HOMEWORK 3

Due on Wed, April 22 in class.

Exercise 1. Let $f(x) = x^2 e^{x^2}$, $x \in \mathbb{R}$. Show that f^{-1} exists and is differentiable on $(0, \infty)$ and compute $(f^{-1})'(e)$.

Exercise 2. Show that $(\arctan(x))' = \frac{1}{1+x^2}$ for $x \in (-\infty, \infty)$.

Exercise 3. Let $f \in C^1(I)$ for some open interval I with $f'(x) \neq 0$ for all $x \in I$.

(1) Prove that f is injective on I and that $f(I) = J$ is an interval.
(2) Show that f^{-1} exists and is differentiable on J.
(3) Give an example that shows that (2) is false if the assamption $f'(x) \neq 0$ fails at even one point.

Exercise 4. Let $f : (a, b) \to \mathbb{R}$ be continuous for some bounded interval (a, b).

(1) Show that if f is differentiable on (a, b) and there is $\epsilon > 0$ such that $|f'(x)| \geq \epsilon$ for all $x \in (a, b)$ then f^{-1} exits and is bounded on $f(a, b)$.
(2) Show that if $f \in C^1(a, b)$ and $f'(x_0) > 0$ at $x_0 \in (a, b)$ then there are open intervals $I \subset (a, b)$, $J \subset f(a, b)$ such that $f : I \to J$ is a bijection and $f^{-1} \in C^1(J)$.

Exercise 5.
Let f be differentiable on a closed bounded interval I. Prove that if f' is injective on I then it is strictly monotone there.

Exercise 6.
Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a strictly increasing function and $E = f(I) \subset \mathbb{R}$ its image. Show that $f : I \to E$ is a bijection and that $f^{-1} : E \to I$ is continuous.

Exercise 7. Use the Teylor series of $\log(x)$ around $x_0 = 1$ to show that

$$\log(2) = \sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots.$$

Exercise 8. Show that for any $y \in [0, 1]$

$$y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} \leq \log(1 + y) \leq y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{64}$$

Exercise 9. Let

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(1) Show that $f \in C^\infty(\mathbb{R})$ and that for any $n \geq 1$ its n'th derivative is given by

$$f^{(n)}(x) = \begin{cases} Q_n(\frac{1}{x})e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases},$$

where $Q_n(t)$ is a polynomial of degree $3n$.
(2) Compute the degree n Teylor polynomial of f around $x_0 = 0$. Do these polynomials approximate $f(x)$ as $n \to \infty$?