HOMEWORK 4

Due on Wed, April 29 in class.

Exercise 1. For each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ compute the limits $\lim_{x\to 0}(\lim_{y\to 0}) f(x,y)$ and $\lim_{y\to 0}(\lim_{x\to 0} f(x,y)$. Determine which of these functions has a limit as $(x,y) \to (0,0)$ in $\mathbb{R}^2$ (justify your answers).

$$(1)\ f(x,y) = \frac{x^2+y^4}{x^2+3y^4} \quad (2)\ f(x,y) = \frac{\sin(x)\sin(y)}{x^2+y^2}, \quad (3) f(x,y) = \frac{xy^2}{x^2+y^4}.$$

Exercise 2. For each of the following functions compute the first order partial derivatives and determine where they are continuous

(1) $f(x,y) = \sqrt{x^2+y^2}$

(2) $f(x,y,z) = \frac{xy}{1+z}$

(3) $f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^{1/3}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Exercise 3. For each of the following functions compute all second order partial derivatives

$$(1)\ f(x,y) = xe^y \ \ (2)\ f(x,y) = \cos(xy), \ \ (3) f(x,y) = \frac{x+y}{x^2+1}.$$

Exercise 4. Let $f(x) = \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}$. Let $a \in \mathbb{R}^n$ and assume that for any $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,m\}$ the partial derivatives $\frac{\partial f_i}{\partial x_j}(a)$ and $\frac{\partial g}{\partial x_j}(f(a))$ exist. Show that the partial derivatives of $g \circ f : \mathbb{R}^n \to \mathbb{R}$ exist at a and satisfy

$$\frac{\partial g \circ f}{\partial x_j}(a) = \sum_{i=1}^{m} \frac{\partial g}{\partial x_i}(f(a)) \frac{\partial f_i}{\partial x_j}(a).$$

Exercise 5. For each of the following functions prove that f is differentiable on its domain and compute Df.

(1) $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (\sin(x), xy, \cos(y))$

(2) $f : (0,\infty) \to \mathbb{R}^2$, $f(t) = (\log(t), \frac{1}{1+t})$

(3) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(r,\theta) = (r\cos(\theta), r\sin(\theta))$

Exercise 6. For each of the following functions determine if $f(x,y)$ is differentiable at $(0,0)$

(1) $f(x,y) = \sqrt{|xy|}$

(2) $f(x,y) = \begin{cases} |xy|\log(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

(3) $f(x,y) = \begin{cases} \frac{x^2+y^2}{\sin(\sqrt{x^2+y^2})} & 0 < \|(x,y)\| < \pi \\ 0 & (x,y) = (0,0) \end{cases}$

Exercise 7. Show that $f(x,y) = \begin{cases} \frac{x^3-xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is continuous on $\mathbb{R}^2$, has first order partial derivatives at every point but it is not differentiable at $(0,0)$.

Exercise 8. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that it is differentiable at any $a \in \mathbb{R}^n$ and compute its derivative $DT(a)$.

Exercise 9. Let $f : V \to \mathbb{R}^m$ with $V \subset \mathbb{R}^n$ open. For any unit vector $u \in \mathbb{R}^n$ with $\|u\| = 1$ the directional derivative of f at a in the direction of u is defined by

$$D_u f(a) = \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}.$$

(1) Prove that if the directional derivatives $D_u f(a)$ exist for all unit vectors u then the partial derivatives $\frac{\partial f}{\partial x_j}(a)$ exists for all $j \in \{1, \ldots, n\}$.
(2) Find an example to show that the converse is not true. That is find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at $(0, 0)$ but do not exist for all u.
(3) Prove that if f is differentiable at a then the directional derivatives $D_u f(a)$ exist for all unit vectors u.
(4) Use the function from Exercise 1(3) to show that the converse is not true.