mathcal{O}(S_0)|$. By the equalities (5), (6) and (8), we have

$$\begin{aligned} \sum_{S \in \mathrm{Cov}_{d,\mu}} \int_{G/\Gamma} \hat{f}(gS) d\eta(g) &= \sum_{S_0 \in \Delta} \sum_{S \in \mathcal{O}(S_0)} \int_{G/\Gamma} \hat{f}(gS) d\eta(g) \\ &= \sum_{S_0 \in \Delta} \int_{G/\Gamma(S_0)} \hat{f}(gS) d\eta(g) \\ &= \sum_{S_0 \in \Delta} |\mathcal{O}(S_0)| c_{\text{area}}(S_0) \int_{\mathbb{R}^2} f \\ &= \sum_{S_0 \in \Delta} \sum_{S \in \mathcal{O}(S_0)} w(S) \int_{\mathbb{R}^2} f \\ &= M_{d,\mu} \int_{\mathbb{R}^2} f. \end{aligned} \tag{9}$$

By the argument of [EO, (3.2)], for such a function $\hat{f}$ on $\mathcal{H}_1(\mu)$ and any $g \in G$, we have

$$\lim_{d \to \infty} \frac{1}{N_{d,\mu}} \sum_{S \in \mathrm{Cov}_{d,\mu}} \hat{f}(gS) = \frac{1}{\nu(\mathcal{H}_1(\mu))} \int_{\mathcal{H}_1(\mu)} \hat{f}(S) d\nu(S). \tag{10}$$

By (9), (10) and $\eta(G/\Gamma) = 1$, we have

$$\begin{aligned} \lim_{d \to \infty} \frac{M_{d,\mu}}{N_{d,\mu}} \int_{\mathbb{R}^2} f &= \lim_{d \to \infty} \frac{1}{N_{d,\mu}} \sum_{S \in \mathrm{Cov}_{d,\mu}} \int_{G/\Gamma} \hat{f}(gS) d\eta(g) \\ &= \int_{G/\Gamma} \Big(\lim_{d \to \infty} \frac{1}{N_{d,\mu}} \sum_{S \in \mathrm{Cov}_{d,\mu}} \hat{f}(gS) \Big) d\eta(g) \\ &= \frac{1}{\nu(\mathcal{H}_1(\mu))} \int_{G/\Gamma} d\eta(g) \int_{\mathcal{H}_1(\mu)} \hat{f} d\nu(S). \\ &= \frac{1}{\nu(\mathcal{H}_1(\mu))} \int_{\mathcal{H}_1(\mu)} \hat{f} d\nu(S). \end{aligned} \tag{11}$$

The interchange of the limit and integral in (11) is justified by the dominated convergence theorem [EM, Theorem 2.2]. Now comparing (7) with (11), we obtain that

$$\lim_{d \to \infty} \frac{M_{d,\mu}}{N_{d,\mu}} = c_\mu.$$

6. Appendix B: Limits of slopes for $3 \le g \le 5$

By a computer program, Zorich calculated the Lyapunov exponents and Siegel-Veech constants for small g. The corresponding table can be found in [EKZ, Appendix A]. Applying Theorem 1.8, we list the limit of slopes

$$s_\mu = \lim_{d\to\infty} s(\mathcal{T}_{d,\mu})$$

in each stratum for small g. In Tables 2, 3 and 4, the first column lists connected components of strata $\mathcal{H}(\mu)$, and the second and third give exact values of and decimal approximations to s_μ, respectively.

Strata	s_μ	Approximations
$\mathcal{H}^{\rm hyp}(4)$	28/3	9.33333
$\mathcal{H}^{\rm odd}(4)$	9	9.
$\mathcal{H}^{\rm hyp}(2,2)$	28/3	9.33333
$\mathcal{H}^{\rm odd}(2,2)$	44/5	8.8
$\mathcal{H}(3,1)$	9	9.
$\mathcal{H}(2,1,1)$	98/11	8.90909
$\mathcal{H}(1,1,1,1)$	468/53	8.83019

Table 2. $g = 3$

Strata	s_μ	Approximations
$\mathcal{H}^{\rm hyp}(6)$	9	9.
$\mathcal{H}^{\rm even}(6)$	60/7	8.57143
$\mathcal{H}^{\rm odd}(6)$	108/13	8.30769
$\mathcal{H}^{\rm hyp}(3,3)$	9	9.
$\mathcal{H}^{\rm non-hyp}(3,3)$	33/4	8.25
$\mathcal{H}^{\rm even}(4,2)$	17/2	8.5
$\mathcal{H}^{\rm odd}(4,2)$	236/29	8.13793
$\mathcal{H}^{\rm even}(2,2,2)$	693/83	8.38554
$\mathcal{H}^{\rm odd}(2,2,2)$	8	8.
$\mathcal{H}(5,1)$	25/3	8.33333
$\mathcal{H}(4,1,1)$	3118/379	8.22691
$\mathcal{H}(3,2,1)$	41/5	8.2
$\mathcal{H}(3,1,1,1)$	65/8	8.125
$\mathcal{H}(2,2,1,1)$	8178/1009	8.10505
$\mathcal{H}(2,1,1,1,1)$	1052/131	8.03053
$\mathcal{H}(1,1,1,1,1,1)$	6675/839	7.9559

Table 3. $g = 4$

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Strata	s_μ	Approximations
$\mathcal{H}^{\text{hyp}}(8)$	44/5	8.8
$\mathcal{H}^{\text{even}}(8)$	8	8.
$\mathcal{H}^{\text{odd}}(8)$	148/19	7.78947
$\mathcal{H}^{\text{hyp}}(4,4)$	44/5	8.8
$\mathcal{H}^{\text{even}}(4,4)$	102/13	7.84615
$\mathcal{H}^{\text{odd}}(4,4)$	670629036/88307837	7.59422
$\mathcal{H}^{\text{even}}(6,2)$	1402948/178429	7.86278
$\mathcal{H}^{\text{odd}}(6,2)$	176/23	7.65217
$\mathcal{H}^{\text{even}}(4,2,2)$	811372/104943	7.73155
$\mathcal{H}^{\text{odd}}(4,2,2)$	2022416/269051	7.51685
$\mathcal{H}^{\text{even}}(2,2,2,2)$	998/131	7.61832
$\mathcal{H}^{\text{odd}}(2,2,2,2)$	2630076/355309	7.40222
$\mathcal{H}(7,1)$	18876/2423	7.79034
$\mathcal{H}(6,1,1)$	456084222/59332837	7.68688
$\mathcal{H}(5,3)$	209/27	7.74074
$\mathcal{H}(5,2,1)$	34386/4493	7.65324
$\mathcal{H}(5,1,1,1)$	2344/309	7.58576
$\mathcal{H}(4,3,1)$	3350523/438419	7.64229
$\mathcal{H}(4,2,1,1)$	4880938/646039	7.55518
$\mathcal{H}(4,1,1,1,1)$	4797996/640763	7.48794
$\mathcal{H}(3,3,2)$	466796/61307	7.61407
$\mathcal{H}(3,3,1,1)$	358044/47435	7.5481
$\mathcal{H}(3,2,2,1)$	45537/6049	7.52802
$\mathcal{H}(3,2,1,1,1)$	20893/2800	7.46179
$\mathcal{H}(3,1,1,1,1,1)$	15537/2101	7.39505
$\mathcal{H}(2,2,2,1,1)$	54114/7271	7.44244
$\mathcal{H}(2,2,1,1,1,1)$	1967748/266761	7.37645
$\mathcal{H}(2,1,1,1,1,1,1)$	37451/5123	7.31037
$\mathcal{H}(1,1,1,1,1,1,1,1)$	569332/78587	7.24461

TABLE 4. $g = 5$

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