MT442 Homework 9

Solutions

Exercise 1. Consider the free wave equation in 2 dimensions corresponding to an impulse function at the origin, given by the initial value problem

\[ u_{tt} = c^2 \nabla^2 u, \quad \bar{x} \in \mathbb{R}^2, \quad t > 0 \]

with initial conditions \( u(x, y, 0) = 0 \) and \( u_t(x, y, 0) = \delta(x, y) \) where the function \( \delta(x, y) \) is characterized by the property that \( \int_{\mathbb{R}^2} f(\bar{x}) \delta(\bar{x}) d\bar{x} = f(0) \).

(a) Apply the formula we obtained by method of descent to find an explicit formula for \( u(\bar{x}, t) \).

(b) What does the solution look like for \( |\bar{x}| > ct \) and for \( |\bar{x}| < ct \).

(c) What is the strength of the signal at distance \( R \) from origin after time \( t = 2R/c \).

Solution. The formula we saw in class for the free wave equation in 2 dimension with initial condition \( u(x, y, 0) = 0 \) and \( u_t(x, y, 0) = g(x, y) \) is given by

\[ u(x, y, t) = \frac{1}{2\pi c} \int_0^ct \int_0^{2\pi} \frac{g(x + c\tau \cos \theta, y + c\tau \sin \theta)}{\sqrt{(ct)^2 - \tau^2}} \tau \, d\tau \, d\theta, \]

that we rewrite as

\[ u(x, y, t) = \frac{1}{2\pi c} \int_0^\infty \int_0^{2\pi} g(x + c\tau \cos \theta, y + c\tau \sin \theta) f_0(\tau) \tau \, d\tau \, d\theta, \]

with

\[ f_0(\tau) = \begin{cases} \frac{1}{\sqrt{(ct)^2 - \tau^2}} & t < ct \\ 0 & t > ct \end{cases} \]

Now make a change of variables of the integral to cartesian coordinates: writing \( \bar{v} = (v_1, v_2) = (\tau \cos(\theta), \tau \sin(\theta)) \) and \( \bar{x} = (x, y) \) we have \( d\bar{v} = \tau d\theta d\tau \) and \( \tau = |\bar{v}| \) so that

\[ u(x, y, t) = \frac{1}{2\pi c} \int_{\mathbb{R}^2} g(\bar{x} + \bar{v}) f_0(|\bar{v}|) d\bar{v} = \frac{1}{2\pi c} \int_{\mathbb{R}^2} g(\bar{v}) f_0(|\bar{v} - \bar{x}|) d\bar{v}. \]

where we made a change of variables \( \bar{v} \mapsto \bar{v} - \bar{x} \). Now since \( g(\bar{x}) = \delta(\bar{x}) \) we see that

\[ u(\bar{x}) = f_0(|0 - \bar{x}|) = \begin{cases} \frac{1}{\sqrt{(ct)^2 - |\bar{x}|^2}} & |\bar{x}| < ct \\ 0 & |\bar{x}| > ct \end{cases} \]

In particular we see that for \( |bar{x}| < ct \) the solution is zero and for \( |\bar{x}| > ct \) it decays like \( \frac{1}{\sqrt{(ct)^2 - |\bar{x}|^2}} \) (the solution blows up when \( \bar{x} \) is of distance exactly \( ct \) from the origin. In particular for \( t = 2R/c \) at a point \( \bar{x} \) at distance \( R \) from origin the soutane will be \( \frac{1}{\sqrt{4R^2 - R^2}} = \frac{1}{\sqrt{3}R} \). \]

Exercise 2. Use separation of variables to find the harmonic function on the square \( 0 \leq x, y \leq 1 \) which \( = 1 \) on the bottom edge of the square \( (y = 0) \) and is zero on the other three sides.
Solution. Setting \( u(x, y) = F(x)G(y) \) we get

\[
F''/F = -G''/G = k,
\]
a constant. The solution to these equations are \( F(x) = e^{\pm \sqrt{k}x} \) and \( G(y) = e^{\pm \sqrt{-k}y} \) and we note that one is oscillatory and the other exponential depending on whether \( k \) is positive or negative. A quick check shows that if \( k \geq 0 \) the BC cannot be satisfied, so \( k = -\lambda^2 \), for some \( \lambda > 0 \), and we can write

\[
F(x) = A \cos(\lambda x) + B \sin(\lambda x), \quad G(y) = Ce^{\lambda y} + De^{-\lambda y}.
\]
To vanish on the sides \( x = 0, 1 \) we must have \( A = 0 \) and \( \lambda = n\pi \), for some \( n = 1, 2, 3, \ldots \). We can now rewrite the solution \( u(x, y) \) as

\[
(A_n e^{n\pi y} + B_n e^{-n\pi y}) \sin(n\pi x).
\]
To vanish at \( y = 1 \) we must have

\[
A_n e^{n\pi} + B_n e^{-n\pi} = 0, \quad \text{(1)}
\]
for all \( n \). To satisfy the final BC, we must add all these solutions:

\[
u(x, y) = \sum_{n=1}^{\infty} (A_n e^{n\pi y} + B_n e^{-n\pi y}) \sin(n\pi x).
\]
At \( y = 0 \) we have

\[
\sum_{n=1}^{\infty} (A_n + B_n) \sin(n\pi x) = 1,
\]
so

\[
A_n + B_n = 2 \int_{0}^{1} 1 \cdot \sin(n\pi x) = \begin{cases} 4/n\pi & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \quad \text{(2)}
\]
Solving equations (??) and (??) we get \( A_n = B_n = 0 \) if \( n \) is even and

\[
A_n = \frac{4}{n\pi(1 - e^{2n\pi})}, \quad B_n = \frac{4}{n\pi(1 - e^{-2n\pi})},
\]
if \( n \) is odd. Hence the solution is

\[
u(x, y) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{e^{n\pi y}}{1 - e^{2n\pi}} + \frac{e^{-n\pi y}}{1 - e^{-2n\pi}} \right) \sin(n\pi x).
\]

Exercise 3. Show that the Poisson kernel \( P(r, \theta) = \frac{1 - r^2}{2\pi \ln(\cos(\theta) + r)} \) is itself a Harmonic function on the disc that is unbounded on the boundary. [Hint: Show that it is the solution to the Laplace equation \( \nabla^2 u = 0 \) on the disc of radius one with BC \( u(1, \theta) = \delta(\theta) \) with \( \delta \) the Dirac delta function defined by the property that \( \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) \delta(\theta) d\theta = f(0) \).]

Solution. Let \( u(r, \theta) \) be the solution to the Laplace equation

\[
\begin{cases}
\nabla^2 u = 0, & 0 < r < 1 \\
u(1, \theta) = \delta(\theta)
\end{cases}
\]
Then according to the Poisson integral formula we have for any \( 0 < r < 1 \) and \( \theta \in [0, 2\pi] \)

\[
u(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} P(r, \theta - \alpha) \delta(\alpha) d\alpha = P(r, \theta).
\]
Since \( \nabla^2 u = 0 \) is harmonic on the disc so is \( P(r, \theta) \), on the boundary \( u(1, \theta) = \delta(\theta) \) is unbounded (it is infinite at \( \theta = 0 \)).
**Exercise 4.** Fix $r$. Find the Fourier coefficients of $P(r,\theta)$, regarded as a periodic function of $\theta$. (These coefficients will depend on $r$. Hint for the integrals: remember how we showed that $(1/2\pi) \int_0^{2\pi} P(r, \phi) d\phi = 1$.)

**Solution.** Since $P(r, \theta)$ is an even function of $\theta$, we have

$$P(r, \theta) = \sum_{n=0}^{\infty} A_n(r) \cos(n\theta),$$

where the Fourier coefficients $A_n(r)$ are given by

$$A_0(r) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \phi) d\phi = 1,$$

and for $n \geq 1$,

$$A_n(r) = \frac{2}{2\pi} \int_0^{2\pi} P(r, \phi) \cos(n\phi) d\phi.$$

This is twice the Poisson integral formula for the harmonic function whose boundary values are $\cos(n\theta)$. But we have already seen the harmonic function $r^n \cos(n\theta)$, which clearly equals $\cos(n\theta)$ when $r = 1$. So we have

$$r^n \cos(n\theta) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \phi - \theta) \cos(n\phi) d\phi.$$

(This is another case of computing the Poisson integral by knowing in advance the harmonic function it computes.) Plugging $\theta = 0$ above it follows that

$$A_n = 2 \cdot r^n \cos(n \cdot 0) = 2r^n$$

and the Fourier expansion of $P(r, \theta)$ is given by

$$P(r, \theta) = 1 + 2 \sum_{n=1}^{\infty} r^n \cos(n\theta).$$

If we use the complex Fourier expansion, via $2 \cos(n\theta) = e^{in\theta} + e^{-in\theta}$, this formula becomes simpler:

$$P(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$