MT441 Homework 2

Due Friday Feb 1, 2019

Exercise 1. Solve the PDE \( u_t = u_{xx} \) for \( 0 \leq x \leq 1 \) with conditions
\[
 u(0, t) = 0, \quad u(1, t) = 1, \quad u(x, 0) = \sin(\pi x) + x.
\]
by first separating the solution into a stable solution and a transient one.

Solution.

Exercise 2. Consider the PDE \( u_t = u_{xx} \) for \( 0 \leq x \leq L \) with BC
\[
 u(0, t) = T_1, \quad u(L, t) + u_x(L, t) = T_2.
\]
Write down the proper choice of a stable solution \( S(x) \) such that the transient solution \( V(x, t) = U(x, t) - S(x) \) will satisfy homogenous BC (there is no need to find the solution for \( V(x, t) \)).

Solution.

Exercise 3. Fix a constant \( h \) and let \( 0 < \lambda_1 < \lambda_2 < \cdots \) be the positive solutions of the equation \( \tan(\lambda) = h\lambda \).

(a) Show that the functions \( \sin(\lambda_n x) \) are orthogonal on \([0, 1] \) and that
\[
 \int_0^1 \sin^2(\lambda_n x) \, dx = \frac{1 - h \cos^2(\lambda_n)}{2}.
\]

(b) Using integration by parts together with the relation \( \tan(\lambda_n) = h\lambda_n \), show that
\[
 \int_0^1 x \sin(\lambda_n x) \, dx = \frac{(h - 1) \cos(\lambda_n)}{\lambda_n}.
\]

Solution.

Exercise 4. Assume that \( h \neq 1 \). Use the previous problem to solve the heat equation \( u_t = u_{xx} \) with boundary conditions
\[
 u(0, t) = 0, \quad u(1, t) - hu_x(1, t) = 0
\]
and initial condition \( u(x, 0) = x \).

Solution.

Exercise 5. Use the method of Eigenfunction Expansions to solve the PDE \( u_t = u_{xx} + \sin(\pi x) \) with BC conditions
\[
 u(0, t) = u(1, t) = 0
\]
and initial conditions
\[
 u(x, 0) = 0.
\]

Solution.

\[ \text{Hint: try } S(x) = A(x - \frac{L}{2}) + B \frac{x}{L} \text{ for suitable } A \text{ and } B \]