Exercise 1. Solve the Dirichlet problem on an Annulus with Dirichlet boundary conditions
\[ \nabla^2 u = 0 \quad 1 < r < 2, \quad 0 < \theta < 2\pi \\
\begin{align*}
u(1, \theta) &= \sin(2\theta) & 0 < \theta < 2\pi \\
u(2, \theta) &= \sin(2\theta) & 0 < \theta < 2\pi
\end{align*} \]

Exercise 2. Solve the Dirichlet problem on an Annulus with mixed Dirichlet and Neumann boundary conditions
\[ \nabla^2 u = 0 \quad 1 < r < 2, \quad 0 < \theta < 2\pi \\
\begin{align*}
u(1, \theta) &= 1 & 0 < \theta < 2\pi \\
u_r(2, \theta) &= \sin(\theta) & 0 < \theta < 2\pi
\end{align*} \]
(a) Is the solution unique?
(b) Will there still be a solution if we change the BC on the outer boundary to \( u_r(2, \theta) = 2 \cos^2(\theta) \)\

Exercise 3. Find the solution to the following inhomogenous BVP
\[ \nabla^2 u = 3r^4 \sin(2\theta) \quad 0 < r < 1, \quad 0 < \theta < 2\pi \\
\begin{align*}
u(1, \theta) &= 0 & 0 < \theta < 2\pi
\end{align*} \]
Hint: Compute the Laplacian of \( f(x, y) = (xy)^3 \) and represent it in polar coordinates.

Exercise 4. Legendre’s polynomials \( P_n(s) \) are solutions to Legendre’s differential equation of order \( n \),
\[ (1 - x^2)P'' - 2xP' + n(n + 1)P = 0, \quad -1 < x < 1, \]
and are given by Rodrigues’ formula
\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n], \]
Use this formula it to compute \( P_0, P_1, P_2, P_3 \), and verify that they indeed solve the corresponding Legendre’s equation.

Exercise 5. Show that if \( P_n(x) \) are Legendre’s polynomials defined above, then the function \( K(\phi) = P_n(\cos(\phi)) \) is a solution to the ODE
\[ [\sin(\phi)K']' + n(n + 1)\sin(\phi)K = 0, \quad -1 < x < 1, \]

Exercise 6. Solve the Dirichlet problem in spherical coordinates in a ball of radius 1, given by
\[ \nabla^2 u = 0 \quad 0 < r < 1 \\
u(1, \theta, \phi) = \cos(3\phi) \quad 0 < \phi < \pi \]
Hint: First use trig formulas to express \( \cos(3\phi) \) as a polynomial in \( \cos(\phi) \), and then as a linear combination of \( P_0(\cos(\phi)), P_1(\cos(\phi)), P_2(\cos(\phi)) \) and \( P_3(\cos(\phi)) \), with \( P_n \) the Legendre polynomials.