HOMEWORK 6

Due on Wed, May 13 in class.

Exercise 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Suppose that for any unit vector $u \in \mathbb{R}^n$ the directional derivatives $D_u f(x)$ exist for any $x \in L(a, a+u)$. Show that for any unit vector $u \in \mathbb{R}^n$ there is t such that $f(a+u) - f(a) = D_u f(a+tu)$.

Exercise 2. Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that $a \in \mathbb{R}^b$ is a local maximum (respectively minimum) for f if there is a small ball $B_r(a)$ such that $f(x) \leq f(a)$ (respectively $f(x) \geq f(a)$) for any $x \in B_r(a)$.

(1) Show that if a is a local maximum then $Df(a) = 0$.
(2) Let $g(x, y) = ax^2 + bxy + cy^2$ for some $a, b, c \in \mathbb{R}$ with $a > 0$ and let $D = b^2 - 4ac$. Prove that if $D < 0$ then $g(x, y) > 0$ for any x, y and if $D > 0$ then g takes both positive and negative values.
(3) Give an example of a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$ with $Df(0) = 0$ but that 0 is not a local maximum or minimum.

Exercise 3. Let $E = \{x, y)|0 < y < x\}$ and let $f(x, y) = (x + y, xy)$ for $(x, y) \in E$.

(1) Prove that f is a bijection from E to $\{(s,t)|t > 0,\ s > 2\sqrt{t}\}$ and find a formula for $f^{-1}(s, t)$.
(2) Use this formula to calculate $D(f^{-1})(s, t)$.
(3) Use the inverse function theorem to compute $D(f^{-1})(f(x, y))$ for $x, y \in E$ and verify that this agrees with the direct calculation.

Exercise 4. For each of the following functions, prove that f^{-1} exists and is differentiable in some nonempty open set containing $(a, b) \in \mathbb{R}^2$, and compute $D(f^{-1})(a, b)$.

(1) $f(u, v) = (3u - v, 2u + 5v)$ at (a, b).
(2) $f(u, v) = (u + v, \sin(u) + \cos(v))$ at $(a, b) = (0, 1)$.
(3) $f(u, v) = (uv, u^2 + v^2)$ at $(a, b) = (2, 5)$.
(4) $f(u, v) = (u^3 - v^2, \sin(u) - \log(v))$ at $(a, b) = (-1, 0)$.

Exercise 5. For each of the following cases, find out if there is a function $z(x, y)$ that solves the equation in some non empty set V containing $(0, 0, 0)$. Is this function differentiable around $(0, 0)$? Calculate the derivative $Dz(0, 0) = \nabla z(0, 0)$ at $(0, 0)$.

(1) $xyz + \sin(x + y + z) = 0$.
(2) $x^2 + y^2 + z^2 + \sqrt[3]{2xy + 3z + 8}$.
(3) $xyz(2\cos(y) - \cos(z)) + z\cos(x) - x\cos(y)$.

Exercise 6. Prove that there exist functions $u(x, y), v(x, y)$ and $w(x, y)$ such that u, v, w are C^1 on some small ball $B_r(1, 1)$ and satisfy $u(1, 1) = v(1, 1) = -w(1, 1) = 1$ and

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1 \end{aligned}$$