MT441 Homework 2

Solutions

Exercise 1. Solve the PDE $u_t = u_{xx}$ for $0 \leq x \leq 1$ with conditions

$$u(0, t) = 0, \quad u(1, t) = 1, \quad u(x, 0) = \sin(\pi x) + x.$$ 

Solution. The steady-state solution is $s(x) = x$. The transient solution $v(x, t) = u(x, t) - s(x)$ satisfies the conditions

$$v(0, t) = v(1, t) = 0, \quad v(x, 0) = \sin(\pi x).$$

Therefore

$$v(x, t) = e^{-\pi^2 t} \sin(\pi x),$$

so

$$u(x, t) = e^{-\pi^2 t} \sin(\pi x) + x.$$ 

Exercise 2. Consider the PDE $u_t = u_{xx}$ for $0 \leq x \leq L$ with BC

$$u(0, t) = T_1, \quad u(L, t) + u_x(L, t) = T_2.$$ 

Write down the proper choice of a stable solution $S(x)$ such that the transient solution $V(x, t) = U(x, t) - S(x)$ will satisfy homogenous BC (there is no need to find the solution for $V(x, t)$).

Solution. We are looking for a steady state solution of the form $S(x) = A(1 - \frac{x}{L}) + B\frac{x}{L}$ that satisfies the same BC, that is $S(0) = T_1$ and $S(L) + S'(L) = T_2$. Now $S(0) = A$ so $A = T_1$ and $S(L) + S'(L) = B + \frac{B-A}{L}$ so $B = \frac{LT_2+T_1}{L+1}$.

Remark: Note that these BC correspond to a rod of length $L$ that on one side is forced to have temp $T_1$ and on the other side is in a bath of temp $T_2$. Close to the left edge on one hand the bath pulls the temp towards $T_1$ and on the other hand heat flows along the rod pulling the temp towards $T_2$. If the rod is very long, the second aspect is less important and so $B = \frac{LT_2+T_1}{L+1}$ is close to $T_2$, but if the rod is very short, the heat will flow fast and $B = \frac{LT_2+T_1}{L+1}$ is close to $T_1$.

Exercise 3. Fix a constant $h$ and let $0 < \lambda_1 < \lambda_2 < \cdots$ be the positive solutions of the equation $\tan(\lambda) = h\lambda$.

(a) Show that the functions $\sin(\lambda_n x)$ are orthogonal on $[0, 1]$ and that

$$\int_0^1 \sin^2(\lambda_n x) \, dx = \frac{1 - h \cos^2(\lambda_n)}{2}.$$ 

(b) Using integration by parts together with the relation $\tan(\lambda_n) = h\lambda_n$, show that

$$\int_0^1 x \sin(\lambda_n x) \, dx = \frac{(h - 1) \cos(\lambda_n)}{\lambda_n}.$$
**Solution.** To simplify notation let’s write \( s_n = \sin(\lambda_n) \) and \( c_n = \cos(\lambda_n) \) and remember that \( s_n = h\lambda nc_n \). For \( n \neq m \) we compute

\[
\int_0^1 \sin(\lambda_n x) \sin(\lambda_m x) \, dx = \frac{1}{2} \int_0^1 \left[ \cos(\lambda_n - \lambda_m)x - \cos(\lambda_n + \lambda_m)x \right] \, dx
\]

\[
= \frac{1}{2} \left[ \frac{\sin(\lambda_n - \lambda_m)}{\lambda_n - \lambda_m} - \frac{\sin(\lambda_n + \lambda_m)}{\lambda_n + \lambda_m} \right]
\]

\[
= \frac{1}{2} \left[ \frac{\sin(\lambda_n c_m - s_m c_n) - \sin(\lambda_n c_m + s_m c_n)}{\lambda_n - \lambda_m} \right]
\]

\[
= \frac{1}{\lambda_n^2 - \lambda_m^2} (\lambda_n \sin(\lambda_n c_m) - \lambda_m \sin(\lambda_m c_m)) = 0,
\]

and

\[
\int_0^1 \sin^2(\lambda_n x) \, dx = \frac{1}{2} \int_0^1 [1 - \cos(2\lambda_n x)] \, dx = \frac{1}{2} \left[ 1 - \frac{2s_n c_n}{2\lambda_n} \right] = \frac{1-hc_n^2}{2}.
\]

Now for the second part

\[
\int_0^1 x \sin(\lambda_n x) \, dx = \int_0^1 x \left[ \frac{\cos(\lambda_n x)}{\lambda_n} \right]' \, dx = \frac{s_n - c_n \lambda_n}{\lambda_n^2}
\]

and plugging in \( s_n = h\lambda nc_n \) we get

\[
\int_0^1 x \sin(\lambda_n x) \, dx = \frac{(h-1)c_n}{\lambda_n} = \frac{(1-1/h)s_n}{\lambda_n^2}
\]

\( \square \)

**Exercise 4.** Assume that \( h \neq 1 \). Use the previous problem to solve the heat equation \( u_t = u_{xx} \) with boundary conditions

\[
u(0, t) = 0, \quad u(1, t) - h u_x(1, t) = 0
\]

and initial condition \( u(x, 0) = x \).

**Solution.** For a separated solution \( u(x, t) = F(x)G(t) \) we have

\[
G' = kG \quad \text{and} \quad F'' = kF
\]

The first equation says \( G = ce^{kt} \). We must have \( k = -\lambda^2 \leq 0 \) since the heat does not go to infinity. If \( \lambda = 0 \) then \( F(x) = Ax + B \). The boundary conditions say that \( 0 = F(0) = B \) while \( 0 = F(1) - hA = A(1 - h) \) since \( h \neq 1 \) we have no nontrivial solution with \( \lambda = 0 \).

If \( \lambda > 0 \) then \( F(x) = A \cos(\lambda x) + B \sin(\lambda x) \). The boundary conditions say that \( A = 0 \) and \( \sin(\lambda) - h \cos(\lambda) = 0 \), so \( \lambda = \lambda_n \) for some \( n = 1, 2, 3, \ldots \). By the previous problem the functions \( \sin(\lambda_n x) \) are orthogonal and the solution satisfying \( u(x, 0) = x \) is

\[
u(x, t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \cdot \sin(\lambda_n x),
\]

where

\[
B_n = \frac{2}{1 - hc_n^2} \cdot \int_0^1 x \sin(\lambda_n x) \, dx = \frac{2}{1 - hc_n^2} \cdot \frac{c_n}{\lambda_n} (h-1).
\]

\( \square \)

**Remark:** Why did we assume that \( h \neq 1 \)?

If \( h = 1 \) then the last calculation shows that \( B_n = 0 \) for all \( n \geq 1 \). But \( u(x, t) = x \) is a solution (!?). Indeed, when we set \( u(x, t) = F(x)G(t) \), and separate variable as \( F''/F = k = G'/G \), the boundary conditions \( u(0, t) = 0 \) and \( u(1, t) = u_x(1, t) \)
imply that $F(0) = 0$ and $F(1) = F'(1)$, so that $F(x) = Cx$, where $C$ is a constant. Hence $F'' = 0$ so $k = 0$ and $G = constant$. Thus we get the new solution $u(x,t) = Cx$, corresponding to $\lambda = 0$.

So if $h = 1$ we must therefore enlarge the family $\{\sin(\lambda_n x) : n = 1, 2, 3, \ldots \}$ to include the function $x$ for $n = 0$. You can check that
\[ \int_0^1 x \sin(\lambda_n x) \, dx = 0 \]
for all $n = 1, 2, \ldots$, so this enlarged family is still orthogonal.

**Exercise 5.** Use the method of Eigenfunction Expansions to solve the PDE $u_t = u_{xx} + \sin(\pi x)$ with conditions
\[ u(0,t) = u(1,t) = u(x,0) = 0. \]

**Solution.** The recipe for solving $u_t = u_{xx} + f(x,t)$ with zero BC and IC is to first find the Fourier coefficients
\[ h_n(t) = 2 \int_0^1 f(x) \sin(n\pi x) \, dx, \]
then
\[ u(x,t) = \sum_{n=1}^\infty H_n(t) \sin(n\pi x), \]
where
\[ H_n(t) = e^{-(n\pi)^2 t} * h_n(t) = e^{-(n\pi)^2 t} \int_0^t e^{(n\pi)^2 \tau} h_n(\tau) \, d\tau. \]

In the problem at hand, the function $f(x,t) = \sin(\pi x)$ is already expressed as a Fourier series; namely $h_1(t) = 1$ and $h_n(t) = 0$ for $n > 1$. So
\[ u(x,t) = H_1(t) \sin(\pi x), \]
where
\[ H_1(t) = e^{-\pi^2 t} \int_0^t e^{\pi^2 \tau} \, d\tau = \frac{1}{\pi^2} \left[ 1 - e^{-\pi^2 t} \right]. \]