HOMEWORK 5

Due on Wed, May 6 in class.

Exercise 1. Let $f, g : \mathbb{R}^3 \to \mathbb{R}^3$ be differentiable at $a \in \mathbb{R}^3$ and suppose $f(a) = (2,1,2),\ g(a) = (1,2,1)$ and

$$Df(a) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},\ Dg(a) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Find the derivative of the dot product $D(f \cdot g)(a)$

Exercise 2. Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}^n$ be a differentiable function. Assume that

$$f(I) \subset \partial B_r(0) = \{x \in \mathbb{R}^n |\ \|x\| = r\},$$

for some $r > 0$. Prove that $Df(t)$ is orthogonal to $f(t)$ for any $t \in I$.

Exercise 3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in \mathbb{R}^n$ and $f(a) \neq 0$.

(1) Show that there is $\delta > 0$ such that if $\|h\| < \delta$ then $f(a+h) \neq 0$.
(2) Consider the linear map $T(h) = \frac{-1}{f^2(a)} Df(a)(h)$ and show that for $\|h\| < \delta$

$$\frac{1}{f(a+h)} - \frac{1}{f(a)} - T(h) = \frac{(f(a+h) - f(a))Df(a)(h)}{f^2(a)f(a+h)} - \frac{f(a+h) - f(a) - Df(a)(h)}{f(a)f(a+h)}.$$

(3) Prove that $\frac{1}{f}$ is differentiable at a and that

$$D\tfrac{1}{f}(a) = -\frac{1}{f^2(a)} Df(a).$$

Exercise 4. For each of the following functions find the tangent plane to the graph of the functions $z = f(x,y)$ at the point $c \in \mathbb{R}^3$.

(1) $f(x,y) = x^2 \sin(y),\ c = (0,0,0)$.
(2) $f(x,y) = x^3 y - xy^3,\ c = (1,1,0)$.

Exercise 5. In this exercise you will prove Taylor's Formula for $\mathbb{R}^n$. Let $V \subset \mathbb{R}^n$ be an open set and $f : V \to \mathbb{R}$ a functions. Assume that all the p'th order partial derivatives exist for any $a \in V$ and let

$$D^{(p)} f(a;h) = \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^p f}{\partial x_{i_1} \ldots \partial x_{i_p}}(a) h_{i_1} h_{i_2} \ldots h_{i_p}.$$

Assume that $a, x \in V$ such that $L(x,a) \subseteq V$ and denote by $h = x - a$.

(1) Show that there is $\delta > 0$ such that for any $t \in (-\delta, 1+\delta)$ the point $a + th \in V$.

(2) Show that the function $F(t) = f(a+th)$ is differentiable on $(-\delta, 1+\delta)$ and satisfies

$$F'(t) = DF(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a+th)h_i$$

(3) Show by induction that F is differentiable p times and

$$F^{(k)}(t) = D^{(k)} f(a+th; h)$$

(4) Use Taylor's formula for the real function F and show that there is $c \in L(x, a)$ such that

$$f(x) = f(a) + \sum_{k=1}^{p-1} \frac{1}{k!} D^{(k)} f(a; h) + \frac{1}{p!} D^{(p)} f(c; h),$$

Exercise 6. Write Taylor's formula of order $p = 3$ for the function $f(x, y) = \sqrt{x} + \sqrt{y}$ around the point $a = (1, 4)$, and for the function $f(x, y) = e^{xy}$ around $a = (0, 0)$.