HOMEWORK 7

Due on Wed, May 20 in class.

Exercise 1. Let $V \subset \mathbb{R}^2$, $a \in V$, and $f : V \to \mathbb{R}$ be C^2 on V. Assume that $f(a)$ is a local minimum for f and prove that for all $h = (h_1, h_2) \in \mathbb{R}^2$.

$$D^{(2)} f(a; h) = \frac{\partial^2 f}{\partial x^2}(a) h_1^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a) h_1 h_2 + \frac{\partial^2 f}{\partial y^2}(a) h_2^2 \geq 0.$$

Exercise 2. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 function and assume that $\nabla F(a, b, c) \neq 0$. Prove that the set $S = \{(x, y, z) | F(x, y, z) = 0\} \subset \mathbb{R}^3$ has a tangent plane at the point (a, b, c) with normal $\nabla F(a, b, c)$. (Notice that the formula for the tangent plane to a graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ we have seen in class is a special case of this where we take $F(x, y, z) = f(x, y) - z$.)

Exercise 3. For each of the following, find the maximum and minimum of f on H

(1) $f(x, y) = x^2 + 2x - y^2$ with $H = \{(x, y) | x^2 + 4y^2 \leq 4\}$.
(2) $f(x, y) = x^2 + 2xy + 3y^2$ with H the region bounded by the triangle with vertices $(1, 0), (1, 2), (3, 0)$.
(3) $f(x, y) = x^3 + 3xy - y^3$ with $H = [-1, 1] \times [-1, 1]$

Exercise 4. For each of the following, use Lagrange multipliers to find all extrema of f subject to the given conditions

(1) $f(x, y) = x + y^2$ with $x^2 + y^2 = 4$.
(2) $f(x, y, z) = xy$ with $x^2 + y^2 + z^2 = 1$ and $x + y + z = 0$..
(3) $f(x, y, z, w) = 3x + y + w$ with $3x^2 + y + 4z^3 = 1$ and $-x^3 + 3z^4 + w = 0$.

Exercise 5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}$ be differentiable at $b = f(a)$. Prove that if $g(b)$ is a local extremum of g then $\nabla(g \circ f)(a) = 0$

Exercise 6. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be C^1 and let $(a, b, c) \in \mathbb{R}^3$ such that $f(a, b, c)$ is an extremum for f subject to the constraint $g(x, y, z) = 0$.

(1) Prove that

$$\frac{\partial f}{\partial x}(a, b, c)\frac{\partial g}{\partial z}(a, b, c) = \frac{\partial f}{\partial z}(a, b, c)\frac{\partial g}{\partial x}(a, b, c),$$

and

$$\frac{\partial f}{\partial y}(a, b, c)\frac{\partial g}{\partial z}(a, b, c) = \frac{\partial f}{\partial z}(a, b, c)\frac{\partial g}{\partial y}(a, b, c).$$

(2) Use this to find all extrema of $f(x, y, z) = 4xy + 2xz + 2yz$ subject to the constraint $xyz = 16$.

Exercise 7. Let $p > 1$ and let $f, g : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \sum_{k=1}^n x_k^2$ and $g(x) = \sum_{k=1}^n |x_k|^p = 1$.

(1) Find all extrema of $f(x)$ subject to the constraint $g(x) = 0$.

(2) Prove that for all $x \in \mathbb{R}^n$ and $1 \leq p \leq 2$,

$$n^{\frac{1}{2}-\frac{1}{p}} \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2} \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}.$$