MT441 Homework 3

Name:

Due Feb 15 in class

In the following exercises the notation $F[f]$ (without subscript $s$ or $c$) denotes the complex Fourier transform

$$F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx.$$  

[See Lesson 11 in the text.]

Exercise 1. Show that $F[f'] = i\omega F[f]$, and that $F[f''] = -\omega^2 F[f]$.

Exercise 2. Let $0 < a < b$ be given positive numbers. Compute the Fourier transforms $F_s[f], F_c[f], F[f]$, for the function

$$f(x) = \begin{cases} 
1 & \text{for } a \leq x \leq b \\
0 & \text{otherwise}
\end{cases}$$

Exercise 3. Find two functions $f(x)$ and $g(x)$ such that $F[f \cdot g] \neq F[f] \cdot F[g]$, where $f \cdot g$ is the usual product of functions given by $(f \cdot g)(x) = f(x)g(x)$.

[Suggestion: Try functions like those Exercise 2, supported on disjoint intervals.]

Exercise 4. The convolution product of two functions $f, g$ on the whole real line is defined by

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi) g(\xi) \, d\xi.$$  

Show that

$$F[f * g] = F[f] \cdot F[g].$$

Exercise 5. The purpose of this exercise is to demonstrate the method of power series solution to ODE’s. We will do it by considering the familiar ODE $u'' + \lambda u = 0$ and find a solution in the form of a power series $u(x) = \sum_{n=0}^{\infty} a_n x^n$.

(a) Show that if $u(x)$ is given by such a power series then

$$u'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \text{ and } u''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$  

(b) Use this to show that if $u(x)$ solves the ODE $u'' + \lambda^2 u = 0$ the coefficients must satisfy the recursion relation

$$a_{n+2} = -\frac{\lambda^2}{(n+1)(n+2)} a_n.$$  

(c) Show (e.g., by induction) that the even coefficients are given by $a_{2n} = (-1)^n \frac{\lambda^{2n}}{(2n)!} a_0$ and the odd ones are given by $a_{2n+1} = (-1)^n \frac{\lambda^{2n}}{(2n+1)!} a_1$.

(d) Write down the most general solution as an explicit power series. Do you identify the functions these power series define?

\footnote{This exercise and the next are propaganda for the convolution product.}