**Math 204 hw1 (Due Wed. Apr. 08).**

(1) Calculate the derivatives of the following functions from the definition:
   (a) $f(x) = x^n$ for $n \in \mathbb{N}$. You may use the Binomial Formula
$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}.$$
   (b) $g(x) = x^\alpha$ for $\alpha = \frac{1}{n}$, $n \in \mathbb{N}$. You may use the formula
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}).$$
   (c) $h(x) = \sin(x)$. You may use the trigonometric formula $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and the limit $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ we proved in class.

(2) In this exercise you will prove that $(e^x)' = e^x$. The function $e^x$ is defined by the following (convergent) series: For any $x \in \mathbb{R}$,
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
   (a) Use the Binomial Formula to show that for any $x, h \in \mathbb{R}$
$$\left| \frac{(x+h)^n - x^n}{h} - nx^{n-1} \right| \leq n(n-1)|h|(|x| + |h|)^{n-2}.$$
   (b) Deduce from this that for any $x, h \in \mathbb{R}$
$$|\frac{e^{x+h} - e^x}{h} - e^x| \leq |h| e^{|x|+|h|}$$
   (c) Conclude that $(e^x)' = e^x$.

(3) Let $I$ be an interval containing $a$ and $f : I \to \mathbb{R}$ be a real function. Show that if there is $\alpha > 1$ such that
$$\forall x \in I,\ |f(x) - f(a)| \leq |x-a|^\alpha$$
then $f$ is differentiable at $a$. Is this also true when $\alpha = 1$?

(4) Let $I$ be an open interval, $f : I \to \mathbb{R}$ a function and $c \in I$. We say that $f$ has a local maximum (respectively minimum) at $c$ if there is some $\delta > 0$ such that $f(x) \leq f(c)$ (respectively $f(x) \geq f(c)$) for all $x \in (c-\delta, c+\delta)$.
   (a) Show that if $f$ has a local maximum at $c$ then there is $\delta > 0$ such that for all $0 < h < \delta$
$$\frac{f(c+h) - f(c)}{h} \leq 0 \text{ and } \frac{f(c+(-h)) - f(c)}{(-h)} \geq 0.$$

(b) Show that if $f$ is differentiable at $c$ and has a local maximum at $c$ then $f'(c) = 0$.
(c) Show that if $f$ is differentiable at $c$ and has a cola minimum at $c$ then $f'(c) = 0$.
(d) Give an example of a function $f$ differentiable at 0 with $f'(0) = 0$ but that $f$ does not have a local maximum or minimum at 0.

(5) Let

$$f_\alpha(x) = \begin{cases} |x|^\alpha \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) For what values of $\alpha$ is this function continuous at 0?
(b) For what values of $\alpha$ is this function differentiable at 0?
(c) For what values of $\alpha$ the derivative is continuous at 0? (You may use the formula $(x^\alpha)' = \alpha x^{\alpha-1}$ for any $0 \neq \alpha \in \mathbb{R}$.)

(6) Let $f$ and $g$ are real-valued functions differentiable at $a \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Prove that the functions $f + g$ and $\alpha \cdot f$ are differentiable at $a$ and $(f+g)'(a) = f'(a)+g'(a)$ and $(\alpha f)'(a) = \alpha f'(a)$.