MT441 Homework 6

Due Friday March 15

**Exercise 1.** Solve the 2-dimensional wave equation

\[ u_{tt} = u_{xx} + u_{yy} \]

on the rectangle \(0 \leq x < 1, 0 < y \leq 2\), where \( u = 0 \) on the boundary of the rectangle and has initial displacement \( u(x, y, 0) = \sin(\pi x) \sin(3\pi y) \) and zero initial velocity.

[Hint: Use separation of variables to find all solution of the form \( u(x, y, t) = X(x)Y(y)T(t) \)]

The point of the rest of the exercises is to introduce the Bessel function \( J_n(x) \), which is the (non singular) solution to the Bessel equation

\[ x^2 J'' + xJ' + (x^2 - n^2)J = 0. \]

and will pay an important part for understanding the solution of the wave equation on the disc.

In the first exercise you will use the method of Frobenius to find the solution \( J_n(x) \) to the Bessel equation as a power series.

\[ J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)(k+n)!} \left( \frac{x}{2} \right)^{2k+n} \]

**Exercise 2.** For method of Frobenius we look for a solution in the form of a shifted power series

\[ J(x) = x^p \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+p}, \]

where \( p \in \mathbb{R} \) is a parameter to be determined later.

a) Compute \( xJ'(x), x^2 J''(x) \) and \( x^2 J(x) \) in terms of the power series.

b) From these power series and the differential equation deduce that \( a_0[p^2 - n^2] = 0, a_1([p + 1]^2 - n^2] = 0 \) and for all \( m \geq 2 \)
\( a_m[(m + p)^2 - n^2] + a_{m-2} = 0. \)

c) Use the first condition together with the assumption that \( J(x) \) is bounded as \( x \to 0 \) to deduce that \( p = n \), and the second condition to deduce that \( a_1 = 0. \)

d) From the last condition (and induction) show that \( a_m = 0 \) for \( m \) odd, and that for \( m = 2k \) even
\[ a_{2k} = a_0 \frac{(-1)^k n!}{4^k k!(k+n)!} \]

e) Find the right normalization for \( a_0 \) that gives the series formula above for \( J_n(x) \).

In the next two exercises you will derive an integral representation of the Bessel function \( J_0(x) \) in terms of an integral. This was Bessel’s original formula for \( J_0(x) \).
Exercise 3. Let \( I_n = \frac{1}{2\pi} \int_0^{2\pi} \cos^n(\theta) \, d\theta \).

a) Use integration by parts to show that
\[
I_n = \frac{n-1}{n} I_{n-2}, \quad \text{for } n \geq 2.
\]

b) Show that \( I_n = 0 \) if \( n \) is odd and
\[
I_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} \quad \text{if } n = 2k. \tag{1}
\]

Exercise 4. Consider the function
\[
J(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \theta} \, d\theta.
\]

a) Show that \( J(x) \) satisfies Bessel’s equation \( xJ'' + J' + xJ = 0 \). [Hint: Do integration-by-parts on \( J'(x) \).]

b) Use the power series for \( e^x \) and the integral formula (1) to derive the power series for \( J(x) \), and show that \( J(x) = J_0(x) \), the Bessel function of index zero.