Exercise 1. Solve the 2-dimensional wave equation

\[ u_{tt} = u_{xx} + u_{yy} \]
on the rectangle \( 0 < x < 1, 0 < y < 2 \), where \( u = 0 \) on the boundary of the rectangle and has initial displacement \( u(x, y, 0) = \sin(\pi x)\sin(3\pi y) \) and zero initial velocity.

Solution. First to find the standing wave solution writing \( u(x, y, t) = X(x)Y(y)T(t) \) we see from separation of variables, as we did in class, the PDE reduced to 3 ODE’s

\[ T'' = k T, \quad X'' = k_1 X, \quad Y'' = k_2 X \]

where \( k = k_1 + k_2 \). For the functions \( X, Y \) we have the BC \( X(0) = X(1) = 0 \) implying that \( k_1 = -(\pi m)^2 \) and \( X(x) = \sin(\pi mx) \) for some integer \( m \) and similarly, \( Y(0) = Y(2) = 0 \) implying that \( k_2 = -(\pi n/2)^2 \) and \( Y(y) = \sin(\pi ny/2) \) for some integer \( n \). We thus get that the general standing wave solution is

\[ \left[ A_{mn} \cos(\lambda_{mn} t) + B_{mn} \sin(\lambda_{mn} t) \right] \sin(m\pi x) \sin(n\pi y/2), \]

with \( \lambda_{mn} = \frac{\pi}{2} \sqrt{4m^2 + n^2} \). The general solution is thus

\[ u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos(\lambda_{mn} t) + B_{mn} \sin(\lambda_{mn} t)] \sin(m\pi x) \sin(n\pi y/2), \]

Zero initial velocity forces all \( B_{mn} = 0 \), and the initial displacement condition forces

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi x) \sin(n\pi y/2) = \sin(\pi x) \sin(3\pi y), \]

so \( A_{16} = 1 \) and the remaining \( A_{mn} = 0 \). Since \( \lambda_{16} = \frac{\pi}{2} \sqrt{40} = \pi \sqrt{10} \), the solution is

\[ u(x, y, t) = \cos(\pi \sqrt{10} \cdot t) \sin(\pi x) \sin(3\pi y). \]

Exercise 2. For method of Frobenius we look for a solution in the form of a shifted power series

\[ J(x) = x^p \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+p}, \]

where \( p \in \mathbb{R} \) is a parameter to be determined later.

a) Compute \( xJ'(x), x^2J''(x) \) and \( x^2J(x) \) in terms of the power series.
b) From these power series and the differential equation deduce that \(a_0[p^2 - n^2] = 0, a_1[(p + 1)^2 - n^2] = 0\) and for all \(m \geq 2\)
\(a_m[(m + p)^2 - n^2] + a_{m-2} = 0\).

c) Use the first condition together with the assumption that \(J(x)\) is bounded as \(x \to 0\) to deduce that \(p = n\), and the second condition to deduce that \(a_1 = 0\).

d) From the last condition (and induction) show that \(a_m = 0\) for \(m\) odd, and that for \(m = 2k\) even
\[
a_{2k} = a_0 \frac{(-1)^k n!}{4^k k!(k + n)!}
\]

e) Find the right normalization for \(a_0\) that gives the series formula above for \(J_n(x)\).

**Solution.**
a) Starting from \(J(x) = \sum_{m=0}^{\infty} a_m x^{m+p}\), and differentiating term by term
\[
x J'(x) = \sum_{m=0}^{\infty} a_m(m + p)x^{m+p}, \quad x^2 J''(x) = \sum_{m=0}^{\infty} a_m(m + p)(m + p - 1)x^{m+p}
\]
and for the last term changing the index of summation gives
\[
x^2 J(x) = \sum_{m=0}^{\infty} a_m x^{m+2+p} = \sum_{m=2}^{\infty} a_{m-2} x^{m+p}.
\]

b) Pugging these into the differential equation and collecting all terms in the power series with the same power we get
\[
x^2 J'' + x J' + (x^2 - n^2) J = \sum_{m=0}^{\infty} [a_m((m + p)(m + p - 1) + (m + p) - n^2)]x^{m+p} + \sum_{m=2}^{\infty} a_{m-2} x^{m+p}
\]
\[
= a_0(p^2 - n^2)x^p + a_2((p + 1)^2 - n^2))x^{p+1} + \sum_{m=2}^{\infty} [a_m((m + p)^2 - n^2) + a_{m-2}]x^{m+p}
\]
For this to be zero, all terms must be zero so indeed \(a_0(p^2 - n^2) = 0\), \(a_2((p + 1)^2 - n^2))x^{p+1}\) and \(a_m((m + p)^2 - n^2) + a_{m-2} = 0\) for all \(m \geq 2\).

c) We may assume \(a_0 \neq 0\) (otherwise replace \(p\) by \(p + 1\)) which forces us to have \(p^2 = n^2\) so \(p = \pm n\). If \(p = -n\) is negative, the series goes to infinity as \(x \to 0\) (as it has negative powers) so we are left with \(p = n\). Next, from \(a_1[(p + 1)^2 - n^2] = 0\), since we already know \(p = n\) we get \((2n + 1)a_1 = 0\) so \(a_1 = 0\).

d) Setting \(p = n\) the last condition reads \(m(m + 2n)a_m = -a_{m-2}\) so \(a_m = -\frac{a_{m-2}}{m(m+2)}\). First, since \(a_1 = 0\), and the recursion relation implies that if \(a_{m-2} = 0\) then \(a_m = 0\) we see that \(a_m = 0\) for all odd \(m\). Next, for \(m = 2k\) even, we have
\[
a_{2k} = -a_{2k-2} = \frac{(-1)^k n!}{4k(k+1)}. \quad \text{The formula} \quad a_{2k} = a_0 \frac{(-1)^k n!}{4^k k!(k + n)!} \quad \text{trivially holds for} \quad k = 0. \quad \text{Assume it holds for} \quad m = 2k - 2 \quad \text{then for} \quad m = 2k \quad \text{we have}
\]
\[
a_{2k} = \frac{-a_{2k-2}}{4k(k+1)} = a_0 \frac{(-1)^{k-1} n!}{4^{k-1} (k - 1)!(k - 1 + n)!} \frac{-1}{4(k+1)} = a_0 \frac{(-1)^k n!}{4^k k!(k + n)!}
\]

e) Plugging in the formula for \(a_m\) in the series (recalling that \(p = n\)) gives
\[
J(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k n!}{4^k k!(k + n)!} x^{2k+n}
\]
\[
= a_0 2^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + n)!} (x/2)^{2k+n}
\]
so taking \(a_0 = \frac{1}{2^n n!}\) gives the standard normalization for the Series of the Bessel function of degree \(n\).
Exercise 3. Let \( I_n = \frac{1}{2\pi} \int_0^{2\pi} \cos^n(\theta) \, d\theta \).

a) Use integration by parts to show that
\[
I_n = \frac{n-1}{n} I_{n-2}, \quad \text{for } n \geq 2.
\]

b) Show that \( I_n = 0 \) if \( n \) is odd and
\[
I_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} \quad \text{if } n = 2k.
\]

Solution.

We write \( c = \cos \theta, \ s = \sin \theta \). Letting \( u = c^{n-1} \) and \( dv = c \, d\theta \), we have \( du = -(n-1)c^{n-2}s \) and \( v = s \), and since \( \sin(0) = \sin(2\pi) = 0 \), we have
\[
\int_0^{2\pi} c^n \, d\theta = (n-1) \int_0^{2\pi} c^{n-2}s^2 \, d\theta = (n-1) \int_0^{2\pi} c^{n-2}(1-c^2) \, d\theta = (n-1) \int_0^{2\pi} c^{n-2} \, d\theta - (n-1) \int_0^{2\pi} c^n \, d\theta.
\]

Solving for \( \int_0^{2\pi} c^n \, d\theta \), we find
\[
\int_0^{2\pi} c^n \, d\theta = \frac{n-1}{n} \int_0^{2\pi} c^{n-2} \, d\theta.
\]

dividing both sides by \( 2\pi \) gives the recursion formula for \( I_n \)

Since \( I_1 = 0 \), it follows that \( I_{2k+1} = 0 \) for all positive integers \( k \) and since \( I_0 = 1 \) we have \( I_2 = 1/2 \) and the formula for \( I_{2k} \) follows by induction on \( k \).

Exercise 4. Consider the function
\[
J(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\cos \theta} \, d\theta.
\]

a) Show that \( J(x) \) satisfies Bessel’s equation \( xJ'' + J' + xJ = 0 \). [Hint: Do integration-by-parts on \( J'(x) \).]

b) Use the power series for \( e^x \) and the integral formula (1) to derive the power series for \( J(x) \), and show that \( J(x) = J_0(x) \), the Bessel function of index zero.

Solution.

For a) we again write \( c = \cos \theta, \ s = \sin \theta \), and let
\[
y = \int_0^{2\pi} e^{ixc} \, d\theta.
\]

Then
\[
y' = \int_0^{2\pi} (ic)e^{ixc} \, d\theta, \quad \text{and} \quad y'' = \int_0^{2\pi} (-c^2)e^{ixc} \, d\theta.
\]

Apply integration by parts to \( y' \), using \( u = e^{ixc} \) and \( dv = ic \) to get
\[
y' = \int_0^{2\pi} (-xs^2)e^{ixc} \, d\theta.
\]

Now
\[
xy'' + y' + xy = \int_0^{2\pi} (-xc^2 - xs^2 + x)e^{ixc} \, d\theta,
\]

and \(-xc^2 - xs^2 + x = xs^2 - xs^2 = 0\), so \( xy'' + y' + xy = 0 \) as desired.
For b) we expand

\[ e^{ixc} = \sum_{n=0}^{\infty} \frac{(ixc)^n}{n!} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \cdot c^n. \]

Applying \( \frac{1}{2\pi} \int_0^{2\pi} \) to both sides and using problem 1, we have

\[ \frac{1}{2\pi} \int_0^{2\pi} e^{ixc} \, d\theta = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \cdot I_n = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1 \cdot 3 \cdots (2k - 1)}{2 \cdot 4 \cdots (2k)} \cdot \frac{x^{2k}}{(2k)!}. \]

Cancelling the odd factors from \((2k)!\) and noting that \(2 \cdot 4 \cdots (2k) = k! \cdot 2^k\), we get

\[ \frac{1}{2\pi} \int_0^{2\pi} e^{ixc} \, d\theta = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(k!)^2 \cdot 2^{2k}}, \]

which is the power series for \( J_0(x) \).

\( \square \)