Exercise 1. Consider the function \( h : \mathbb{R} \to \mathbb{R} \) defined by
\[
h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}
\]
(1) Show that for any \( c \neq 0 \) you can find three sequences \((x_n)_{n \in \mathbb{N}}\) \((y_n)_{n \in \mathbb{N}}\) and \((z_n)_{n \in \mathbb{N}}\) with \( \lim x_n = \lim y_n = \lim z_n = c \) such that \( \lim h(x_n) = 0 \) \( \lim h(y_n) = c \) and the series \( (h(z_n)) \) does not converge.

Proof. Let \( c \neq 0 \). For each \( n \in \mathbb{N} \) consider the open interval \( I_n = (c - \frac{1}{n}, c + \frac{1}{n}) \). Since the irrationals and rationals numbers are both dense in \( \mathbb{R} \) there is an irrational number \( x_n \in I_n \) and a rational number \( y_n \in I_n \). We thus get two sequences \((x_n)\) and \((y_n)\) and since \( |x_n - c| < \frac{1}{n} \) we get that \( x_n \to c \) and similarly \( y_n \to c \). Now \( h(x_n) = 0 \) for all \( n \) so \( \lim h(x_n) = 0 \) and \( h(y_n) = y_n \) so \( \lim h(y_n) = \lim y_n = c \). Finally let \( z_n = \begin{cases} x_n & n \text{ is odd} \\ y_n & n \text{ is even} \end{cases} \). Then \( z_n \to c \) as well but \( \lim h(z_n) \) does not exist since it has two subsequences \((h(z_{2n}))\) and \((h(z_{2n+1}))\) converging to different limits. \( \square \)

(2) On the other hand, show that any sequence \((x_n)_{n \in \mathbb{N}}\) with \( \lim x_n = 0 \) satisfies that \( \lim_{n \to \infty} h(x_n) = 0 \).

Proof. Assume that \( x_n \to 0 \). Then for any \( \epsilon > 0 \) there is \( N \in \mathbb{N} \) such that \( |x_n| < \epsilon \) for all \( n > N \). But then \( |h(x_n)| < \epsilon \) for all \( n > N \) (since if \( x_n \) is rational then \( |h(x_n)| = |x_n| < \epsilon \) and if \( x_n \) is irrational then \( |h(x_n)| = 0 < \epsilon \)). \( \square \)

Exercise 2. In each of the following cases prove the following limit formulas directly from the definition

(1) \( \lim_{x \to 2} (2x + 4) = 8 \)
For any \( \epsilon > 0 \) let \( \delta = \frac{\epsilon}{2} \) then if \( |x - 2| < \delta \) then \( |2x + 4 - 8| = |2(x - 2)| < 2\delta = \epsilon. \)

(2) \( \lim_{x \to 0} x^3 = 0 \)
For any \( \epsilon > 0 \) let \( \delta = \epsilon^{1/3} \). Then if \( |x| < \delta \) then \( |x^3 - 0| = |x|^3 < \delta^3 = \epsilon. \)

(3) \( \lim_{x \to 2} x^3 = 8 \)
For any \( \epsilon > 0 \) let \( \delta = \min \{1, \epsilon/12\} \). Using the identity \( x^3 - 8 = (x - 2)(x^2 + 2x + 4) \) and noting that if \( |x - 2| < 1 \) then \( 1 < x < 3 \) and \( x^2 + 2x + 4 < 12 \) we get that if \( |x - 2| < \delta \) then \( |x^3 - 8| < 12\delta = \epsilon \).

(4) \( \lim_{x \to \pi} x = 3 \) where the floor function \( \lfloor x \rfloor \) denotes the greatest integer smaller or equal to \( x \) (for example \( \lfloor 2.2 \rfloor = 2 \)).
Recall that \( 3.14 < \pi < 3.15 \) in particular, for any \( \epsilon > 0 \) if \( |x - \pi| < \delta = 0.1 \) then \( 3.04 < x < 3.05 \) and \( |x| = 3 \) so \( |\lfloor x \rfloor - 3| = 0 < \epsilon \).
Exercise 3. Use the sequential criterion for functional limits to show that following limits do not exist

1. \( \lim_{x \to 0} \frac{|x|}{x} \)
   - Let \( x_n = \frac{1}{n} \) and \( y_n = -\frac{1}{n} \) then \( \lim x_n = \lim y_n = 0 \) but \( \lim \frac{|x_n|}{x_n} = 1 \) and \( \lim \frac{|y_n|}{y_n} = -1 \) so the limit \( \lim_{x \to 0} \frac{|x|}{x} \) does not exist.

2. \( \lim_{x \to 1} h(x) \)
   - This follows from the fact that there is a sequence \( z_n \to 1 \) where \( \lim h(z_n) \) does not exist.

Exercise 4. Let \( f, g : A \to \mathbb{R} \) be two functions and let \( c \in A' \) a limit point of \( A \). Assume that \( f(x) \geq g(x) \) for all \( x \in A \) and that \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \). Show that \( L \geq M \).

Proof. Let \( (x_n) \) be a sequence in \( A \) with \( x_n \not\to c \) and \( \lim x_n = c \). Then \( \lim f(x_n) = L \) and \( \lim g(x_n) = M \). Since \( f(x_n) \geq g(x_n) \) for all \( n \) we get that \( L \geq M \). \( \square \)

Exercise 5. Let \( f, g : A \to \mathbb{R} \) be two functions and let \( c \in A' \) a limit point of \( A \). Assume that \( f(x) \) is bounded, that is, that there is some \( M > 0 \) with \( |f(x)| \leq M \) for all \( x \in A \). Show that if \( \lim_{x \to c} g(x) = 0 \) then \( \lim_{x \to c} (f(x)g(x)) = 0 \) as well.

Proof. For any \( \epsilon > 0 \) let \( \delta > 0 \) be such that if \( |x - c| < \delta \) then \( |g(x)| < \epsilon/M \). Then if \( |x - c| < \delta \) then \( |f(x)g(x) - 0| \leq M|g(x)| < \epsilon \). \( \square \)

Exercise 6 (Challenge question). Consider the function \( t : \mathbb{R} \to \mathbb{R} \) defined by

\[
t(x) = \begin{cases} 
1 & x = 0 \\
\frac{1}{q} & x = \frac{p}{q} \text{ with } p \in \mathbb{Z}, \ q \in \mathbb{N} \text{ in lowest terms} \\
0 & x \not\in \mathbb{Q}
\end{cases}
\]

You will now prove that for any \( x \in \mathbb{R} \) one has \( \lim_{x \to c} t(x) = 0 \).

1. For any \( M > 0 \) let
   \[
   \{ Q_M = \{ \frac{p}{q} : 0 < q < M \text{ and } |p| < (|c| + 1)q \text{ in lowest terms} \}
   \]
   Show that this set has finitely many elements.
   
   Proof. Since any \( \frac{p}{q} \) in \( Q_M \) satisfies \( 0 < q < M \) and \( |p| < (|c| + 1)q < (|c| + 1)M \) then \( Q_M \) has at most \( M^2(|c| + 1) \) elements. \( \square \)

2. Let \( \delta_1 = \min\{|r - c| : r \in Q_M, \ r \not\in c\} \) and explain why \( \delta_1 > 0 \) must be strictly positive.
   
   Proof. This is a finite set of positive elements (since \( |r - c| > 0 \) whenever \( r \not\in c \)) so their minimum is positive. \( \square \)
(3) Let $\delta = \min\{\delta_1, 1\}$ and show that if $|x - c| < \delta$ and $x \neq c$ then $|t(x)| < \frac{1}{M}$

Proof. Let $x \in \mathbb{R}$ and assume that $0 < |x - c| < \delta$. If $x$ is irrational then $|t(x)| = 0 < \frac{1}{M}$ so it is enough to consider the case where $x \in \mathbb{Q}$. For $x \in \mathbb{Q}$ write $x = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in reduced form. If $q > M$ then $t(q) = \frac{1}{q} < M$ and we are done. We will now show that the assumption $0 < q \leq M$ leads to a contradiction. Indeed since $|\frac{p}{q} - c| < \delta \leq 1$ we have that $|\frac{p}{q}| < |c| + 1$ so if $0 < q < M$ then $\frac{p}{q} \in Q_M$. But this contradicts the bound $|\frac{p}{q} - c| < \delta \leq \delta_1 = \min\{|r - c| : r \in Q_M, r \neq c\}$.\hfill $\Box$

(4) Show that indeed $\lim_{x \to c} t(x) = 0$ (using the $\epsilon, \delta$ definition).

Proof. Given $\epsilon > 0$ let $M > 0$ with $\frac{1}{M} < \epsilon$. Let $\delta > 0$ be as above, then if $|x - c| < \delta$ we showed that $|t(x)| < \frac{1}{M} < \epsilon$ so we are done.\hfill $\Box$

(5) At what points is the function $t(x)$ continuous?

We have that $\lim_{x \to c} t(x) = 0$ at all points so the function is continuous only at points for which $t(x) = 0$, that is it is continuous at all $x \in \mathbb{R} \setminus \mathbb{Q}$.

\[\text{Hint: Consider the cases when } x = \frac{p}{q} \text{ is rational and } x \text{ is irrational separately}\]