ANALYTIC NUMBER THEORY: HOMEWORK 3

Exercise 1. Let $\chi$ denote a non-principal Dirichlet character mod $q$ for some $q \geq 3$

(1) Show that for any $0 < \delta < 1$ for all $s = \sigma + it$ with $\sigma \geq \delta$ we can bound $|L(s, \chi)| \ll (q(1 + |t|))^{1-\delta}$ where the implied constant depends only on $\delta$.

(2) Show that for any $\epsilon < 1$ and $c > 0$ and any real $s > \max\{\epsilon, 1 - \frac{c}{\log(q)}\}$ we can bound $|L(s, \chi)| \ll \log(q)$, where the implied constant depends on $\epsilon$ and $c$.

Exercise 2. In this exercise you will find all primitive real characters modulo $q$.

(1) Let $q = p_1^{r_1} \cdots p_k^{r_k}$. Show that $\chi$ is a real primitive character modulo $q$, if and only if $\chi(n) = \chi_1(n) \cdots \chi_k(n)$ with each $\chi_j$ a primitive real character mod $p_j^{r_j}$.

(2) Let $p$ be an odd prime. Recall that the multiplicative group $(\mathbb{Z}/p^r\mathbb{Z})^*$ is cyclic of order $p^{r-1}(p-1)$. Use this to deduce that there can be a real primitive character mod $p^r$ only when $r = 1$ and that, in that case, there is a unique such character.

(3) Show that there is a unique real primitive character mod 4 and two primitive real characters mod 8, and that these are the only primitive real characters modulo powers of 2.

(4) Conclude that the only values of $q$ for which there is a real primitive character mod $q$ are of the form $q = q_0$, $q = 4q_0$ or $q = 8q_0$ with $q_0$ odd and square free, moreover, in the first two cases this character is unique, while in the last there are two such characters.

(5) Conclude that each of these characters are given by $\chi = (d)$ with $d$ a fundamental discriminant.

Exercise 3. In this exercise you will prove Landau’s result: There is $c > 0$ such that for $\chi_1, \chi_2$ two distinct real primitive characters modulo $q_1, q_2$ for which the corresponding $L$-functions have Siegel zeros $\beta_1, \beta_2 \in (\frac{1}{2}, 1)$ then $\min(\beta_1, \beta_2) \leq 1 - \frac{c}{\log(q_1q_2)}$.

(1) Recall that each character mod $q$ is induced from a unique primitive character mod $q'$ for some $q'|q$ and show that $\chi_1 \chi_2$ is a real, non-principal character mod $q_1 q_2$.

(2) Prove that for any two real characters and any $\sigma > 1$,

$$-\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma, \chi_1) - \frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_1 \chi_2) \geq 0.$$ 

(3) Show that there is $c_1 > 0$ such that for $\sigma > 1$,

$$-\frac{L'}{L}(\sigma, \chi_i) \leq c_1 \log(q_i) - \frac{1}{\sigma - \beta_i}, \quad -\frac{L'}{L}(\sigma, \chi_1 \chi_2) \leq c_1 \log(q_1q_2).$$

(4) Use these inequalities with $\sigma = 1 + \frac{\delta}{\log(q_1q_2)}$ for sufficiently small $\delta$, to show that at least one of $\beta_1, \beta_2$ must satisfy that $\beta < 1 - \frac{c}{\log(q_1q_2)}$ for some $c > 0$. 


Exercise 4. In this exercise you will show that the set of $q$’s for which there can be Siegel zeros is very sparse

(1) Let $k > 0$. Show that there is $c_k > 0$ such that if $q_1, q_2, q_3 \ldots$ is a sequence such that for each $j$ there is a primitive real characters mod $q_j$ with $L$-function having a Siegel zero $\beta_j \geq 1 - \frac{c_k}{\log(q_j)}$, then $q_{j+1} \geq q_j^k$.

(2) Use this (with, say $k = 2$) to show that the number of such $q$’s up to $T$ is $O(\log \log(T))$. 
