Math 204 hw2 (Due Wed. Apr. 15).

(1) Use the chain rule to calculate the derivatives of $f(x) = x^x$ and $g(x) = e^{\sin(x^2)}$. You may use the fact that $(\log(x))' = \frac{1}{x}$.

(2) Let f, g be real functions differentiable at 2 and 3 and satisfying $f'(2) = a,\ f'(3) = b,\ g'(2) = c$ and $g'(3) = d$. Assume that $f(2) = 1,\ f(3) = 2,\ g(2) = 3$ and $g(3) = 4$ and calculate $(fg)'(2),\ (f/g)'(3), (f \circ g)'(3)$ and $(g \circ f)'(2)$.

Remark 0.1. There is a mistake in the last two derivatives It should have been $(f \circ g)'(2)$ and $(g \circ f)'(3)$.

(3) Let f be a function differentiable on all of $\mathbb{R}$
  (a) Show that $f'(x)$ is the constant function if and only if $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.
  (b) Show that if $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$ then $|f(x)| \leq 1 + |x|$.
  (c) Show that if f' is continuous at 0 and $f'(0) > 0$ then there is $\delta > 0$ such that for any $-\delta < x < y < \delta,\ f(x) < f(y)$ (i.e., f is strictly increasing in a neighborhood of 0).
  (d) Show that this is not true if we remove the continuity assumption. That is, give an example of a function f differentiable on $\mathbb{R}$ with $f'(0) > 0$ such that for any $\delta > 0$ there are $0 < x < y < \delta$ with $f(x) > f(y)$.

(4) Let f be differentiable on an open interval (a, b). Show that if the derivative f' is bounded then f is uniformly continuous on (a, b).

(5) Let f be twice differentiable on (a, b) and let $a < x_1 < x_2 < x_3 < b$ with $f(x_1) > f(x_2)$ and $f(x_2) < f(x_3)$. Show that there is a point $c \in (a, b)$ with $f''(c) > 0$.

(6) Prove the inequality $\ln(1 + x) \leq x$ for all $x \geq 0$.

(7) Evaluate the following limits
  (a) $\lim_{x \to 0^+} \frac{\cos(x) - e^x}{\ln(1+x^2)}$
  (b) $\lim_{x \to 0^+} x^x$
  (c) $\lim_{x \to 1} \frac{\ln(x)}{\sin(\pi x)}$