HOMEWORK 8

Due on Wed, May 27 in class.

Exercise 1. Let $f : [a, b] \to \mathbb{R}$ be monotone. Show that f is integrable.

Exercise 2. Let $E = \{\frac{1}{n} | n \in \mathbb{N}\}$ and $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases}.$$

Show that f is integrable and that $\int_0^1 f(x)dx = 0$.

Exercise 3. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}, \text{ with } p, q \in \mathbb{N} \text{ co-prime} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

(1) Show that the set of points where f is not continuous is $\mathbb{Q}$.
(2) Show that for any $\epsilon > 0$ there are only finitely many points, $t_1, \dots t_n$, satisfying $f(t_j) < \epsilon$.
(3) Given $\epsilon > 0$ construct a partition such that $U(f, P) < \epsilon$.
(4) Conclude that f is integrable and that $\int_0^1 f(x)dx = 0$.

Exercise 4. Let $P_n = \{\frac{j}{n} | j = 0, \dots, n\}$ be partitions of $[0, 1]$.

(1) Show that for any bounded function $f : [0, 1] \to \mathbb{R}$, f is integrable if and only if
$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = I_0(f),$$
In which case $I_0(f) = \int_0^1 f(x)dx$.
(2) Compute $U(f, P_n)$ and $L(f, P_n)$ and $\int_0^1 f(x)dx$ for $f(x) = x$ and for $f(x) = e^x$.

Exercise 5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

(1) Show that if f is continuous at x_0 and $f(x_0) \neq 0$ then $\underline{\int_a^b} |f(x)|dx > 0$.
(2) Assume that f is continuous on $[a, b]$. Show that $\int_a^b |f(x)|dx = 0$ iff $f(x) = 0$ for all $x \in [a, b]$.
(3) Is this true without the continuity assumption?

Exercise 6. Let f be integrable on $[a, b]$ and $E \subset [a, b]$ a finite set. Show that if $g : [a, b] \to \mathbb{R}$ is a function satisfying $f(x) = g(x)$ for all $x \notin E$ then g is integrable and $\int_a^b f(x)dx = \int_a^b g(x)dx$.

Exercise 7. Suppose that $g_n : [0, 1] \to \mathbb{R}$ are positive integrable functions such that $\lim_{n\to\infty} g_n(x) = 0$ pointwise for any $x \in [0, 1]$.

(1) Is it true that $\lim_{n\to\infty} \int_0^1 g_n(x)dx = 0$? Prove or give a counterexample.
(2) Assume that $\lim_{n\to\infty} \int_0^1 g_n(x)dx = 0$ and show that $\lim_{n\to\infty} \int_0^1 f(x)g_n(x)dx = 0$ for any integrable f.

Exercise 8. Let $f : [a, b] \to \mathbb{R}$ be an integrable function and let $\{x_k\}_{k=0}^{\infty}$ be a sequence in $[a, b]$ with $\lim_{k\to\infty} x_k = b$. Show that

$$\lim_{n\to\infty} \int_a^{x_n} f(x)dx = \int_a^b f(x)dx.$$