MT441 Homework 4

Solution

Exercise 1. Solve the heat equation \( u_t = u_{xx} \) for a rod which is infinite in both directions with initial condition
\[
\phi(x) = \begin{cases} 
1 & \text{for } -1 \leq x \leq 1 \\
0 & \text{otherwise.}
\end{cases}
\]
Express your solution in terms of the error function
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi.
\]

Solution. Let \( U(t) = U(t, \omega) = \mathcal{F}[u] \), where the Fourier transform is taken with respect to \( x \). Applying \( \mathcal{F} \) to both sides of the equation \( u_t = u_{xx} \) and using Exercise 4, we get
\[
U'(t) = -\omega^2 U(t),
\]
so
\[
U(t) = U(0) e^{-\omega^2 t} = \mathcal{F}[\phi] \cdot e^{-\omega^2 t}.
\]
From the tables (we have seen in class) we have
\[
e^{-\omega^2 t} = \mathcal{F} \left[ \frac{1}{\sqrt{2\pi}} e^{-(x^2/4t)} \right],
\]
so
\[
\mathcal{F}[u] = U(t) = \mathcal{F}[\phi] \cdot \mathcal{F} \left[ \frac{1}{\sqrt{2\pi}} e^{-(x^2/4t)} \right] = \mathcal{F} \left[ \phi * \frac{1}{\sqrt{2\pi}} e^{-(x^2/4t)} \right].
\]
Taking the inverse transform of both sides gives
\[
u(x, t) = \phi(x) * \frac{1}{\sqrt{2\pi}} e^{-(x^2/4t)},
\]
for a general initial condition \( \phi(x) \). For the \( \phi(x) \) given in this problem we compute the solution
\[
u(x, t) = \phi(x) * \frac{1}{\sqrt{2\pi}} e^{-(x^2/4t)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\xi) \cdot \frac{1}{\sqrt{2\pi}} e^{-(x-\xi)^2/4t} d\xi = \frac{1}{2\sqrt{\pi t}} \int_{-1}^{1} e^{-(x-\xi)^2/4t} d\xi
\]
\[
= \frac{1}{\sqrt{\pi}} \int_{(x-1)/2\sqrt{t}}^{(x+1)/2\sqrt{t}} e^{-s^2} ds = \frac{1}{2} \left[ \text{erf} \left( \frac{x+1}{2\sqrt{t}} \right) - \text{erf} \left( \frac{x-1}{2\sqrt{t}} \right) \right].
\]
As an illustration, the solution \( u(x, t) \) is shown for \( t = .001, .01, 1, 3, 6, 1.0 \), cooling from deep red to light blue. Note how the solution for small \( t \) approximates the initial condition.
Exercise 2. Use the sine or cosine transform to solve the following problem

\[
\begin{align*}
  u_t &= u_{xx} & 0 < x < \infty, & 0 < t < \infty \\
  u_x(0, t) &= 0 & 0 < t < \infty \\
  u(x, 0) &= \begin{cases} 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}
\end{align*}
\]

Solution. Recall that the sine and cosine transforms \( F_s \) and \( F_c \) satisfy that

\[
\begin{align*}
  F_s[f''] &= \frac{2}{\pi} \omega f(0) - \omega^2 F_s[f], \\
  F_c[f''] &= -\frac{2}{\pi} \omega f'(0) - \omega^2 F_c[f].
\end{align*}
\]

so since our boundary conditions at zero involve \( u_x(0, t) \) we should use the cosine transform.

Now letting

\[
U(\omega, t) = F_c[u] := \frac{2}{\pi} \int_0^\infty u(x, t) \cos(\omega x) \, dx,
\]

and inverting the equation we get

\[
U_t(\omega, t) = -\frac{2}{\pi} \omega u_x(0, t) - \omega^2 U(\omega, t) = -\omega^2 U(\omega, t),
\]

where we used the BC: \( u_x(0, t) = 0 \). We also invert the IC to get:

\[
\begin{align*}
  U(\omega, 0) &= \frac{2}{\pi} \int_0^\infty u(x, 0) \cos(\omega x) \, dx \\
  &= \frac{2}{\pi} \int_0^1 \cos(\omega x) \, dx = \frac{2 \sin(\omega)}{\pi \omega}.
\end{align*}
\]
We now have the ODE for the function $U(\omega, t)$ in the $t$ variable:

\[
U' = -\omega^2 U
U(0) = \frac{2 \sin(\omega)}{\pi \omega},
\]

whose solution is $U(\omega, t) = \frac{2 \sin(\omega)}{\pi \omega} e^{-\omega^2 t}$.

The last step is inverting this transform. The function $\frac{2 \sin(\omega)}{\pi \omega} e^{-\omega^2 t}$ is not in our table of transforms so we need to work a bit. Using the inversion formula we can write

\[
u(x, t) = \int_0^\infty U(\omega, t) \cos(\omega x) d\omega
= \frac{2}{\pi} \int_0^\infty \frac{e^{-\omega^2 t}}{\omega} \sin(\omega) \cos(\omega x) d\omega
= \frac{1}{\pi} \int_0^\infty \frac{e^{-\omega^2 t}}{\omega} (\sin(\omega(1 + x)) + \sin(\omega(1 - x))) d\omega
= \frac{1}{\pi} \int_0^\infty \frac{e^{-\omega^2 t}}{\omega} \sin(\omega(1 + x)) d\omega + \frac{1}{\pi} \int_0^\infty \frac{e^{-\omega^2 t}}{\omega} \sin(\omega(1 - x)) d\omega.
\]

Note that when $x < 1$ the integrals above are the inverse sine transform of $f(x, t) = \mathcal{F}^{-1}_s \left[ \frac{e^{-\omega^2 t}}{\omega} \right]$ evaluated at $(1 + x)$ and $(1 - x)$ respectively. When $x > 1$ we replace $\sin(1 - x) = -\sin(x - 1)$ and note that the second integral $-f(x - 1, t)$. We thus get that

\[
u(x, t) = \begin{cases} 
\frac{f(1 + x, t) + f(1 - x, t)}{\pi} & x \leq 1 \\
\frac{f(1 + x, t) - f(x - 1, t)}{\pi} & x > 1 
\end{cases}.
\]

We now just need to compute $f(x, t)$ by inverting the sine transform $\mathcal{F}^{-1}_s \left[ \frac{e^{-\omega^2 t}}{\omega} \right]$ as follows:

\[
f(x, t) = \int_0^\infty \frac{e^{-\omega^2 t}}{\omega} \sin(\omega x) d\omega
= \int_0^\infty \frac{e^{-\omega^2 t}}{\omega} \sin(\omega x) d\omega + \int_0^\infty \frac{\sin(\omega x)}{\omega} d\omega
= \frac{\pi}{2} \left( 1 - \text{erfc} \left( \frac{x}{2 \sqrt{t}} \right) \right),
\]

and plugging this in above gives

\[
u(x, t) = \begin{cases} 
1 - \frac{1}{2} \left( \text{erfc} \left( \frac{1 + x}{2 \sqrt{t}} \right) + \text{erfc} \left( \frac{1 - x}{2 \sqrt{t}} \right) \right) & x \leq 1 \\
\frac{1}{2} \left( \text{erfc} \left( \frac{x - 1}{2 \sqrt{t}} \right) - \text{erfc} \left( \frac{x + 1}{2 \sqrt{t}} \right) \right) & x > 1 
\end{cases}.
\]

The solution $u(x, t)$ is shown for $t = .001, .01, .1, 1$, cooling from deep red to light blue. Note how the solution for small $t$ approximates the initial condition.
Exercise 3. Let $a > 0$ be a positive constant. Use the convolution formula (see eq. (12.9) in the text), along with the integral formula (for constants $A, B, C$ with $A > 0$)

$$
\int_{-\infty}^{\infty} e^{-(Ax^2+2Bx+C)} \, dx = \sqrt{\frac{\pi}{A}} \cdot e^{(B^2-AC)/A}
$$
to solve the initial-value problem

$$
\begin{align*}
&u_t = u_{xx}, \quad u(x, 0) = e^{-(x/a)^2}, \\
&\text{for } -\infty < x < \infty \text{ and } t \geq 0.
\end{align*}
$$

Solution. According to the convolution formula, the solution is given by

$$
u(x, t) = \phi(x) * \left( \frac{1}{\sqrt{2\pi t}} \cdot e^{-(x^2/4t)} \right) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \phi(\xi) e^{-(x-\xi)^2/4t} \, d\xi = \frac{1}{2\sqrt{\pi}t} \int_{-\infty}^{\infty} \phi(\xi) e^{-(\xi^2/4t)} \, d\xi
$$

for a general initial condition $\phi(x)$. Taking $\phi(x) = e^{-(x/a)^2}$, we get

$$
u(x, t) = \frac{1}{2\sqrt{\pi t}} \cdot \sqrt{\pi} \cdot \frac{2a\sqrt{t}}{a^2+4t} \cdot e^{-x^2/(a^2+4t)} = \frac{a}{\sqrt{a^2+4t}} \cdot e^{-x^2/(a^2+4t)}.
$$

Exercise 4. Consider the following problem

$$
\begin{align*}
u_t &= u_{xx}, \quad 0 < x < \infty, \quad 0 < t < \infty \\
u(0, t) &= \sin(t), \quad 0 < t < \infty \\
u(x, 0) &= 0, \quad 0 \leq x < \infty
\end{align*}
$$

(a) What is the physical interpretation of this problem?

(b) Use the Laplace transform to solve this problem (consider in which variable $x$ or $t$ you rather apply the transform).

Solution.

(a) The equation describes the temperature of a half infinite rod that has constant temperature zero at time $t = 0$ and the temperature in one end is made to oscillate like $\sin(t)$.

(b) We apply the Laplace transform in the $t$ variable, setting $\mathcal{L}[u] = U(x, s)$ we get the ODE

$$
\begin{align*}
U_{xx} &= sU(x) \\
U(0) &= \frac{1}{s^2+1}
\end{align*}
$$

(where we used that $u(x, 0) = 0$ in the first equation and that $\mathcal{L}[u(0, t)] = \mathcal{L}[\sin(t)] = \frac{1}{1+s^2}$ for the second. The solution to this ODE with these IC is

$$
U(x, s) = \frac{e^{-\sqrt{s}x}}{s^2+1} \frac{s}{s^2+1}.
$$
where we multiplied and divided by $s$ to make it easier to find the inverse transform. From the inversion table we see that $\mathcal{L}^{-1}\left[\frac{s}{s^2 + 1}\right] = \cos(t)$ and $\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{s}}}{s}\right] = \text{erfc}\left(\frac{x}{2\sqrt{t}}\right)$, hence, using the finite convolution

$$u(x, t) = \int_0^t \cos(t - \tau) \text{erfc}\left(\frac{x}{2\sqrt{\tau}}\right) d\tau.$$