Imperfect Common Knowledge and Real Exchange Rate Dynamics*

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Abstract†

Large and persistent fluctuations in real exchange rates have long puzzled international macroeconomists. I develop a two-country New-Keynesian framework with dispersed information among price-setters to study this puzzle. In the model, the interaction of strategic complementarity in price-setting and sluggish adjustment of higher-order expectations drives volatility and endogenous persistence of the real exchange rate. I show that a standard calibration of the model captures the volatility and persistence of the real exchange rate observed in the data when the economy is driven by aggregate demand shocks. The analytical tractability of the framework allows for fast Bayesian estimation and testing of the empirical predictions for the real exchange rate. In particular, the test will involve the comparison of the model’s impulse responses to those coming from a structural VAR.

Keywords: PPP Puzzle, Strategic Complementarities, Dispersed Information, Bayesian Estimation

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1 Introduction

Purchasing power parity (PPP) states that consumption price levels, once converted into the same currency, should be equal across countries. It follows that the real exchange rate (RER) - which is the ratio of price levels expressed in a common currency - should be constant and equal to unity. PPP is strongly rejected by the data: real exchange rate fluctuations are large and long-lived for virtually all country pairs. Kenneth Rogoff (1996) in his survey of the empirical evidence finds that real exchange rates are highly volatile and that the half-life of deviations from PPP are in the range of three to five years.\(^1\) While he suggests that a monetary model with sticky prices could explain the ample volatility of the RER, these types of models generally fail to generate the persistence in PPP deviations unless one makes unreasonable assumptions about the length of price stickiness.\(^2\) The difficulty in accounting for both the volatility and the extremely slow reversal to parity of the real exchange rate constitutes a long-standing puzzle in international macroeconomics - the PPP puzzle.

In my research project, I investigate whether dispersed information can account for the dynamic properties of the real exchange rate by building an open-economy DSGE model with imperfect information and estimating it with the data. In particular, I develop a two-country, monetary model where dispersed information about aggregate demand generates incomplete adjustment of prices to nominal shocks, resulting in large and persistent fluctuations in real variables. Firms receive a private signal about aggregate demand and face strategic complementarity in setting prices so that they want to raise prices when they believe everyone else does. Strategic complementarity requires that agents form higher-order expectations (HOE) i.e. beliefs about other agents’ beliefs about aggregate demand. Beliefs, in turn, update very slowly as agents only know the history of their private signals and never learn the true state of aggregate demand. Slow movements in higher-order expectations translate into endogenously slow moving prices, which trigger large and persistent fluctuations in real variables from aggre-

\(^1\)The half-life is defined as the largest time \(T\) such that \(IR(T - 1) \geq 0.5\) and \(IR(T) < 0.5\), where \(IR(T)\) denotes the time-\(T\) response of the RER to a unitary impulse.

\(^2\)A strand of the literature argues that the slow reversal could be explained by the aggregation bias that arises when all the relative relative prices that comprise the real exchange rate do not converge to PPP at the same speed. See Imbs et al. (2005) and Carvalho and Nechio (2011).
gate demand shocks. In an open-economy setting, this intuition implies that changes in relative aggregate demands trigger large and persistent movements in relative real variables, such as the real exchange rate.

I contribute to the literature by analytically deriving real exchange rate dynamics in an open-economy model with dispersed information under different asset market structures. To the best of my knowledge, this is the first paper that incorporates this type of information friction to study the aggregate real exchange rate.\(^3\) From a methodological point of view, imperfect information models generally do not allow for a finite state-space representation and practitioners typically solve these types of models via numerical approximations to the state transition equation. In contrast, in my setup only two specific weighted averages of HOE matter for the dynamics of the model. This insight allows me to characterize a finite state vector and solve the model using the traditional approach of the “sums” versus “differences” in general equilibrium open-economy models (Aoki, 1981).

The model predicts that the extent of these fluctuations depends not only on standard open-economy parameters such as the degree of home bias in consumption and the elasticity of substitution between home and foreign goods but also on the degree of strategic complementarity and the relative precision of agents’ signals. In particular, both the persistence and the volatility increase (i) the higher is the degree of strategic complementarity and (ii) the lower the precision of the signal. Volatility also increases together with (i) the degree of home bias and (ii), if international trade in assets is unavailable, with the complementarity between home and foreign goods. I show that under a standard calibration the model can reproduce the persistence of the real exchange rate and its volatility relative to output observed in the data. Furthermore, with some persistence in monetary shocks, the real exchange rate exhibits the hump-shaped response found in the data (Steinsson, 2008). These promising results show that information frictions can be an important factor in explaining the puzzle.

Moving forward, I want to test whether an estimated version of the model can match the empirical properties of the real exchange rate. The empirical analysis would proceed in two

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\(^3\)Crucini et al. (2010) study good-level real exchange rate using a sticky information model. In another paper, Crucini et al (2012) study the deviation of the law of one price using dispersed information model where the state of the world is revealed after one period.
steps, in the spirit of Melosi (2014). First, I would conduct a Bayesian estimation of the model using only data on real GDP and GDP deflators for the US and an aggregate of European countries. Excluding RER from the observables in the estimation stage would ensure that the model parameters are not pinned down with the intention of fitting the time series of the RER. Then, I would compare the impulse response of the RER of the estimated model to those coming from an identified response of the RER in the data. This would constitute a valid test as no data on the RER are used in the estimation of the parameters of the model.

More specifically, for the Bayesian estimation I would augment the model with technology shocks to allow the number of structural shocks (monetary and productivity) to match the number of observables (real GDP and GDP deflators). The analytical tractability of the model will be of great advantage at this stage, allowing for fast computation of the likelihood function and quick draws from the posterior distribution. As to the identified response of RER in the data, I intend to estimate a bivariate structural VAR of nominal and real exchange rates. The choice of the variables is justified by the model’s prediction that monetary shocks have long-run effects on nominal exchange rates but not on real exchange rates. This prediction can be used to identify the monetary shock in the VAR using a standard long-run restriction à la Blanchard Quah (1989). The impulse responses of the RER to a monetary shock from the model could then be compared with those coming from the VAR.

2 The Model

The model is a two-country open-economy monetary model that follows the international macroeconomic tradition initiated by Obstfeld and Rogoff (1995). The setup is similar to Corsetti et al (2010). The world economy consists of two countries, called H (Home) and F (Foreign). Every country is populated by a continuum of agents residing in an interval [0,1]. Each country specializes in one type of tradable goods, produced in a number of varieties or brands with measure equal to the population size. All goods produced are traded.

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4 More specifically, shocks to money differentials.
2.1 Preferences and Households

The utility function of the representative household in country H is

$$E_t \left\{ \sum_{t=0}^{\infty} \beta^t \left[ \log(C_t) - \int_0^1 \frac{L_{it}^{1+1/\psi}}{1 + 1/\psi} di \right] \right\}$$  \hspace{1cm} (1)$$

The representative household has full information, $E(.)$ denotes the statistical expectations operator, and $\beta$ is the discount factor. Households receive utility from consumption $C_t$ and disutility from working, where $L_{it}$ is hours of labor input in the production of domestic variety $i$. $\psi$ represents the Frisch elasticity of labor supply. Following Woodford (2003, Ch. 3), labor markets for each variety are segmented to generate more strategic complementarities in price setting.\(^5\) Risk is pooled internally to the extent that all domestic agents receive the same consumption level.

Households consume both types of traded goods. The consumption of these goods is $C_{Ht}$ and $C_{Ft}$. For each type of goods, one brand is an imperfect substitute for all the other brands and $\gamma$ is the elasticity of substitution between brands. Consumption of Home and Foreign goods by Home agents are defined as

$$C_{Ht} \equiv \left( \int_0^1 C_t(h) \frac{2-1}{\gamma} di \right)^{1/\gamma} \hspace{1cm} C_{Ft} \equiv \left( \int_0^1 C_t(f) \frac{2-1}{\gamma} di \right)^{1/\gamma} \hspace{1cm} \gamma > 1$$

The overall consumption basket, $C_t$, is defined as

$$C_t \equiv \left[ \alpha^{1/\lambda} (C_{Ht})^{1/\omega} + (1 - \alpha)^{1/\lambda} (C_{Ft})^{1/\omega} \right]^{\omega/\lambda} \hspace{1cm} \omega > 0$$

where $\alpha$ is the weight of the home consumption good and $\omega$ is the elasticity of substitution between home and foreign goods, which I alternatively refer to as the trade elasticity. The utility-based CPI is

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\(^5\) Pricing decisions are strategic complements if, when other firms raise their prices, then a given firm $i$ wishes to raise its price as well. It is closely related to the concept of “real rigidity” in that it depends solely upon real factors: the structure of production costs and of demand. For a discussion see Ball and Romer (1990) and Woodford (2003, Ch 3).
\[
P_t = \left[ \alpha P_{Ht}^{1-\omega} + (1 - \alpha) P_{Ft}^{1-\omega} \right] \frac{1}{1-\omega}
\]

where \( P_{Ht} \) and \( P_{Ft} \) are the price sub-indices for the home- and foreign-produced goods, expressed in domestic currency

\[
P_{Ht} = \left( \int_0^1 p_t(h)^{1-\gamma} \right) \frac{1}{1-\gamma} \\
P_{Ft} = \left( \int_0^1 p_t(f)^{1-\gamma} \right) \frac{1}{1-\gamma}
\]

Foreign prices, denoted with an asterisk like all the foreign variables, are similarly defined. So, the Foreign CPI is

\[
P^*_t = \left[ (1 - \alpha)(P^*_H)^{1-\omega} + \alpha (P^*_F)^{1-\omega} \right] \frac{1}{1-\omega}
\]

Let \( Q_t \) denote the real exchange rate, that is, the relative price of consumption: \( Q_t \equiv \frac{\epsilon_t P^*}{P_t} \), where \( \epsilon_t \) is the nominal exchange rate (domestic currency per foreign currency). Even if the law of one price (LOP) holds at the individual good level (i.e. \( P_t(h) = \epsilon_t P_t(h) \)), the presence of home bias in consumption, that is \( \alpha > 1/2 \), implies that the price of consumption will not be equalized across countries. Put differently, purchasing-power-parity (PPP), i.e. \( Q_t = 1 \) will not hold. The terms of trade are defined as the price of imports in terms of exports, i.e. \( T_t = \frac{P_{Ft}}{\epsilon_t P_{Ht}} \). Furthermore, if the LOP holds, the real exchange rate will be proportional to the terms of trade

\[
q_t = (2\alpha - 1) t_t
\]

where lower case letters denote percentage deviations from steady state.\(^6\) Equation (2) implies that an improvement in the terms of trade always appreciate the real exchange rate. This is in line with the empirical evidence (Obstfeld and Rogoff, 2000). Minimizing expenditure over goods, we can derive the domestic household demand for a generic good \( h \), produced in country H, and the demand for a good \( f \), produced in country F:

\[
C_t(h) = \left( \frac{P_t(h)}{P_{Ht}} \right)^{-\gamma} \left( \frac{P_{Ht}}{P_t} \right)^{-\omega} \alpha C_t \\
C_t(f) = \left( \frac{P_t(f)}{P_{Ft}} \right)^{-\gamma} \left( \frac{P_{Ft}}{P_t} \right)^{-\omega} (1 - \alpha) C_t
\]

\(^6\)This result assumes symmetric initial conditions.
Assuming that the LOP holds, total demand for a generic good $h$ or $f$ can be written as

\[ Y_t^d(h) = \left( \frac{P_t(h)}{P_{ht}} \right)^{-\gamma} \left( \frac{P_{ht}}{P_t} \right)^{-\omega} \left[ \alpha C_t + (1 - \alpha) Q^*_{ct} C^*_t \right] \quad (3) \]

\[ Y_t^d(f) = \left( \frac{P_t(f)^*}{P_{ft}^*} \right)^{-\gamma} \left( \frac{P_{ft}^*}{P_t^*} \right)^{-\omega} \left[ \alpha Q^*_{ct} C_t + (1 - \alpha) C^*_t \right] \quad (4) \]

### 2.2 Budget Constraints

The generic home household’s budget constraint can be written as

\[ P_t C_t + B_t + \int q_{ht, s_{t+1}} B_{ht, s_{t+1}} ds_{t+1} \leq \int_0^1 W_{it} L_{it} di + B_{t-1} + B_{ht} + P_t \int_0^1 X_{it} di \quad (5) \]

$B_t$ are nominal domestic bonds held between $t - 1$ and $t$ which are zero in net supply and pay off a risk-free nominal rate $R_t$ at the beginning of time $t$. $B_{ht}$ is the holding of state-contingent claims that pay off one unit of domestic currency in the realized state of the world at time $t, s_t$. $q_{ht, s_{t+1}}$ is the time-$t$ price of a state-contingent claim that pays off one unit of domestic currency if tomorrow’s state is $s_{t+1}$. $W_{it}$ is the wage for the $i$th type of labor input and $X_{it}$ are the real profits of domestic firm $i$. Maximizing (1) subject to (5) gives the static first order condition:

\[ C_t^{1/\psi} = \frac{W_{it}}{P_t} \quad (6) \]

and the following Euler equation

\[ \frac{1}{C_t} = \beta (1 + R_{t+1}) \mathbb{E}_t \left[ \frac{1}{C_{t+1}} \frac{P_{t+1}}{P_{t+1}} \right] \quad (7) \]

### 2.3 Price-setting decisions

Firms do not observe the state of aggregate demand but at each date they receive a private signal about it. Prices are set in the producer’s currency and there are no barriers to trade so that the LOP always holds. Firm $h$’s expected real profit in period $t$, conditional on the history
of signals observed by that firm at time $t$, are given by

$$X_{ht} = \mathbb{E}_{ht} \left[ \frac{P_t(h)}{P_t(H_t)} Y^d_t(h) - \frac{W_{ht}}{P_t} H_{ht} \right]$$

(8)

where $\mathbb{E}_{ht}$ is the expectation operator conditional on the history of firm $h$’s signals, $Z_h^t = \{Z_{h,\tau}\}_{\tau = -\infty}^t$. The production function is given by

$$Y_t(h) = AL_{ht}$$

(9)

Firms receive two signals, $z_{h,t}^m$ and $z_{h,t}^{\ast}$, about Home and Foreign monetary policy

$$Z_{h,t} = \begin{bmatrix} z_{h,t}^m \\ z_{h,t}^{\ast} \end{bmatrix} = \begin{bmatrix} m_t \\ m_t^{\ast} \end{bmatrix} + \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_v^{\ast 2} \end{bmatrix} \begin{bmatrix} v_{h,t}^m \\ v_{h,t}^{\ast} \end{bmatrix}$$

(10)

where $v_{h,t}^m, v_{h,t}^{\ast} \sim \mathcal{N}(0, 1)$. $m_t = \ln M_t$ and $m_t^{\ast} = \ln M_t^{\ast}$ represent the money supplies and the signal noises $v_{h,t}^m$ and $v_{h,t}^{\ast}$ are assumed to be independently and identically distributed across firms and time. Foreign firms receive similar signals with the same distributions. In every period $t$, firms observe the history of their signals $Z_h^t$ (that is, their information set is $I_{ht} = \{Z_{h,\tau}\}_{\tau = -\infty}^t$) and maximize (8) subject to (9) and (4). The first order condition yields

$$\mathbb{E}_{ht} \left[ \left( \frac{P_t(h)}{P_{ht}} \right)^{-\gamma} \left( \frac{P_{ht}}{P_t} \right)^{-\omega} C_t^{W} \right] = \gamma \mathbb{E}_{ht} \left[ \left( \frac{P_t(h)}{P_{ht}} \right)^{-\gamma - 1} \left( \frac{P_{ht}}{P_t} \right)^{-\omega} C_t^{W} W_{ht} \right]$$

where $C_t^W \equiv \alpha C_t + (1 - \alpha) \mathcal{Q}_t^* C_t^*$. Following the tradition in this literature, we log-linearize the price-setting equation around the deterministic steady state so that the transition equation of average price is linear. We assume that firms use the log-linearized model, rather than the original nonlinear model, when addressing their signal-extraction problem. This assumption greatly simplifies the analysis, because it allows the use of the Kalman filter to characterize the dynamics of firms’ belief. Finally we assume that at the beginning of time firms are endowed with an infinite history of signals. This implies that the Kalman gain matrix is time-invariant and identical across firms (Melosi, 2014).
Appendix A shows that, under the PCP assumption the log-linearized first-order condition for a generic $h$ and $f$ firm, combined with equation (2), reads

$$p_t(h) = E_{it}\{p_{Ht} + (\gamma + \psi)^{-1}[(1 - \alpha)(\psi + 2\alpha\omega)t_t + (\alpha + \psi)c_t + (1 - \alpha)c_t^*]\}$$ (11)

$$p_t^*(f) = E_{it}^*\{p_{Ft}^* - (\gamma + \psi)^{-1}[(1 - \alpha)(\psi + 2\alpha\omega)t_t - (1 - \alpha)c_t - (\alpha + \psi)c_t^*]\}$$ (12)

### 2.4 Monetary Policy

Following Woodford (2001), we assume monetary policy ensures that nominal incomes follow exogenous stochastic processes

$$M_t = P_{Ht}Y_{Ht} \quad \quad M_t^* = P_{Ft}^*Y_{Ft}$$

$$\Delta M_t = \ln M_t - \ln M_{t-1} + u_t^m$$

$$\Delta M_t^* = \ln M_t^* - \ln M_{t-1}^* + u_t^{m*}$$

where $\Delta M_t \equiv \ln M_t - \ln M_{t-1}$ and the monetary shocks $u_t^m$ and $u_t^{m*}$ are iid, distributed as $N(0, \sigma_u^2)$ and uncorrelated across countries. Under financial autarky, nominal income must equal nominal consumption expenditure so that

$$M_t = P_{Ht}Y_{Ht} = P_tC_t \quad \quad M_t^* = P_{Ft}^*Y_{Ft} = P_t^*C_t^*$$ (15)

Under complete markets, we assume a standard cash-in-advance constraint for consumption expenditure which ensures that equations (15) hold as well.

### 2.5 Exchange rate determination

The exchange rate behavior will differ between the financial market economy and the complete market one.
2.5.1 Financial Autarky

Given that agents cannot trade assets internationally, trade must be balanced in every period. That is the value of imports must be equal to the value of exports

\[ P_{Ft}C_{Ft} = \epsilon_t P_{Ht}C_{Ht} \]

which can be rewritten as

\[ T^{1-\omega}_t C_t = Q_t^* C_t^* \quad (16) \]

It is this equation, together with the monetary policy rules and the optimal prices, that determines nominal and real exchange rates in financial autarky.

2.5.2 Complete Markets

If asset markets are internationally complete, the following risk-sharing condition must hold

\[ \frac{C_t}{C_{t+1} P_{t+1}} = \frac{C_t^*}{C_{t+1}^* \epsilon_t P_t^*} \]

This equation relates the cross-country differential in the growth rate of consumption to the depreciation of the exchange rate. Assuming symmetric initial conditions, this can be rewritten as

\[ \frac{C_t}{C_t^*} = \frac{\epsilon_t P_t^*}{P_t} \quad (17) \]

Equation (17) is an efficiency condition that equates the marginal rate of substitution between home and foreign consumption to their marginal rate of transformation, expressed as equilibrium prices i.e. the real exchange rate. A key consequence is that home consumption can rise relative to foreign only if the real exchange rate depreciates. Equation (17), combined with the monetary policy rules and the optimal prices, determines real and nominal exchange rates under complete markets.
3 Model Solution

The solution of models with imperfect common knowledge in general depends on the dynamics of higher-order beliefs. At least two approaches have been developed to solve these kind of models. A numerical approach consists in guessing and verifying the laws of motion for the vector of higher-order beliefs. Since this vector is infinite-dimensional, in practice it is truncated at a sufficiently high order.\footnote{For an example see Lorenzoni (2009).}

Another approach developed by Woodford (2001) exploits the fact that in some cases only a \textit{weighted average} of these higher-order expectations matters for the solution of the model. The advantage of this approach is that the dimensionality of the state vector becomes much smaller and there is no need to truncate it. The model developed here meets the conditions for the applicability of this method. Understanding exactly which weighted average of higher-order expectations matters requires first working with the pricing equations (11) and (12) and expressing them in terms of higher order expectations of exogenous variables, in our case the money supplies $m_t$ and $m^*_t$. Below we will work through the derivation for the case of financial autarky and leave the derivation of the complete markets case for the Appendix B.

3.1 Solution under Financial Autarky

Log-linearizing equation (16) yields

$$(1 - \omega) t_t + c_t = \omega q_t + c^*_t$$

(18)

Combining this with the equation (2), the equation for the price indices, the process for nominal income (15) and its foreign counterpart, into equation (11) yields

$$p_t(h) = \mathbb{E}_{ht}[(1 - \xi)p_{ht} + \xi m_t]$$

(19)

where $\xi = \frac{1+\psi}{\gamma+\psi} > 1$. Repeating the process for the foreign pricing equation yields

$$p_t^*(f) = \mathbb{E}_{ft}[(1 - \xi)p_{ft}^* + \xi m^*_t]$$

(20)
The parameter $1 - \xi$ represents the degree of strategic complementarities in price setting i.e. it tells by how much the optimal price of an individual firm changes when all the other competitors are changing the prices. Note that in financial autarky with flexible exchange rates and PPP there is no dependence of optimal domestic prices on foreign variables, because exchange rates adjust in such a way that domestic nominal prices do not need to respond to foreign ones.

Integrating (19) over domestic agents and (20) over foreign agents and noting that the log-linear price indices read as $p_{Ht} = \int_0^1 p_t(h)dh$ and $p_{Ft} = \int_0^1 p_t(f)df$ we obtain

$$p_{Ht} = \bar{\mathbb{E}}_t[(1 - \xi)p_{Ht} + \xi m_t]$$ (21)

$$p_{Ft}^* = \bar{\mathbb{E}}_t[(1 - \xi)p_{Ft}^* + \xi m_t^*]$$ (22)

where $\bar{\mathbb{E}}_t = \int_0^1 \mathbb{E}_h dh = \int_0^1 \mathbb{E}_f df$. Note that Home and Foreign average expectations will be identical given that home and foreign agents are of the same measure and receive exactly the same type of signals. This insight becomes crucial for the solution in the complete markets case.

We can substitute equation (21) and (22) into (19) and (20), integrate the resulting equation and repeat the process to find the equilibrium prices

$$p_{Ht} = \xi \sum_{k=1}^{\infty} (1 - \xi)^{k-1} \bar{\mathbb{E}}^{(k)}_t m_t$$ (23)

$$p_{Ft}^* = \xi \sum_{k=1}^{\infty} (1 - \xi)^{k-1} \bar{\mathbb{E}}^{(k)}_t m_t^*$$ (24)

Equations (23) and (24) say that the domestic and foreign optimal prices will be a weighed average of the average higher order expectations about domestic or foreign monetary policy, similar to Woodford (2001). If agents had full information, the expectation operator would coincide with the full-information rational expectation operator and we would obtain $p_{Ht} = m_t$ and $p_{Ft}^* = m_t^*$. In other words, prices would fully adjust to monetary shocks and there would be no effects on real variables. This is most clear if one solves for outputs using $m_t = p_{Ht} + y_{Ht}$.
and $m_t^* = p_{Ft}^* + y_{Ft}$, which yields

$$y_{Ht} = \xi \sum_{k=1}^{\infty} (1 - \xi)^{k-1} \left( m_t - \bar{E}_t^{(k)} m_t \right)$$

(25)

$$y_{Ft} = \xi \sum_{k=1}^{\infty} (1 - \xi)^{k-1} \left( m_t^* - \bar{E}_t^{(k)} m_t^* \right)$$

(26)

Under full-information rational expectations $\bar{E}_t^{(k)} m_t = m_t$ and $\bar{E}_t^{(k)} m_t^* = m_t^*$ for every $k$, which implies that outputs do not respond to monetary shocks. When information is imperfect, outputs respond to the extent that average higher-order expectations about monetary policies deviate from their true value. Finally, we can use equation (18) with (2) to express the consumption differential as a function of the terms of trade

$$c_t - c_t^* = (2\alpha \omega - 1) t_t$$

(27)

Using the monetary policy equations, the definition of real exchange rate and equation (2) again we obtain

$$\varepsilon_t = m_t - m_t^* - 2\alpha (\omega - 1) t_t$$

(28)

where $\varepsilon_t$ is the percentage deviation of $\epsilon_t$ from the steady state. With the definition of terms of trade and the solutions for the optimal prices (23) and (24), we obtain the expression for the nominal exchange rate

$$\varepsilon_t = \frac{1}{1 + 2\alpha (\omega - 1)} (m_t^P) + \frac{2\alpha (\omega - 1)}{1 + 2\alpha (\omega - 1)} \xi \sum_{k=1}^{\infty} (1 - \xi)^{k-1} \left( \bar{E}_t^{(k)} m_t^P \right)$$

(29)

where $m_t^P \equiv m_t - m_t^*$. Finally, substituting the optimal prices and (29) into (2) gives an expression for the real exchange rate

$$q_t = \frac{2\alpha - 1}{1 + 2\alpha (\omega - 1)} \xi \sum_{k=1}^{\infty} (1 - \xi)^{k-1} \left( m_t^P - \bar{E}_t^{(k)} m_t^P \right)$$

(30)
The intuition behind equations (29) and (30) is straightforward. First, the exchange rate, being a relative price, is only affected by shocks to relative money supplies, $m_t - m_t^*$. Above we have said that under full-information rational expectations, prices adjust completely to nominal shocks and real quantities are not affected. In terms of exchange rates, this means that the nominal exchange rate completely absorbs the relative monetary shock and the real exchange rate is left unaffected. In that case equation (29) would read $\varepsilon_t = m_t^P$. Under imperfect common knowledge instead, as long as we have home bias, the real exchange rate responds to shocks to the extent that higher-order expectations deviates from full-information rational expectations. As a result, the nominal exchange rate adjusts on impact by less than the relative monetary shock. To sum up the response of the real exchange rate depends on three factors: (i) the response of HOE (ii) the degree of home bias and (iii) the trade elasticity. The higher the degree of home bias and the lower the degree of substitution between goods the larger is the adjustment in relative prices of home and foreign goods to clear the markets. Given that prices are “sticky”, the larger the response of the real exchange rate. Section 4 explores in more details the role of HOE in determining real exchange rate fluctuations.

3.1.1 State-space representation

To solve the model we follow Woodford (2001) and guess that the state of the system includes the exogenous state variables plus the two specific weighted averages of high-order expectations implied by the optimal prices (23) and (24). In particular, we define $F_t \equiv \xi \sum_{k=1}^{\infty} (1 - \xi) X_t^{(k)}$ where $X_t = [m_t, m_{t-1}, m_t^*, m_{t-1}^*]'$ is the vector of exogenous state variables. The transition equation for the model can be shown to be

$$\dot{X}_t = \bar{c} + \bar{B} \dot{X}_{t-1} + \bar{b} u_t$$ (31)
where

\[ \bar{X}_t = \begin{bmatrix} X_t \\ F_t \end{bmatrix}, \bar{c} = \begin{bmatrix} (1 - \rho_m)g \\ 0 \\ (1 - \rho_m)g \\ 0 \\ d_{4\times1} \end{bmatrix}, \bar{B} = \begin{bmatrix} B_{4\times4} & 0 \\ G_{4\times4} & H_{4\times4} \end{bmatrix}, \bar{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ m_{4\times1} \end{bmatrix}, \bar{u}_t = \begin{bmatrix} u^m_t \\ u^{m*}_t \end{bmatrix} \]

Equation (31) is the state transition equation of the system, whereas equation (10) and its foreign counterpart are the observation equations. The \( \bar{B} \) matrix is given by the exogenous processes for monetary policy whereas \( d, G, H \) and \( m \) are to be determined by solving the signal-extraction problem of the firms using the Kalman filter equations. One can show that these matrices are functions of the parameters of the model (in particular \( \Theta = \{ \rho_m, \sigma^2_v, \sigma^2_v^*, \sigma^2_u, \sigma^2_u^*, \xi \} \)) and the Kalman gain matrix associated with the firms’ signal extraction problem. Iteration over the Riccati’s equation allows to solve the model very quickly (0.11 seconds). For details see the Appendix C.

4 Analytical Results under Random Walks

To gain intuition about the cyclical properties of the real exchange rate let us study the simple case when money supplies follow a random walk. Precisely, for this section we assume:

\[ m_t = m_{t-1} + u_t \]

(32)

\[ m^*_t = m_{t-1} + u^*_t \]

(33)

which is obtained as a special case from equation (13) setting \( g = \rho_m = 0 \). With random walks in nominal spending and linear updating implied by the signal-extraction problem we can establish Proposition 1.

Proposition 1 Assuming random walk processes for nominal spending in financial autarky
the real exchange rate follows an AR(1) process

\[ q_t = \nu^{FA} q_{t-1} + \frac{2\alpha - 1}{1 + 2\alpha(\omega - 1)} \nu^{FA} u^D_t \]

where \( u^D_t = u_t - u^*_t \) and \( 1 - \nu^{FA} = \xi \times \kappa_1 + (1 - \xi) \times \kappa_2 \in (0,1) \) and \( \kappa_1, \kappa_2 \) are the non-zero elements of the Kalman gains matrix. The autocorrelation and variance of the real exchange rate are

\[ \rho_\hat{Q} = \nu^{FA} \quad \sigma^2_\hat{Q} = \left(\frac{2\alpha - 1}{1 + 2\alpha(\omega - 1)}\right)^2 \left(\frac{\nu^{FA}}{1 - \nu^{FA}}\right)^2 (\sigma_u^2 + \sigma_{u^*}^2) \]

Furthermore, given that output follows the AR(1) process \( y_{Ht} = \nu^{FA} y_{Ht-1} + \nu^{FA} u_t \) the variance of the real exchange rate relative to output is

\[ \frac{\sigma^2_\hat{Q}}{\sigma^2_{Y_H}} = \left(\frac{2\alpha - 1}{1 + 2\alpha(\omega - 1)}\right)^2 \left(\frac{\sigma_u^2 + \sigma_{u^*}^2}{\sigma_u^2}\right) \]

Under the assumption of complete markets the real exchange rate follows a very similar process (details in the Appendix B).

**Proposition 2** Assuming random walk processes for nominal spending and complete asset markets (CM) the real exchange rate follows an AR(1) process

\[ q_t = \nu^{CM} q_{t-1} + (2\alpha - 1)\nu^{CM} u^D_t \]

where \( \varphi \equiv \frac{(1+\psi)+4\alpha(1-\alpha)(\omega-1)}{\gamma+\psi} \), \( 1 - \nu^{CM} = \varphi \times \kappa_1 + (1 - \varphi) \times \kappa_2 \in (0,1) \) and \( \kappa_1, \kappa_2 \) are the non-zero elements of the Kalman gains matrix. The autocorrelation and variance of the real exchange rate are

\[ \rho_\hat{Q} = \nu^{CM} \quad \sigma^2_\hat{Q} = (2\alpha - 1)^2 \left(\frac{\nu^{CM}}{1 - \nu^{CM}}\right)^2 (\sigma_u^2 + \sigma_{u^*}^2) \]

Finally if \( \omega = 1 \) then \( \xi = \varphi \) and \( \nu^{FA} = \nu^{CM} \).

The last part of Proposition 2 simply reiterates a well-known results in international macroeconomics due to Cole and Obstfeld (1991). In these type of models, when the trade elasticity
is equal to unity, terms of trade fluctuations ensure complete international risk-sharing even in the case of financial autarky. In other words, there is no need for international trade in assets to insure countries against idiosyncratic risks because terms of trade movements already offer a natural, perfect insurance mechanism.

In contrast to the financial autarky case, equations (36) and (37) in Appendix B stress the interdependence between the home optimal price and the foreign counterpart through the terms of trade. This is the reason why the degree of strategic complementarities in the case of complete markets depends also on the trade elasticity $\omega$, as highlighted in Proposition 2. Intuitively, for a given change in demand, the more substitutable home and foreign goods are, the smaller is the change in price that a firm $i$ would implement, if it could change its price without affecting the other firms’ decision. This implies that the higher the trade elasticity the smaller the firms’ reaction to changes in aggregate demand conditions i.e. the larger the degree of strategic complementarity.

Figures 1 and 2 offer a graphical explanation of the results in Proposition 1. In particular, Figure 1 shows the contour sets for the volatility of real exchange rate relative to output as the parameters $\alpha$ and $\omega$ vary in the financial autarky case. The relative volatility of the RER rises with $\alpha$ and falls with $\omega$. The interpretation is pretty straightforward. For a given movement in relative prices ($t_t$), a higher degree of home bias increases the volatility of the real exchange rate. This happens because, if the price of the home good rises relative to the foreign good, the home consumption basket will increase in price relative to the foreign one, given the higher weight being placed on home goods. As to the elasticity of substitution, the higher the substitution between home and foreign goods, the lower is the relative price change required to equilibrate demands.

Figure 2 depicts the contour sets for persistence of the real exchange rate. Similarly to Woodford (2001), it can be shown that the parameter $\nu$ governing the persistence in the RER depends on the underlying parameters $\xi$ and $\sigma^2_t/\sigma^2_u$. A lower $\xi$ indicates a higher degree of strategic complementarities, which means that agents put a larger weight on the their beliefs about others’ action (and beliefs about others’ beliefs about others’ actions) relative to their belief about the current state of money supply (see equation (23)). This implies that higher-
order beliefs receive a higher weight than lower-order beliefs. With high-order beliefs moving more sluggishly than low-order beliefs\(^8\), prices move more sluggishly, which in turn implies larger movements in the RER. Finally, when the relative precision of the signal falls \((\sigma_v^2/\sigma_u^2 \downarrow)\) agents will weight their prior more than their signal, failing to change prices and slowly updating their beliefs when monetary shocks hit the economy. The slow movement in prices again triggers slow reversion to PPP.

Finally, notice from Propositions 1 and 2, that the a higher \(\nu\) not only affects the persistence of the exchange rate but also its volatility. To understand this, think about the response of prices when a monetary shock hits the Home economy. The higher \(\nu\) the smaller the adjustment of home prices at the impact of the shock for the same reasons discussed above. Slow impact response of prices drives the amplification of monetary shocks onto the RER.

To have a feeling of the quantitative properties of this model we explore a baseline calibration of the financial autarky model to see whether it can match the empirical properties of the real exchange rate in the data. Here we focus on matching the relative volatility of the exchange rate to output and its autocorrelation. We take the figures from Table Chari, Kehoe and McGrattan (2002). The real exchange rate is 4.36 times as volatile as output and its autocorrelation is 0.83.\(^9\)

Table 1 summarizes the calibration used for this exercise. The empirical literature has not settled on the value of the trade elasticity \(\omega\). Here we follow Rabanal et al. (2008) and assume a relative low value such that Home and Foreign goods are complements. We choose a degree of home bias 0.9 consistent with the literature. These reasonable parameter values allows us to match the relative volatility of the RER.

The parameter \(\xi\) depends on \(\gamma\) and \(\psi\). We set these to 7 and 0.5 following Mankiw Reis (2010). For this exercise we set \(\rho_m = 0\). The signal to noise \((\sigma_v^2/\sigma_u^2)\) remains to be calibrated. Given the poor prior information we have on this parameter we choose it so as to match the persistence of the exchange rate found in the data. We settle on the value of 1/6. This is the same value Woodford (2001) chooses. It is the value that in a closed economy matches

\(^8\)See Woodford (2002) or Melosi (2014) for further explanation and graphical examples.

\(^9\)The statistics are bases on the RER between the US and a European aggregate of Italy, Germany, France and the UK over the period 1973:1-1994:4
the inflation and output dynamics of the imperfect information model when compared to a Calvo model where one-third of all prices are revised each quarter, consistent with empirical evidence of Blinder et al. (1998). Quite surprisingly, this sensible calibration of the signal to noise ratio provides enough persistence in the RER to match the empirical counterpart, even without assuming autocorrelated monetary shocks.

5 Impulse Responses

In this section we allow the monetary processes to be autocorrelated so that $\rho_m \neq 0$ while keeping $g = 0$. We study the effects of persistent increases in nominal spending via impulse responses.

Figure 3 shows the impulse responses of key variables for the financial autarky model, under different values of $\rho_m$. In response to a home monetary shock, home output rises as a result of incomplete price adjustment. Consumption rises in both countries, more so in Home given the presence of home bias. The nominal exchange rate depreciates as a result of the monetary expansion. Home goods’ prices rise by less than the nominal exchange rate, resulting in an worsening (upward movement) of the terms of trade. The real exchange rate depreciates, as it is proportional to the terms of trade. Finally, domestic inflation as both the price of the home goods and foreign goods rise in domestic currency. Conversely, Foreign inflation falls as foreign goods’ prices are unchanged and home goods’ prices fall in domestic currency.

Introducing persistence in nominal spending does not alter the direction of the responses but does change their shape. In particular, most variables now display a hump-shaped response to the monetary shock. The hump in the impulse response of the real exchange rate is consistent with the empirical literature (for instance, see Steinsson(2008)). Interestingly, domestic inflation displays a hump for $\rho_m = 0.6$ but not for $\rho = 0.3$. Hence, this model would be consistent with the concept of “sticky inflation” given enough persistence in monetary shocks. Finally, increasing $\rho_m$ “shifts out” the hump for all the variables as agents in early periods receive (on average) signals that indicate positive growth in money supply but it then takes them some time to realize the end of the monetary stimulus.

Figure 4 similarly shows the impulse responses for the case of complete markets. The re-
sponses are similar in nature to the previous case, except for a couple of things. First, foreign output now responds to the home monetary shock, and it increases given the complementarity of home and foreign goods. Second, the amplification on the real exchange rate is a little muted compared to the financial autarky case as our baseline calibration implied $2\alpha(\omega - 1) < 0$.

6 Empirical Analysis

This section has still to be developed. We would like to take to the data a variant of the model that includes technology shocks and investigate whether it can explain the dynamics of the RER. To do so we would not estimate the model the real exchange rate directly but we would rather use other time series to estimate the parameters. In particular, given four shocks (monetary and technology shocks in the two countries), we could estimate the model using Bayesian methods to the US and an aggregate of European countries using four observables: the price levels and real GDP. We could then test the ability of the model to explain the real exchange rate by either comparing second moments or comparing impulse responses from an identified VAR. More precisely, a bivariate SVAR of nominal and real exchange rates could be estimated where the monetary shock is identified via the long-run restriction that it affects only the nominal exchange rate. The model- and data-impulse responses could then be compared. This would constitute a valid test of the model as no RER data would be used for its estimation.

7 Conclusion

I have developed an analytically tractable, two-country New-Keynesian model with dispersed information among price-setters to study the PPP puzzle. The model suggests that strategic complementarity in price-setting and sluggish adjustment of higher-order expectations can be important factors in the explanation of the puzzle. In particular, volatility and persistence of the real exchange in line with the data can be obtained with a very reasonable calibration of the model. Moreover, with some persistence in the monetary process, the model can generate the hump-shaped impulse response of the real exchange rate observed in the data. Given the promising properties of the model, it seems natural to move forward towards its estimation and empirical testing.
8 Appendices

8.1 Appendix A: Obtaining equations (11) and (12)

Available upon request.

8.2 Appendix B: Solution under Complete Markets

Under complete markets, consumption differentials are related to the real exchange rate through equation (17). In log-linear terms

\[ q_t = c_t - c^*_t \]  

(34)

This implies that the nominal exchange rate will be equal to the money differential

\[ \varepsilon_t = q_t + p_t - p^*_t = (c_t + p_t) - (c^*_t + p^*_t) = m_t - m^*_t \]  

(35)

Using these relationships, the monetary policies and the definition of terms of trade into the optimal price equations (23) and (24) one obtains

\[ p_t(h) = E_h \left[ (1 - \xi)p_{Ht} + \frac{2\alpha(1 - \alpha)(\omega - 1)}{(\gamma + \psi)} t_t + \xi m_t \right] \]  

(36)

\[ p^*_t(f) = E_f \left[ (1 - \xi)p^*_{Ft} - \frac{2\alpha(1 - \alpha)(\omega - 1)}{(\gamma + \psi)} t_t + \xi m^*_t \right] \]  

(37)

These equations show the interdependence of the optimal price with their foreign counterpart through the terms of trade. In particular, if home and foreign goods are substitute \((\omega > 1)\), other things equal, a rise in the price of foreign goods (that is a rise in \(t_t\)) causes expenditure switching away from foreign goods towards home goods. The increased demand for home goods makes it optimal to raise \(p_t(h)\). If goods are instead substitutes, a rise in \(t_t\) decreases demand both for foreign and home output, hence the optimal price for a home good \(p_t(h)\) falls.

We can disentangle these two equations following the tradition of the “sum” versus “differences” approach in general equilibrium open-economy models (Aoki, 1981). Averaging (36)
and (37) across home and foreign agents respectively yields

\[ p_{Ht} = \bar{E}_t \left[ (1 - \xi)p_{Ht} + \frac{2\alpha(1 - \alpha)(\omega - 1)}{(\gamma + \psi)} t_t + \xi m_t \right] \] (38)

\[ p_{Ft}^* = \bar{E}_t \left[ (1 - \xi)p_{Ft}^* - \frac{2\alpha(1 - \alpha)(\omega - 1)}{(\gamma + \psi)} t_t + \xi m_t^* \right] \] (39)

The key to the solution is that if signals are symmetric across countries and signal noises have the same distributions, then the average expectations of exogenous and endogenous variables will be the same in Home and Foreign. We can take the sum of (38) and (39)

\[ p_{Ht} + p_{Ft}^* = \bar{E}_t \left[ (1 - \xi)(p_{Ht} + p_{Ft}^*) + \xi (m_t + m_t^*) \right] \] (40)

which yields the solution

\[ p_{Ht} + p_{Ft}^* = \xi \sum_{k=1}^{\infty} (1 - \xi)^{k-1}\bar{E}_t^{(k)} m_t^W \] (41)

where \( m_t^W = m_t + m_t^* \). We can take the difference between (38) and (39)

\[ p_{Ht} - p_{Ft}^* = \bar{E}_t \left[ (1 - \xi)(p_{Ht} - p_{Ft}^*) + \xi (m_t - m_t^*) \right] \] (42)

Using the definition of terms of trade and the solution for the nominal exchange rate this becomes

\[ p_{Ht} - p_{Ft}^* = \bar{E}_t \left[ (1 - \varphi)(p_{Ht} - p_{Ft}^*) + \varphi (m_t - m_t^*) \right] \] (43)

where \( \varphi \equiv \frac{(1 + \psi) + 4\alpha(1 - \alpha)(\omega - 1)}{\gamma + \psi} \). The solution to the above equation yields

\[ p_{Ht} - p_{Ft}^* = \varphi \sum_{k=1}^{\infty} (1 - \varphi)^{k-1}\bar{E}_t^{(k)} m_t^D \] (44)

where \( m_t^D = m_t - m_t^* \). It is immediate to verify that world output and output differentials are
given by

$$y_t^W \equiv y_{Wt} + y_{Ft} = m_t^W - \xi \sum_{k=1}^{\infty} (1 - \xi)^k \bar{m}^{(k)}_t m_t^W$$

$$y_t^D \equiv y_{Wt} - y_{Ft} = m_t^D - \varphi \sum_{k=1}^{\infty} (1 - \varphi)^k \bar{m}^{(k)}_t m_t^D$$

Finally, the real exchange rate is

$$q_t = (2\alpha - 1) \left( m_t^D - \varphi \sum_{k=1}^{\infty} (1 - \varphi)^k \bar{m}^{(k)}_t m_t^D \right)$$

Country-specific variables can be solved using the fact that

$$x_{H} = \left( x_{W} + x_{D} \right) / 2$$

and

$$x_{F} = \left( x_{W} - x_{D} \right) / 2.$$  

### 8.2.1 State-space representation

We follow the approach used in 3.1.1 but define the exogenous state vector as

$$X_t = \{m_t^W, m_{t-1}^W, m_t^D, m_{t-1}^D\}.$$  

The vector $F_t$ and the other matrices involved in the transition equation are defined as above, with the only exception that now $u_t = \{u_t^W, u_t^D\}$. Given the orthogonality of the monetary shocks and the noise shocks across countries, there is a simple mapping between these new “sums” and “differences” shocks and the country specific ones. Hence, firms can be thought of as receiving signals $\{z_{mW}^{ht}, z_{mD}^{ht}\}$ or $\{z_{mW}^{ht}, z_{mD}^{ht}\}$. For ease of exposition we follow the latter approach and define the signals

$$Z_{h,t} = \begin{bmatrix} z_{mW}^{ht} \\
\end{bmatrix}
\begin{bmatrix} m_t^W \\
m_t^D \\
\end{bmatrix}
\begin{bmatrix} \sigma_{vW}^2 & 0 \\
0 & \sigma_{vD}^2 \\
\end{bmatrix}
\begin{bmatrix} v_{mW}^{ht} \\
v_{mD}^{ht} \\
\end{bmatrix}$$

and similarly for foreign firms.

### 8.3 Appendix C: Solving for the matrices $d, G, H$ and $m$

Available upon request.
References


9 Tables & Figures

Table 1: Baseline Calibration

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Figure 1: Relative Volatility of Real Exchange Rate ($\rho = 0$)
Figure 2: Persistence of Real Exchange Rate ($\rho = 0$)
Figure 3: Impulse Responses to a Home Monetary Shock - Financial Autarky
Figure 4: Impulse Responses to a Home Monetary Shock - Complete Markets