CONVERGENCE OF NORMALIZED BETTI NUMBERS IN NONPOSITIVE CURVATURE

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Abstract. We study the convergence of volume-normalized Betti numbers in Benjamini-Schramm convergent sequences of non-positively curved manifolds with finite volume. In particular, we show that if X is an irreducible symmetric space of noncompact type, $X \neq \mathbb{H}^3$, and (M_n) is any Benjamini-Schramm convergent sequence of finite volume X-manifolds, then the normalized Betti numbers $b_k(M_n)/vol(M_n)$ converge for all k.

As a corollary, if X has higher rank and (M_n) is any sequence of distinct, finite volume X-manifolds, the normalized Betti numbers of M_n converge to the L^2 Betti numbers of X. This extends our earlier work with Nikolov, Raimbault and Samet in [1].

Contents

1. Introduction

We begin with a fair amount of general motivation, mostly from Elek [15] and Bowen [11]. The well-versed reader can skip ahead to §1.1 for the statements of our results.

The *normalized Betti numbers* of a space X are the quotients

$$\beta_k(X)/\mathrm{vol}(X), \quad \text{where} \quad \beta_k(X) := \dim H_k(X, \mathbb{R}).$$

All spaces in this paper will be either Riemannian manifolds or simplicial complexes. In the latter case, volume should be interpreted as the number of vertices.

Fix $d > 0$. A simplicial complex K has *degree at most* d if every vertex in K is adjacent to at most d edges. In [15], Elek shows that the normalized Betti numbers of finite simplicial complex K with degree at most d are *testable*, meaning that there is a way to read off approximations of the normalized Betti numbers while only looking at bounded random samples of K. More precisely, given $\epsilon > 0$, there is some $R(\epsilon)$ as follows. Given K, select R vertices of K at random and look at the R-neighborhood of each in K. Testability means there is a way to guess from this data what the normalized Betti numbers of K are, that is correct up to an error of ϵ with probability $1 - \epsilon$.

This is really a continuity result, in the following sense. Consider the topological space

$$\mathcal{K} = \big\{\text{pointed finite degree simplicial complexes } (K,p)\big\}/\sim,$$

where each $p \in K$ is a vertex, two pointed complexes are equivalent if they are isomorphic via a map that takes basepoint to basepoint, and where two complexes are close if for large R, the R-balls around their basepoints are isomorphic. Each finite complex K induces a finite measure μ_K on $\mathcal{K}$, defined by pushing forward the counting measure on the vertex set $V(K)$ under the map

$$V(K) \longrightarrow \mathcal{K}, \quad p \mapsto [(K,p)].$$

A sequence (K_n) in $\mathcal{K}$ *Benjamini-Schramm (BS) converges*[1] if the probability measures $\mu_{K_n}/\mathrm{vol}(K_n)$ weakly converge to some probability measure on $\mathcal{K}$. One can then reformulate the testability of normalized Betti numbers above as saying:

Theorem 1.1 (Elek [15, Lemma 6.1])**.** *If (K_n) is a BS-convergent sequence of simplicial complexes, each with degree at most d, the normalized Betti numbers $\beta_k(K_n)/vol(K_n)$ converge for all k.*

Informally, the relationship with testability is that if we fix $R > 0$ and take $n, m >> 0$, convergence says the measures associated to the two complexes K_n, K_m will be close. So by the definition of the topology on $\mathcal{K}$, we will have that for large R, the distribution of randomly sampled R-balls in K_n will be almost the same as that in K_m, so having a way to accurately guess the normalized Betti numbers from these (nearly identical) data sets means that the normalized Betti numbers of K_n and K_m must be close.

Recently, a number of authors, see e.g. [1, 2, 10, 11, 24], have studied the analogous version of BS-convergence for Riemannian manifolds. Adopting the language of [2],'s set

$$\mathcal{M} = \{\text{pointed Riemannian manifolds } (M,p)\}/\text{pointed isometry},$$

[1]Benjamini-Schramm convergence of graphs was first studied in their paper [9]. See also Aldous–Lyons [3] for a broader picture of BS-convergence in the case of graphs.

endowed with the topology of pointed smooth convergence. See §2.1. A finite volume manifold M induces a finite measure μ_M on $\mathcal{M}$, by pushing forward the Riemannian measure on M via the map $p \mapsto [(M,p)]$, and we say that a sequence (M_n) *Benjamini-Schramm (BS) converges* if the measures $\mu_{M_n}/\mathrm{vol}(M_n)$ weakly converge to some probability measure.

The Riemannian analogue of Theorem 1.1 is not true, since if no geometric constraints are imposed, we can pack as much homology as desired into a part of a manifold with negligible volume. For example: connect sum a small volume genus $g(n)$ surface, say with volume 1, somewhere on a round radius-n sphere. The resulting surfaces will BS-converge to an atomic measure on the single point $[(\mathbb{R}^2, p)] \in \mathcal{M}$, where $p \in \mathbb{R}^2$ is any basepoint. But by choosing $g(n)$ appropriately, we can make the first Betti numbers whatever we like.

In this example, the real problem is injectivity radius. For a Riemannian manifold M and a point $x \in M$ we denote the injectivity radius of M at x by $\mathrm{inj}_M(x)$. Given $\epsilon > 0$, the *ϵ-thick part* and the *ϵ-thin part* of M are

$$M_{\geq\epsilon} = \{x \in M \;:\; \mathrm{inj}_M(x) \geq \epsilon/2\} \text{ and } M_{<\epsilon} = M \setminus M_{\geq\epsilon}.$$

One says that M is *ϵ-thick* if $M = M_{\geq\epsilon}$. Now, under geometric constraints like curvature bounds, there is a standard way to model an ϵ-thick manifold M by a simplicial complex $K(M)$ with comparable volume and bounded degree: one selects an ϵ-net S in M, and lets $N(S)$ be the nerve of the covering of M by ϵ-balls. One can then show:

Theorem 1.2 (Elek–Bowen + ABBG)**.** *If (M_n) is a BS-convergent sequence of compact, ϵ-thick Riemannian manifolds with upper and lower curvature bounds, then the normalized Betti numbers $b_k(M_n)/vol(M_n)$ converge.*

The above is really a special case of a more general result, see §2.3. A word is in order about the attributions: it was originally conceived by Elek, and then written up and published by Bowen [11, Theorem 4.1], but there is a significant error in the very last line of Bowen's proof, which we (ABBG) fix in §2.3. Briefly, the idea is to superimpose a bunch of Poisson processes on M_n to create a random $\epsilon/2$-net $S_n \subset M_n$, and then to prove that the random nerve complexes $N(S_n)$ BS-converge. By a slight generalization of Theorem 1.1 above, the expected normalized Betti numbers $E[b_k(N(S_n))]/|S_n|]$ will converge. Since M_n is ϵ-thick, the Nerve Theorem implies that each $N(S_n)$ is homotopy equivalent to M_n, so the normalized Betti numbers $b_k(M_n)/vol(M_n)$ converge.

1.1. **Results.** Our interest in this paper is whether for certain manifolds of nonpositive curvature, one can control the thin parts well enough so that BS-convergence implies convergence of normalized Betti numbers, without any assumption of thickness.

Although almost all of the real work in this paper is done more generally, we start as follows. Let X be an irreducible symmetric space of noncompact type. An *X-manifold* is a complete Riemannian manifold whose universal cover is isometric to X.

Theorem 1.3. *Suppose that $\dim(X) \neq 3$ and (M_n) is a BS-convergent sequence of finite volume X-manifolds. Then for all k, the sequence $b_k(M_n)/vol(M_n)$ converges.*

Here, the only three-dimensional irreducible symmetric spaces of noncompact type are scales of $\mathbb{H}^3$. In fact, the conclusion of Theorem 1.3 is false when $X = \mathbb{H}^3$. As an example, let $K \subset S^3$ be a knot such that the complement $M = S^3 \setminus K$ admits a hyperbolic metric, e.g. the figure-8 knot. Using meridian–longitude coordinates, let M_n be obtained by Dehn filling M with slope $(1, n)$; then each M_n is a homology 3-sphere. The manifolds $M_n \to M$ geometrically, see [8, Ch E.6], so the measures μ_{M_n} weakly converge to μ_M (c.f. [5, Lemma 6.4]) and the volumes $\mathrm{vol}(M_n) \to \mathrm{vol}(M)$. However, $0 = b_1(M_n) \not\to b_1(M) = 1$, so the normalized Betti numbers of the sequence $M_1, M, M_2, M, \ldots$ do not converge. See also Example 3.1 for a similar counterexample in which volume goes to infinity. In fact, there is a real sense in which the *only* counterexamples come from Dehn filling. See §3.

To illustrate a special case of Theorem 1.3, let's say that (M_n) *BS-converges to* X when the measures μ_{M_n} weakly converge to the atomic probability measure on the point

$$[(X, x)] \in \mathcal{M},$$

where $x \in X$ is any basepoint. Now any X as above admits a (compact, even) X-manifold M, by a theorem of Borel [25, Theorem 14.1]. A theorem of Mal'cev [23] says that $\pi_1 M$ is residually finite. So, we can take a tower of regular covers

$$\cdots \to M_2 \to M_1 \to M$$

corresponding to a nested sequence of normal subgroups of $\pi_1 M$ with trivial intersection, and such a sequence (M_n) will BS-converge to X, see [1] for details.

In this example, if we take M to be compact then convergence of the normalized Betti numbers of (M_n) follows from an earlier theorem of DeGeorge and Wallach [13]$\beta_k^{(2)}(X)$ of the symmetric space X (see also Lück [22] for a more general result). We refer the reader to [1, 22] for more information on L^2 Betti numbers. Here, let us just remark that any sequence of manifolds that BS-converges to X can be interleaved with the example above, and the result still BS-converges. So, the following is a direct consequence of Theorem 1.3.

Corollary 1.4. *Suppose that (M_n) is a sequence of finite volume X-manifolds that BS-converges to X. Then for all $k \in \mathbb{N}$, we have $b_k(M_n)/vol(M_n) \to \beta_k^{(2)}(X)$.*

With Nikolov, Raimbault and Samet, we proved this in [1] for sequences of compact, ϵ-thick manifolds, using analytic methods. One could also prove it in the thick case by using Theorem 1.2 above (the Bowen–Elek simplicial approximation technique) and interleaving with a covering tower. In the thin case, we were able to push our analytic methods far enough to give a proof for $X = \mathbb{H}^d$, see [1, Theorem 1.8]. Hence, there is no problem in allowing $X = \mathbb{H}^3$ in Corollary 1.4, even though Theorem 1.3 does not apply.

While we were finishing this paper, Alessandro Carderi sent us an interesting preprint where, among other things, he proves the same result as Corollary 1.4 if either $k = 1$, or k is arbitrary and the symmetric space $X = G/K$ is of higher rank and M_n is non compact, or in most cases when X is of rank 1. His proof is quite different, he considers the ultraproduct of the sequence of actions of G on G/Γ_n. He then identifies the L^2-Betti numbers of the resulting G-action with the L^2-Betti numbers of the group G.

Corollary 1.4 is particularly powerful when X has real rank at least two. In this case, we proved with Nikolov, Raimbault and Samet that *any* sequence of distinct finite volume X-manifolds BS-converges to X, see [1, Theorem 4.4]. So, Corollary 1.4 implies:

Corollary 1.5. *Suppose that* $\mathrm{rank}_{\mathbb{R}} X \geq 2$ *and* (M_n) *is any sequence of distinct finite volume X-manifolds. Then for all* $k \in \mathbb{N}$*, we have* $b_k(M_n)/vol(M_n) \to \beta_k^{(2)}(X)$.

In the two corollaries above, we can identify the limit of the normalized Betti numbers when the BS-limit is X. In general, one can think of Theorem 1.3 as giving a definition of 'L^2-Betti numbers' for arbitrary limits of BS-convergent sequences. The measures on $\mathcal{M}$ that arise as such limits have a special property called *unimodularity*, see [2], and it would be interesting to find a good intrinsic definition of the 'L^2-Betti numbers' of a unimodular measure that is compatible with Theorem 1.3.

1.2. **The proof, and generalities in nonpositive curvature.** To prove Theorem 1.3, we split into cases depending on $\mathrm{rank}_{\mathbb{R}} X$. When the rank is one, we need to deal with general BS-convergent sequences, but the thin parts of rank one locally symmetric spaces are easy to understand. And when the rank is at least two, the only possible BS-limit we need to consider is X. We now give two theorems that handle these two cases. We state them very generally, without any assumption of symmetry.

Theorem 1.6 (Pinched negative curvature, arbitrary BS-limits)**.** *Let* (M_n) *be a BS-convergent sequence of finite volume Riemannian d-manifolds, with* $d \neq 3$*, and with sectional curvatures in the interval* $[-1, \delta]$*, for some* $-1 \leq \delta < 0$*. Then the normalized Betti numbers* $b_k(M_n)/vol(M_n)$ *converge for all* k*.*

Theorem 1.7 (Nonpositive curvature, with a thick BS-limit)**.** *Let* $\epsilon > 0$ *and let* (M_n) *be a sequence of real analytic, finite volume Riemannian d-manifolds with sectional curvatures in the interval* $[-1, 0]$*, and assume the universal covers of the* M_n *do not have Euclidean de Rham-factors. If* (M_n) *BS-converges to a measure* μ *on* $\mathcal{M}$ *that is supported on* ϵ*-thick manifolds, the normalized Betti numbers* $b_k(M_n)/vol(M_n)$ *converge for all* k*.*

Let's see how to deduce Theorem 1.3 from these results. Suppose X is an irreducible symmetric space of noncompact type, $\dim(X) \neq 3$. When X has rank one, X has pinched negative curvature, so therefore Theorem 1.3 follows from Theorem 1.6. When X has higher rank, [1, Theorem 4.4] says that any BS-convergent sequence (M_n) of X-manifolds BS-converges to X, as mentioned above. Since X is actually ϵ-thick for any ϵ, Theorem 1.7 applies, and Theorem 1.3 follows.

The reader may wonder where we use $d \neq 3$ in the proof of Theorem 1.6. When $d = 2$, one can deduce the claim from Gauss–Bonnet. In general, the point is that the boundary of a Margulis tube is homeomorphic to an S^{n-2}-bundle over S^1. When $d \geq 4$, this bundle is not aspherical, so it can be distinguished from a cusp cross section, which prevents one from doing Dehn filling as in our problematic 3-dimensional example. More to the point, one can show that when $d \geq 4$, Margulis tubes with very short cores have boundaries with large volume, see Proposition 3.1, which implies that the number of Margulis tubes with short cores one can see in a manifold is sublinear in volume. Hence, the contribution of the tubes to homology cannot affect the normalized Betti numbers much.

The key to Theorem 1.7 is a celebrated theorem of Gromov, see [6], that bounds the Betti numbers of an analytic manifold with sectional curvatures in $[-1, 0]$ and no local Euclidean deRham factors linearly in terms of its volume. Delving a bit into its proof, one can show that in the setting of Theorem 1.7, the Betti numbers of the thin parts of the M_n grow sublinearly with $\mathrm{vol}(M_n)$. One can then combine the proof of Theorem 1.2 (the Bowen–Elek simplicial approximation argument), which handles the thick parts of the M_n, with Mayer–Vietoris to get Theorem 1.7, although one has to be a bit careful in controlling the boundary of the thin part, where the thick and thin parts are glued.

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2. Spaces of spaces and simplicial approximation

In this section, we discuss the topology on $\mathcal{M}$ and a similar topology on the space $\mathbb{M}$ of all pointed metric measures spaces. We then state and prove a generalization of the Bowen–Elek theorem on the convergence of Betti numbers of thick spaces, which was stated in a weak form in the introduction as Theorem 1.2.

2.1. **The smooth topology.** In the introduction, we introduce the space

$$\mathcal{M} = \{\text{pointed Riemannian manifolds } (M, p)\}/\text{pointed isometry},$$

endowed with the topology of pointed smooth convergence. Here, a sequence (M_n, p_n) converges *smoothly* to (M_∞, p_∞) if there is a sequence of smooth embeddings

$$\phi_n : B_{M_\infty}(p_\infty, R_n) \longrightarrow M_n \tag{1}$$

with $R_n \to \infty$ and $\phi_n(p_\infty) = p_n$, such that $\phi_n^* g_n \to g_\infty$ in the C^∞-topology, where g_n are the Riemannian metrics on M_n. We call (ϕ_n) a *sequence of almost isometric maps coming from smooth convergence*. Note that each metric $\phi_n^* g_n$ is only partially defined on M_∞, but their domains of definition exhaust M_∞, so it still makes sense to say that $\phi_n^* g_n \to g_\infty$ on all of M_∞, even if the language is a bit abusive. Álvarez López, Barral Lijó and Candel [4] have shown that $\mathcal{M}$, with the smooth topology, is a Polish space.

2.2. **Metric measure spaces.** A *metric measure space* (or *mm*-space) is a complete, separable proper metric space M equipped with a Radon measure vol. Let

$$\mathbb{M} = \{\text{pointed mm-spaces } (M, \mathrm{vol}, p)\}/\text{pointed measure preserving isometry}.$$

Following Bowen [11, Definitions 28 and 29], an (ϵ, R)-*relation* between pointed mm-spaces $(M_1, \mathrm{vol}_1, p_1)$ and $(M_2, \mathrm{vol}_2, p_2)$ is a pair of isometric embeddings

$$M_i \longrightarrow Z, \quad i = 1, 2$$

into some common metric space Z having the following properties:

(a) $d_Z(p_1, p_2) < \epsilon$,
(b) $B_{M_1}(p_1, R) \subset (M_2)_\epsilon$ and $B_{M_2}(p, R) \subset (M_1)_\epsilon$,
(c) for all (closed, say) subsets $F_i \subset B_Z(p_i, R)$, we have

$$\mathrm{vol}_1(F_1) < (1+\epsilon)\mathrm{vol}_2(\,(F_1)_\epsilon\,) + \epsilon, \quad \mathrm{vol}_2(F_2) < (1+\epsilon)\mathrm{vol}_1(\,(F_2)_\epsilon\,) + \epsilon.$$

In other words, (a), (b) and (c) say respectively that the images of the base points are close, the images of M_1, M_2 are close in the Chabauty topology, and the push forwards of the measures are close in the weak topology[2].

We endow $\mathbb{M}$ with the topology generated by (ϵ, R)-relations. That is, a basic open set in $\mathbb{M}$ is obtained by picking some element of $\mathbb{M}$ and some $\epsilon, R > 0$, and considering all other elements of $\mathbb{M}$ that are (ϵ, R)-related to the first. Bowen shows that $\mathbb{M}$ is separable and metrizable, see [11, Theorem 3.1]. (We should note that the way Bowen initially defines the topology on $\mathbb{M}$ in [11, Definition 5] is slightly different, but he proves that the two points of view are equivalent in [11, Proposition B.4].)

Lemma 2.1. *Suppose that (M_i, p_i), $i = 1, 2$, are pointed Riemannian d-manifolds and for some $R > 0$ there is an embedding $\phi : B_{M_1}(p_1, R) \longrightarrow M_2$ with $\phi(p_1) = p_2$ and*

$$(1-\delta)|v| \le |d\phi(v)| \le (1+\delta)|v|, \quad \forall v \in TB_{M_1}(p_1, R). \tag{2}$$

Then if $\delta = \delta(\epsilon, d)$ is small, the triples (M_i, vol_i, p_i) are (ϵ, R)-related, where vol_i is the Riemannian measure on M_i.

Proof. Take $\delta < \epsilon$ and let ϕ be as in the statement of the lemma. We want to produce an (ϵ, R) relation between M_1 and M_2. Define the common space Z as the disjoint union

$$Z = M_1 \sqcup M_2,$$

endowed with a metric that restricts to the given metrics on M_1, M_2, and where for $x \in M_1, y \in M_2$,

$$d(x, y) = \inf\{d(x, x') + \delta + d(\phi(x'), y) \mid x' \in B_{M_1}(p_1, R+1)\}.$$

We now verify that Z gives an (ϵ, R)-relation. First, $d_Z(p_1, p_2) = \delta < \epsilon$. Second, if $x \in M_1 \cap B_Z(p_1, R) = B_{M_1}(p_1, R)$, then $d(x, \phi(x)) = \delta < \epsilon$, so $x \in (M_2)_\epsilon$. Third, if $F_1 \subset B_Z(p_1, R)$ is a closed subset, then we have

$$\mathrm{vol}_1(F_1) = \mathrm{vol}_1(F_1 \cap M_1) \le (1+\delta)^d \mathrm{vol}_2(\phi(F_1 \cap M_1)) \le (1+\delta)^d \mathrm{vol}_2(\,(F_1)_\delta\,),$$

[2] We should mention that Bowen does not include multiplicative factors of $(1+\epsilon)$ in his version of (c), but this difference does not affect the resulting topology on $\mathbb{M}$ below.

where the first inequality follows from (2), and the second follows from the fact that $d(x,\phi(x)) = \delta$. So, as long as δ is small, the right side will be at most $(1+\epsilon)\mathrm{vol}_2((F_1)_\epsilon)$. The two remaining parts of properties (a) and (b) follow similarly. ☐

As an immediate corollary, we get the following:

Corollary 2.2. *The natural inclusion $\mathcal{M} \longrightarrow \mathbb{M}$ from the space of pointed Riemannian manifolds (with the smooth topology) to the space of pointed mm-spaces is continuous.*

We will need a slight variant of $\mathbb{M}$ for our work below. Let

$$\mathbb{M}^{ext} = \{(M, \mathrm{vol}, p, E) \mid (M, \mathrm{vol}, p) \in \mathbb{M},\ E \supset M \text{ a super-metric space}\}/\sim,$$

where a super-metric space is just a proper, separable metric space that contains M as a submetric space. We call a quadruple (M, vol, p, E) an *extended* pointed mm-space; two quadruples are identified in $\mathbb{M}^{ext}$ if there is a pointed isometry between the super-metric spaces E that restricts to a measure preserving isometry from one mm-space M to the other. The topology on $\mathbb{M}^{ext}$ is similar to that on $\mathbb{M}$: we say that $(M_i, \mathrm{vol}_i, p_i, E_i)$, $i = 1, 2$, are (ϵ, R)-*related* if there are isometric embeddings

$$E_i \longrightarrow Z, \quad i = 1, 2$$

that restrict to give an (ϵ, R)-relation between the triples $(M_i, \mathrm{vol}_i, p_i)$, and where also

$$B_{E_1}(p_1, R) \subset (E_2)_\epsilon, \quad B_{E_2}(p_2, R) \subset (E_1)_\epsilon. \tag{3}$$

We then have the following variant of Lemma 2.1.

Lemma 2.3. *Suppose that (M_i, p_i), $i = 1, 2$, are pointed Riemannian d-manifolds with distinguished subsets $T_i \subset M_i$ and that for some $R > 0$ there is an embedding*

$$\phi : B_{M_1}(p_1, R) \longrightarrow M_2$$

with $\phi(p_1) = p_2$ that satisfies the following three properties:

(1) $(1-\delta)|v| \le |d\phi(v)| \le (1+\delta)|v|, \quad \forall v \in TB_{M_1}(p_1, R)$.
(2) $\phi^{-1}(T_2) \subset (T_1)_\delta$, and $\phi(T_1 \cap B_{M_1}(p_1, R)) \subset (T_2)_\delta$,
(3) $vol_1(\phi^{-1}(T_2) \bigtriangleup T_1) < \delta$,

where $\bigtriangleup$ is the symmetric difference. Then if $\delta = \delta(\epsilon, d)$ is sufficiently small, the quadruples $(T_i, vol_i|_{T_i}, p_i, M_i)$ are (ϵ, R)-related, where here vol_i is the Riemannian measure on M_i.

Proof. The proof is similar to that of Lemma 2.1. With $Z = M_1 \sqcup M_2$ and d the metric defined in Lemma 2.1, equation (3) above follows exactly as before as long as $\delta < \epsilon$. So, we just need to verify that Z gives an (ϵ, R)-relation between the subsets T_1, T_2. Property (a) is immediate from the definition of the metric on Z. For (b), if $x \in B_{T_1}(p_1, R_1)$ then $\phi(x) \subset (T_2)_\delta$, so $d_Z(x, T_2) < 2\delta$. So, (b) holds if $\delta \le \epsilon/2$, as the proof of the other part is similar. For (c), suppose $F \subset B_Z(p_1, R)$ is closed. Then

$$\begin{aligned} \mathrm{vol}_1|_{T_1}(F_1) &= \mathrm{vol}_1(F_1 \cap T_1) \\ &\le \mathrm{vol}_1(F_1 \cap \phi^{-1}(T_2)) + \mathrm{vol}_1(\phi^{-1}(T_2) \bigtriangleup T_1) \end{aligned}$$

$$< (1+\delta)^d \mathrm{vol}_2(\phi(F_1) \cap T_2)) + \delta$$
$$= (1+\delta)^d \mathrm{vol}_2|_{T_2}(\phi(F_1)) + \delta$$

So since $\phi(F_1) \subset (F_1)_\delta$, (c) holds if $(1+\delta)^d \leq (1+\epsilon)$. The other part of (c) is similar. □

2.3. **Normalized Betti numbers of mm-spaces.** If (M, vol) is a finite volume mm-space, let $\mu_{(M,\mathrm{vol})}$ be the measure on $\mathbb{M}$ obtained by pushing forward the vol under

$$M \longrightarrow \mathbb{M}, \ p \longmapsto (M, \mathrm{vol}, p).$$

A sequence of finite volume mm-spaces (M_n, vol_n) *Benjamini-Schramm (BS) converges* if the associated measures $\mu_{(M_n,\mathrm{vol}_n)}/\mathrm{vol}_n(M_n)$ weakly converges to some probability measure on $\mathbb{M}$.

An mm-space M is *special* if M has finitely many path components[3], the measure vol is non-atomic and fully supported, and metric spheres have measure zero. In [11], Bowen claims the following theorem, and justifies it by fleshing out an argument of Elek.

Theorem 2.4 (see [11, Theorem 4.1])**.** *Suppose (M_n) is a BS-convergent sequence of finite volume special mm-spaces and that there are constants r, v_0, v_1 such that*

(1) all $r/2$-balls in M_n have volume at least v_0,
(2) all $20r$-balls have volume at most v_1,
(3) all ρ-balls in M_n with $\rho < 10r$ are strongly convex*, meaning that for any two points x, y in a ρ-ball B, there is a unique point $z \in B$ with $d(x,z) = d(y,z) = 1/2d(x,y)$.*

Then the normalized Betti numbers $b_k(M_n)/vol(M_n)$ converge for all k.

As mentioned in the introduction, there is a significant error in the very last line of Bowen's proof of this theorem. Below, we will prove a slightly more (and less) general theorem, Theorem 2.5. While it does not strictly imply Theorem 2.4, it can be used in all Bowen's applications. See the beginning of the proof of Theorem 2.5at the end of the section for an explanation of the error in Bowen's argument.

To motivate the more general result, recall that Elek–Bowen's approach involves approximating each M_n by the nerve complex N associated to an open cover by balls centered at points of a suitable net $S \subset M$. Condition (3) is only used to say that the nerve is homotopy equivalent to M_n, so we should be able to state a version of Theorem 1.2 in which (3) is omitted, if we talk about the Betti numbers of the nerve complexes instead of the Betti numbers of the M_n. To make a result that is compatible with the machinery of Gelander described in §3.2, it is also important for us to take nets in the M_n, but construct the corresponding nerves using balls in larger spaces E_n. In other words, we need to work with the *extended mm-spaces* of §2.2.

To that end, we say that an extended mm-space $\mathfrak{M} = (M, \mathrm{vol}, E)$ is *finite volume* or *special* if the mm-space M is. When $\mathfrak{M}$ has finite volume, we can construct a finite measure

[3]Bowen requires M to be path connected in his definition of special, but finitely many components suffices everywhere below.

$\mu_{\mathfrak{M}}$ on $\mathbb{M}^{ext}$ by pushing forward vol under the map

$$p \in M \mapsto (M, \mathrm{vol}, p, E).$$

If $\mathfrak{M}_n = (M_n, \mathrm{vol}_n, E_n)$ is a sequence of extended mm-spaces, then we say that $(\mathfrak{M}_n)$ *BS-converges* if the sequence of measures $\mu_{\mathfrak{M}_n}/\mathrm{vol}_n(M_n)$ weakly converges.

We define an (r_0, r_1)*-net* in $\mathfrak{M}$ to be a subset $S \subset M$ such that

(1) S is r_0*-separated,* i.e. $d(x, y) > r_0$ for all $x \neq y \in S$,
(2) S r_1*-covers* M, i.e. for every $p \in M$, there is some $x \in S$ with $d(p, x) < r_1$,

and an $[r_2, r_3]$*-weighted* (r_0, r_1)*-net* is a (r_0, r_1)-net S with a function

$$\rho : S \longrightarrow [r_2, r_3],$$

where here $r_0 < r_1 \leq r_2 < r_3$. Given any weighted net (S, ρ) in $\mathfrak{M}$, we let $N_E(S, \rho)$ be the nerve complex associated to the collection of E-balls $B_E(x, \rho(x))$, where $x \in S$.

Theorem 2.5. *Fix k, let $\mathfrak{M}_n = (M_n, vol_n, E_n)$ be a BS-convergent sequence of extended finite volume special mm-spaces and pick constants $v_{min} > 0$, $r_1 > r_0 > 0$, and $r_3 > r_2 \geq 2r_1$, and a function $v_{max} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that*

(1) all $r_0/2$-balls in every M_n have volume at least v_{min},
(2) for all $r \in \mathbb{R}_+$, every r-ball in M_n has volume at most $v_{max}(r)$,
(3) for every sequence of $[r_2, r_3]$-weighted $(r_0, 2r_1)$-nets (S_n, ρ_n) in M_n,

$$\frac{\left|b_k(N_{E_n}(S_n, \rho_n)) - B_n\right|}{vol_n(M_n)} \to 0.$$

Then the ratios $B_n/vol(M_n)$ converge.

As mentioned above, our theorem does not actually imply Theorem 2.4, the reason being that our (2) requires a uniform upper bound on the volumes of all functions in M_n, not just on ones with a particular radius. In basically all applications, however, the upper bound in Theorem 2.4 (2) comes from a curvature lower bound, which also implies our (2). In contrast, note that Bowen's condition (3) implies ours, by the nerve theorem.

Before starting the proof, we record one elementary measure theoretic lemma.

Lemma 2.6. *Suppose that (M, vol) is a mm-space and that every ball in M with radius in the interval $[r_0/2, r_0]$ has volume between v_{min} and v_{max}. Set $c = v_{min}^2/(2v_{max})$, $c' = v_{min}^2/(2v_{max}^2)$. Then for every measurable subset $A \subseteq M$, we have that*

$$vol\Big(\left\{ x \in A \;\middle|\; vol(B_{r_0}(x) \cap A) \geq c \cdot \frac{vol(A)}{vol(M)} \right\} \Big) \geq c' \cdot \frac{vol(A)}{vol(M)} \cdot vol(A).$$

Here, the ball $B_{r_0}(x)$ is the metric ball in M. Note that if $\mathrm{vol}(A)/\mathrm{vol}(M)$ is bounded away from zero, the lemma says that a definite proportion of A is taken up by points $x \in A$ such that A takes up a definite proportion of $B_{r_0}(x)$.

Proof. Note that

$$\begin{aligned}
\int_A \mathrm{vol}(B_{r_0}(x)\cap A)\,dx &= \mathrm{vol}^2(\{(a,b)\in A^2 \mid d(a,b)<r_0\})\\
&\geq \frac{1}{v_{max}}\mathrm{vol}^3(\{(x,a,b)\in M\times A^2 \mid d(a,x)<r_0/2 \text{ and } d(a,b)<r_0\})\\
&\geq \frac{1}{v_{max}}\mathrm{vol}^3(\{(x,a,b)\in M\times A^2 \mid d(a,x)<r_0/2 \text{ and } d(x,b)<r_0/2\})\\
&= \frac{1}{v_{max}}\int_M \mathrm{vol}(B_{r_0/2}(x)\cap A)^2\,dx\\
&\geq \frac{1}{v_{max}\mathrm{vol}(M)}\left(\int_M \mathrm{vol}(B_{r_0/2}(x)\cap A)\,dx\right)^2\\
&= \frac{1}{v_{max}\mathrm{vol}(M)}\left(\int_A \mathrm{vol}(B_{r_0/2}(x))\,dx\right)^2\\
&\geq \frac{v_{min}^2\mathrm{vol}(A)^2}{v_{max}\mathrm{vol}(M)}\\
&= 2\cdot\left(c\cdot\frac{\mathrm{vol}(A)}{\mathrm{vol}(M)}\right)\cdot\mathrm{vol}(A).
\end{aligned}$$

From this the lemma follows immediately, since if $f : A \longrightarrow [0, max]$ is a function then

$$\int_A f \geq 2\cdot\epsilon\cdot\mathrm{vol}(A) \implies \mathrm{vol}\{x\in A \mid f(x)\geq\epsilon\}\geq\frac{\epsilon}{max}\mathrm{vol}(A). \quad\square$$

Also, for convenience below, we record the following variation of Theorem 2.5.

Corollary 2.7. *Fix k, let $\mathfrak{M}_n = (M_n, vol_n, E_n)$ be a sequence of extended finite volume special mm-spaces and assume that for some sequence of constants V_n, the measures $\mu_{\mathfrak{M}_n}/V_n$ weakly converge to some finite measure μ on $\mathbb{M}^{ext}$. Pick constants $v_{min} > 0$, $r_1 > r_0 > 0$, and $r_3 > r_2 \geq 2r_1$, and a function $v_{max} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that*

1. *all $r_0/2$-balls in every M_n have volume at least v_{min},*
2. *for all $r \in \mathbb{R}_+$, every r-ball in M_n has volume at most $v_{max}(r)$,*
3. *for every sequence of $[r_2, r_3]$-weighted $(r_0, 2r_1)$-nets (S_n, ρ_n) in M_n,*

$$\frac{\left|b_k(N_{E_n}(S_n,\rho_n)) - B_n\right|}{V_n} \to 0.$$

Then the ratios B_n/V_n converge.

Note that here, the measures $\mu_{\mathfrak{M}_n}/V_n$ and their limit may not be probability measures.

Proof of Corollary 2.7 given Theorem 2.5. Let μ be the weak limit of $\mu_{\mathfrak{M}_n}/V_n$. Then

$$\lim_{n\to\infty}\frac{\mathrm{vol}_n(M_n)}{V_n} = \lim_{n\to\infty}\int 1\ d(\mu_{\mathfrak{M}_n}/V_n) = \mu(\mathbb{M}^{ext}) \in [0,\infty).$$

Suppose first that $\mu(\mathbb{M}^{ext}) = 0$. By (1) and (2), the number of points in any $(r_0, 2r_1)$-net in M_n is proportional to $\mathrm{vol}_n(M_n)$, so the Betti numbers in (3) are $O(\mathrm{vol}_n(M_n))$. Combining (3) and the triangle inequality, we have $B_n/V_n \to 0$.

If $\mu(\mathbb{M}^{ext}) > 0$, then $V_n/\mathrm{vol}_n(M_n)$ has a finite limit, so the probability measures

$$\frac{\mu_{\mathfrak{M}_n}}{\mathrm{vol}_n(M_n)} = \frac{V_n}{\mathrm{vol}_n(M_n)} \cdot \frac{\mu_{\mathfrak{M}_n}}{V_n} \to \frac{\mu}{\mu(\mathbb{M}^{ext})}.$$

In other words, the extended mm-spaces $\mathfrak{M}_n$ BS-converge. Also, $\mathrm{vol}_n(M_n)$ and V_n are proportional, so (3) holds with $\mathrm{vol}_n(M_n)$ instead of V_n. Theorem 2.5 then says that $B_n/\mathrm{vol}_n(M_n)$ converges, from which it follows that B_n/V_n converges too. □

2.4. **The proof of Theorem 2.5.** Naively, an ideal approach to the theorem would be as follows. One would first construct a random weighted net (S_n, ρ_n) in each M_n, and then prove that the associated random nerve complexes $N_{E_n}(S_n, \rho_n)$ BS-converge. Then one would apply Elek's Theorem 1.1 (or rather, a version for random complexes from Bowen's paper), to get that the sequence of expected normalized Betti numbers

$$\frac{E[b_k(N_{E_n}(S_n, \rho_n))]}{E[|S_n|]} = \frac{b_k(M_n)}{E[|S_n|]} \tag{4}$$

converges. Here, the equality is by condition (3) in the statement of the theorem. Moreover, if the random nets are constructed in some canonical, local way (Bowen builds his random nets from Poisson processes), one could then prove that the ratio $E[|S_n|]/\mathrm{vol}_n(M_n)$ converges, and the theorem would follow.

The problem with this approach is that one really needs to make the 'random nerve complex' construction a continuous map

$$\mathbb{M}^{ext} \longrightarrow \mathcal{P}(\mathcal{K}), \tag{5}$$

where $\mathcal{P}(\mathcal{K})$ is the space of probability measures on the space of pointed complexes. For then we can produce the sequence of random nerve complexes by pushing forward the weakly convergent sequence of measures $\mu_{\mathfrak{M}_n}/\mathrm{vol}_n(M_n)$ on $\mathbb{M}^{ext}$, and hence the random nerves will also be weakly convergent. To make the map (5), one needs to convert the base point of a pointed space (M, p) into a base point for the corresponding random nerve complex, i.e. to a point in the random net $S \subset M$. Essentially, one just takes the closest point in S to p, but then there are problems because in order to apply Elek's result we need to produce measures on $\mathcal{K}$ in which there is an *equal probability* of choosing any base point, and the Voronoi cells corresponding to the points in a net $S \subset M$ may not all have the same measure. One avoids this problem by conditioning on the event that the base point $p \in M$ lies in some small, fixed-volume ball around a net point. This works fine, but then it turns out that the expectations in (4) are now taken with respect to the *conditioned* measures. (Bowen's mistake is that he forgot this fact[4].) While the numerator

[4]The mistake is in the very last line of the proof, in which he uses that $\mathbb{E}[\mathrm{vol}(M_i(v_0/2))]/\mathrm{vol}(M_i) = \lambda_i(\mathbb{MS}(v_0/2))$, the point being that the expectation should really be taken with respect to λ_i', while he is assuming it is taken with respect to λ_i.

is still $b_k(M_n)$, this change alters the denominator in such a way that one now needs prove that $E[|S_n|^2]/\mathrm{vol}_n(M_n)^2$ converges, instead of $E[|S_n|]/\mathrm{vol}_n(M_n)$. (Here, we are writing both expectations with respect to the original measures, not the conditioned ones.) This turns out to be quite subtle, and requires a complete restructuring of the proof.

The way we fix the proof is as follows. Bowen constructs his random nets $S_n \subset M_n$ iteratively, by starting with a Poisson process, throwing out points that are too close together, and then moving on to a new Poisson process to fill in the gaps. Repeating this *infinitely many* times guarantees that a net is produced almost surely. Instead of doing this, we just go through finitely many Poisson processes. While this doesn't always produce a net, we can show that it produces some discrete subset that with high probability can be extended to a net without using too many extra points. Most of Bowen's argument goes through when we replace his nets with our 'almost-nets', and it turns out that using our construction, it is much easier to estimate the quantity analogous to $E[|S_n|^2]$.

In the rest of the section, we formalize the argument sketched above.

2.4.1. *Random 'almost-nets'.* Fix a finite volume special mm-space (M, vol) and real numbers $0 < r_0 < r_1$. For each $j \in \mathbb{N}$, let P^j be a Poisson process on M with intensity 1 and let $f^j : P^j \longrightarrow [0,1]$ be a random function whose values are chosen independently according to Lebesgue measure. Each function f^j is almost surely injective, and when it is, it induces a linear order $<$ on P^j via $s < t \iff f^j(s) < f^j(t)$. Set

$$P^j(< s) = \{t \in P^j \mid t < s\}.$$

Pick some r with $r_0 < r < r_1$ and choose a continuous function

$$\phi : [0, \infty) \longrightarrow [0,1], \text{ where } \phi(t) = 0 \text{ if } t \le r_0,\ \phi(t) = 1 \text{ if } t \ge r.$$

For each pair $s, t \in P$, let $X(s,t)$ be a Lebesgue-random element of $[0,1]$, where the $X(s,t)$ are independent as s, t are varied. We now recursively define subsets

$$S^j \subset P^j, \quad S^{\le j} := S^1 \cup \cdots \cup S^j, \quad S^{<j} := S^1 \cup \cdots \cup S^{j-1},$$

where given $S_1, \ldots, S_{j-1}$, the rule is that for $s \in P^j$, we say that $s \in S^j$ if

$$\text{for all } t \in P^j(< s) \cup S^{<j}, \quad \phi(d(s,t)) \ge X(s,t).$$

In other words, go through all the elements s of a Poisson process P^1 one by one, in some random order. For each s, backtrack through all previously considered t, flip for each a $[0,1]$-valued coin, and add s to S^1 if for each t, the value $\phi(d(s,t))$ is bigger than the result of the coin flip. After finishing with all available s, switch over to a new Poisson process, and add points to S^2 using a similar rule, comparing them against previous points in P^2 and also against all points in S^1. Then repeat this with a third Poisson process to define S^3, and a fourth to define S^4, etc...

For later use, we record:

Claim 2.8. *There is some $c = c(r_0, r_1, v_{min}, v_{max}) > 0$ such that for all j, if M satisfies conditions (1) and (2) in the statement of the theorem, we have $E[|S^j|] \ge c \cdot vol(M)$.*

Proof. Certainly, it suffices to set $j = 1$. Let $B_1, \dots, B_k$ be a maximal collection of disjoint r_0-balls in M, and note that $k \geq c_1 \cdot \mathrm{vol}(M)$ for some uniform c_1, by (1) and (2). For each i, there is some definite probability that the Poisson process P_1 will intersect the r_1-neighborhood of B_i in a single point x that lies in B_i. When this happens, this x will automatically be included in S^1. So, $E[|S^1 \cap B_i|] \geq c_2$ for some uniform $c_2 > 0$. Hence, $E[|S^1|] \geq c_1 \cdot c_2 \cdot \mathrm{vol}(M)$ by linearity of expectation. □

By the definition of ϕ, we will almost never add s to S^j if $d(s,t) \leq r_0$ for some previously considered t. So, for each j, the subset $S^{\leq j}$ is almost surely r_0-separated. On the other hand, we cannot ensure that any particular $S^{\leq j}$ is an (r_0, r_1)-net, since it may not r_1-cover M. (You do get a net if you take $j = \infty$, as in Bowen's proof.) However, define

$$n^j = \min \left\{ |T \setminus S^{\leq j}| \;\middle|\; T \text{ is a } (r_0, 2r_1)\text{-net in } M \text{ with } T \supset S^{\leq j} \right\},$$

a random integer associated to each choice of M and j. We prove:

Proposition 2.9. *Given $\epsilon > 0$, there is some $j = j(\epsilon, r_0, r_1, v_{min}, v_{max})$ such that for any M satisfying (1) and (2) in the statement of the theorem, we have*

$$\frac{E[n^j]}{vol(M)} < \epsilon.$$

Proof. For each j, write R^j for the complement of the r_1-neighborhood of $S^{\leq j} \subset M$, and let R^j_2 be the complement of the $2r_1$-neighborhood. If X is any maximal r_0-separated set in the space R^j_2, then the M-balls $B_{r_0}(x), x \in X$ are disjoint and contained in R^j, so

$$v_{min} \cdot |X| \leq \mathrm{vol}(R^j).$$

Since the union $X \cup S^j$ is a $(r_0, 2r_1)$-net in M, this means $n^j \leq \mathrm{vol}(R^j)/v_{min}$, so to prove the proposition it suffices (after adjusting ϵ) to find j such that

$$\frac{E[\mathrm{vol}(R^j)]}{\mathrm{vol}(M)} < \epsilon. \tag{6}$$

Claim 2.10. *There is some $\delta = \delta(\epsilon, r_0, r_1, v_{min}, v_{max}) < 1$ as follows. Suppose a fixed $S^{\leq j}$, and hence R^j, is given, and that $vol(R^j)/vol(M) \geq \epsilon/2$. Then*

$$E[vol(R^{j+1}) \,|\, R^j] \leq \delta \cdot vol(R^j).$$

Here, we write $E[\mathrm{vol}(R^{j+1}) \,|\, R^j]$ to indicate that this is the expected volume of R^{j+1}, conditioned on our particular choice of a fixed R^j. This is to remove some ambiguity when we apply the claim later. In the *proof* of Claim 2.10, though, we will always consider R^j as fixed and just write $E[\,\cdot\,]$, omitting any reference to R^j. Also, to avoid a proliferation of constants in the following proof, we will use the notation $x \preceq y$ to mean that $x \leq Cy$ for some constant $C > 0$ depending only on $\epsilon, r_0, r_1, v_{min}, v_{max}$.

Proof. Fix c, c' as in Lemma 2.6. Let $R^j_\circ$ be the subset of R^j consisting of all points $x \in R^j$ such that $\mathrm{vol}(B_{r_1}(x) \cap R^j) \geq c \cdot \epsilon/2$, and let $R^j_{\circ\circ}$ be the further subset consisting of all points $x \in R^j_\circ$ such that $\mathrm{vol}(B_{r_1}(x) \cap R^j_\circ) \geq c \cdot \epsilon/2$. Then Lemma 2.6 says that

$$\mathrm{vol}(R^j_\circ) \geq \mathrm{vol}(R^j_{\circ\circ}) \succeq \mathrm{vol}(R^j).$$

Pick a maximal r_0-separated (say) subset $Z \subset R^j_{\circ\circ}$. By assumption, there is an upper bound for the volumes of all the r_0-balls around points $z \in Z$, so since $R^j_{\circ\circ}$ is contained in the union of all such balls, our lower bound on the volume of $R^j_{\circ\circ}$ implies that

$$|Z| \succeq \mathrm{vol}(R^j). \tag{7}$$

For each $z \in Z \subset R^j_{\circ\circ}$, the volume of $B_{r_1}(z) \cap R^j_\circ$ is bounded below and the volume of $B_{2r_1}(z)$ is bounded above, so if P^{j+1} is the Poisson process used in defining S^{j+1}, we have

$$P^{j+1} \cap B_{r_1}(z) \cap R^j_\circ \neq \emptyset \text{ and } |P^{j+1} \cap B_{2r_1}(z)| = 1 \tag{8}$$

with some definite probability. Hence, by linearity of expectation and (7),

$$E\Big[\,\big|\{z \in Z \mid (8) \text{ holds for } z\}\big|\,\Big] \succeq \mathrm{vol}(R^j).$$

But for every z such that (8) holds, the single point of P^{j+1} that is in $B_{r_1}(z) \cap R^j_\circ$ is at least r_1-away from every other point of P^{j+1}, and is also at least r_1-away from $S^{\leq j}$, since $z \in R^j$. Hence, this single point lies not just in P^{j+1}, but in S^{j+1}. It follows that

$$E\big[\,|\{x \in S^{j+1} \cap R^j_\circ \mid B_{r_1}(x) \cap S^{j+1} = \{x\}\}\,|\big] \succeq \mathrm{vol}(R^j). \tag{9}$$

Note that $R^{j+1} = R^j \setminus \bigcup_{x \in S^{j+1}} B_{r_1}(x)$. By definition of $R^j_\circ$, the r_1-ball around each $x \in S^{j+1} \cap R^j_\circ$ intersects R^j in a set with volume bounded below, and if we only look at those x where $B_{r_1}(x) \cap S^{j+1} = \{x\}$, all the balls $B_{r_1}(x)$ are disjoint. So by (9),

$$E\left[\mathrm{vol}\left(R^j \cap \bigcup_{x \in S^{j+1}} B_{r_1}(x)\right)\right] \succeq \mathrm{vol}(R^j),$$

and the claim follows. □

We now complete the proof of the proposition. Let σ^j be the law of $S^{\leq j}$. Then conditioning on whether $\mathrm{vol}(R^j)/\mathrm{vol}(M) \geq \epsilon/2$ or not, we have

$$\frac{E[\mathrm{vol}(R^{j+1})]}{\mathrm{vol}(M)} \leq \frac{\epsilon}{2} + \int_{\mathrm{vol}(R^j)/\mathrm{vol}(M) \geq \epsilon/2} \frac{E[\mathrm{vol}(R^{j+1}) \mid R^j]}{\mathrm{vol}(M)} \, d\sigma^j. \tag{10}$$

Let's call the second term on the right in (10) X^{j+1}. Then by Claim 2.10, we have

$$X^{j+1} \leq \delta \cdot \int_{\mathrm{vol}(R^j)/\mathrm{vol}(M) \geq \epsilon/2} \frac{\mathrm{vol}(R^j)}{\mathrm{vol}(M)} \, d\sigma^j \leq \delta \cdot \int_{\mathrm{vol}(R^{j-1})/\mathrm{vol}(M) \geq \epsilon/2} \frac{E[\mathrm{vol}(R^j)]}{\mathrm{vol}(M)} \, d\sigma^{j-1} \leq \delta X^j$$

for all j, where the middle inequality uses the inclusion $R^j \subset R^{j-1}$ to say that the condition on $\mathrm{vol}(R^j)$ is *at least* as restrictive as the condition on $\mathrm{vol}(R^{j-1})$. Since $\delta < 1$ is fixed, there is some uniform $j = j(\epsilon, r_0, r_1, v_{min}, v_{max})$ such that $X^j < \epsilon/2$, and then

$$\frac{E[\mathrm{vol}(R^j)]}{\mathrm{vol}(M)} < \epsilon/2 + \epsilon/2 = \epsilon$$

as desired in (6). □

We will also need the following variance estimate.

Lemma 2.11. *Given j, there is some $C = C(j, r_0, r_1, v_{min}, v_{max})$ such that for any M satisfying (1) and (2) in the statement of the theorem, we have*

$$Var[|S^{\leq j}|] := E[|S^{\leq j}|^2] - E[|S^{\leq j}|]^2 < C vol(M).$$

Proof. Recall from the construction that $S^{\leq j} = S^1 \cup \cdots \cup S^j$, where each S^j is the set of all points s in a Poisson process P^j on M such that

$$\forall t \in P^j(< s) \cup \bigcup_{i<j} S^i, \quad \phi(d(s,t)) \geq X(s,t). \tag{11}$$

The main ingredient is the following claim.

Claim 2.12. *Suppose that $A, B \subset M$ and that $d(A,B) \geq 2 \cdot (j+1) \cdot r_1$. Then the random subsets $S^{\leq j} \cap A$ and $S^{\leq j} \cap B$ are independent.*

Proof. For s, t as in (11), if $d(s,t) \geq r_1$ then $\phi(d(s,t)) = 1 \geq X(s,t) \in [0,1]$. So, *the decision whether to include s in S^j only depends upon the points t that lie in an r_1-neighborhood of s.* So, suppose $d(A,B) \geq (j+1) \cdot r_1$. The intersections $P^j \cap N_{r_1}(A)$ and $P^j \cap N_{r_1}(B)$ are independent, since P^j is a Poisson process, and the order of the elements in the first set is also independent of the order in the second, since the order on P^j is determined by picking independent, random values in $[0,1]$ for each $s \in P^j$. Assuming inductively that $S^{<j} \cap N_{r_1}(A)$ and $S^{<j} \cap N_{r_1}(B)$ are independent, the result follows for j. □

Fix some positive $v < v_{min}$. For $x \in M$, let $B(x)$ be the unique volume-v ball around x. Here, we are using that M is special and satisfies condition (1) in the theorem; (1) also implies that $B(x)$ has radius less than $r_0/2$. Given $S^{\leq j}$, let

$$S^{\leq j}(v) = \cup_{x \in S^{\leq j}} B(x) \subset M.$$

Note that since $S^{\leq j}$ is r_0-separated, this is a disjoint union. Then if χ^j is the characteristic function of $S^{\leq j}(v)$, and σ^j is the law of $S^{\leq j}$, we have

$$v^2 E[|S^{\leq j}|^2] = \int \int_{M \times M} \chi^j(x) \chi^j(y) \, dx \, dy \, d\sigma^j = \int_{M \times M \setminus} \int \chi^j(x) \chi^j(y) \, d\sigma^j \, dx \, dy,$$

by Fubini. Let $D \subset M \times M$ be the subset consisting of all pairs (x,y) such that

$$d(x,y) \leq 2 \cdot (j+1) \cdot r_1 + 2r_0.$$

(Compare with Claim 2.12.) Then we can split the outer two integrals above into an integral over D, and an integral over $M \times M \setminus D$. By condition (2) in the statement of the theorem, we have $\mathrm{vol}(D) = O(\mathrm{vol}(M))$, so since $|\chi^j(x)\chi^j(y)| \leq 1$, we have

$$v^2 E[|S^{\leq j}|^2] = \int_{M \times M \setminus D} \left(\int \chi^j(x) \chi^j(y) \, d\sigma^j \right) dx dy + O(\mathrm{vol}(M))$$

But for a fixed x, we have $\chi^j(x) = 1$ exactly when there is some point of $S^{\leq j}$ in the set A_x of all the centers of volume-v balls in M that contain x. Now $A_x \subset B_{r_0}(x)$, so if $(x,y) \in D$ then the distance $d(A_x, A_y)$ is bigger than the constant from Claim 2.12, and hence the random variables $\chi^j(x)$ and $\chi^j(y)$ are independent. Thus, the above

$$= \int_{M\times M\setminus D} \left(\int \chi^j(x)\, d\sigma^j\right)\left(\int \chi^j(y)\, d\sigma^j\right) dxdy + O(\mathrm{vol}(M)).$$

But using again that $\mathrm{vol}(D) = O(\mathrm{vol}(M))$, the above

$$\begin{aligned} &= \int_{M\times M} \left(\int \chi^j(x)\, d\sigma^j\right)\left(\int \chi^j(y)\, d\sigma^j\right) dxdy + O(\mathrm{vol}(M)) \\ &= v^2 E[|S^{\leq j}|]^2 + O(\mathrm{vol}(M)), \end{aligned}$$

and the Lemma follows. □

Finally, in the proof of Theorem 2.5, it will be crucial that the random $S^{\leq j} \subset M$ depend continuously on M in the appropriate sense. To say precisely what this means, let

$$\begin{aligned} \mathbb{M}^{ext}_{sp} &= \{ \text{ pointed, extended special mm-spaces } (M, \mathrm{vol}, p, E) \text{ } \}/\sim \ \subset \ \mathbb{M}^{ext}, \\ \mathbb{MS}^{ext} &= \{ \text{ 5-tuples } (M, \mathrm{vol}, p, E, S), \text{ where } S \subset M \text{ is discrete } \}/\sim, \end{aligned}$$

where here *sp* stands for special, and in the second line $\sim$ is generated by isomorphisms of pointed extended mm-spaces that preserve the distinguished subsets. If $\mathfrak{M} = (M, \mathrm{vol}, p, E)$ is an extended special, pointed mm-space and j is fixed, we can choose a random element of $\mathbb{MS}^{ext}$ by choosing a random $S^{\leq j} \subset M$ as above. The law of this random element is a measure on $\mathbb{MS}^{ext}$ depending on $\mathfrak{M}$, and this defines a *continuous* function

$$\sigma : \mathbb{M}^{ext}_{sp} \longrightarrow \mathcal{P}(\mathbb{MS}^{ext}). \tag{12}$$

Here, continuity follows from exactly the same arguments as [11, Lemma 4.2]. The only difference is that our mm-spaces are extended, but since the extensions E do not appear anywhere above, their presence does not affect the proof, except in notation.

2.4.2. *Random nerve complexes.* Above, we defined the space $\mathbb{MS}^{ext}$ of pointed, extended special mm-spaces with distinguished discrete subsets. Here, we explain how to construct a random nerve complex from certain elements of $\mathbb{MS}^{ext}$, in a way that depends continuously on the input. Consider the subset

$$(\mathbb{MS}^{ext})' \subset \mathbb{MS}^{ext}$$

consisting of all tuples $(M, \mathrm{vol}, p, E, S)$ such that *there is a unique element of S that is closest to p.* Given some such tuple, construct a simplicial complex by choosing independently and Lebesgue-randomly a number $\rho(x) \in [r_2, r_3]$ for each $x \in S$ and taking the nerve $N_E(S, \rho)$ of the collection of balls $B_E(x, \rho(x))$, where $x \in S$. Note that the balls are in E, not in M. The unique element of S closest to p is a natural base point for the nerve, and if K is the *connected component* of $N_E(S,\rho)$ containing p then we have a map

$$(\mathbb{MS}^{ext})' \longrightarrow \mathcal{P}(\mathcal{K}), \tag{13}$$

where $\mathcal{K}$ is the space of all pointed, finite degree simplicial complexes, $\mathcal{P}(\cdot)$ denotes the space of probability measures, and the map sends $(M, \mathrm{vol}, p, E, S)$ to the (law of the) random pointed complex (K,p) described above. Note that properties (1) and (2) in the statement of the theorem imply that there is a universal degree bound for all constructed K, that is independent of n.

Claim 2.13. *The map in* (13) *is continuous.*

Proof Sketch. In his Claim 1 at the beginning of the proof of Theorem 4.1, Bowen shows that a variant of (13) is continuous. Here are the discrepancies with our version. First, Bowen works only with $[r_2, r_3] = [5r_0, 6r_0]$, but his argument obviously generalizes. He also does not use extended mm-spaces, and so the map he constructs is from the subset $\mathbb{MS}' \subset \mathbb{MS}$ of all (M, p, vol, S) where there is a unique closest point in S to p, and the balls he uses in constructing the nerve are just in M, not in some larger space. However, the quick proof of continuity works verbatim in our extended setting: basically all that is used is that the topology comes from pointwise Hausdorff convergence of the mm-spaces M, so since we are putting the same topology on the super-sets E, the argument extends. □

2.4.3. *Convergence of normalized Betti numbers.* We now begin on the main argument for the proof of Theorem 2.5. Much of the work below is from [11], altered so that we use our $S^{\leq j}$ instead of his random nets.

Let $\mathfrak{M}_n = (M_n, \mathrm{vol}_n, E_n)$, be as in the statement of Theorem 2.5, and fix $\epsilon > 0$. Pick $j = j(\epsilon, r_0, r_1, v_{min}, v_{max})$ as in Proposition 2.9, and let σ_n be the law of the random subset $S^{\leq j} \subset M_n$ constructed above. For simplicity in notation, we'll drop the subscript j below and just write S_n for a σ_n-random subset of M_n.

For each n, let λ_n be the measure on $\mathbb{MS}^{ext}$ obtained by pushing forward the product measure $(\mathrm{vol}_n/\mathrm{vol}_n(M_n)) \times \sigma_n$ under the map

$$M_n \times \{ \text{ discrete } S \subset M_n \} \longrightarrow \mathbb{MS}^{ext}, \quad (p,S) \mapsto (M_n, \mathrm{vol}_n, p, E_n, S) \tag{14}$$

But λ_n can also be obtained by pushing forward $\mu_{\mathfrak{M}_n}/\mathrm{vol}_n(M_n)$ via the continuous map $\sigma : \mathbb{M}^{ext}_{sp} \longrightarrow \mathcal{P}(\mathbb{MS}^{ext})$ from (12) to get a *measure* on $\mathcal{P}(\mathbb{MS}^{ext})$, and then taking the expected value. In symbols,

$$\mu_{\mathfrak{M}_n} \in \mathcal{P}(\mathbb{M}^{ext}_{sp}) \overset{\sigma_*}{\longmapsto} \sigma_*(\mu_{\mathfrak{M}_n}) \in \mathcal{P}(\mathcal{P}(\mathbb{MS}^{ext})) \overset{\int}{\longmapsto} \lambda_n \in \mathcal{P}(\mathbb{MS}^{ext}).$$

Since both the above maps are continuous and $(\mu_{\mathfrak{M}_n})$ is weakly convergent, it follows that (λ_n) converges weakly to some probability measure λ_∞ on $\mathbb{MS}^{ext}$.

Fix $v > 0$. If $S \subset M$ is a discrete set, let $S(v)$ be the union of all volume-v closed balls in M that are centered at points of S. Let

$$\mathbb{MS}^{ext}(v) \subset \mathbb{MS}^{ext}$$

be the subset consisting of all tuples $(M, \mathrm{vol}, p, S, E)$ such that $p \in S(v)$. Fixing n and $0 \le v \le v_{min}/2$, say, the volume-v balls around the points of an (r_0, r_1)-net in M_n are all disjoint, since by hypothesis balls of radius $r_0/2$ have volume at least v_{min}. Hence, a λ_n-random element of the set $\mathbb{MS}^{ext}(v)$ can be described by first picking $S_n \subset M_n$ randomly according to σ_n, and then choosing a random point from one of the $|S_n|$-many disjoint volume-v balls centered at points of S_n. It follows that

$$\lambda_n(\mathbb{MS}^{ext}(v)) = v \cdot \frac{E[|S_n|]}{\mathrm{vol}_n(M_n)}, \tag{15}$$

where S_n is chosen σ_n-randomly. So, we have that for $0 < v < v' \le v_{min}/2$ and all n,

$$\frac{\lambda_n(\mathbb{MS}^{ext}(v'))}{\lambda_n(\mathbb{MS}^{ext}(v))} = v'/v.$$

It then follows just like in the proof of Bowen's Claim 2 that

$$\lambda_\infty(\partial\mathbb{MS}^{ext}(v_{min}/2)) = 0, \quad \text{and} \quad \lambda_n(\mathbb{MS}^{ext}(v_{min}/2)) \to \lambda_\infty(\mathbb{MS}^{ext}(v_{min}/2)). \tag{16}$$

Finally, note that by Claim 2.8, we have some uniform $c = c(r_0, v_{min}, v_{max})$ such that $E[|S_n|] \ge c \cdot \mathrm{vol}_n(M_n)$. This gives a lower bound for (15), and taking the limit gives

$$\lambda_\infty(\mathbb{MS}^{ext}(v_{min}/2)) \ge \frac{1}{2} c v_{min} > 0. \tag{17}$$

Restrict each λ_n to $\mathbb{MS}^{ext}(v_{min}/2)$ and normalize to produce probability measures

$$\lambda'_n \in \mathcal{P}(\mathbb{MS}^{ext}(v_{min}/2)).$$

By (16) and the Portmanteau Theorem, we have $\lambda'_n \to \lambda'_\infty \in \mathcal{P}(\mathbb{MS}^{ext}(v_{min}/2))$. Note that since all our nets are r_0-separated, and $r_0/2$-balls have volume at least v_{min}, we have

$$\mathbb{MS}^{ext}(v_{min}/2) \subset (\mathbb{MS}^{ext})'.$$

Hence, we can apply (13) to the λ'_n to get a sequence ν_n of probability measures on the space $\mathcal{K}$ of all pointed, finite degree complexes. Note that each ν_n is supported on the finite subset of $\mathcal{K}$ consisting of all pointed complexes with at most, say, $\mathrm{vol}(M_n)/v_{min}$ vertices.

Now for each n, when we pick a λ'_n-random element of $\mathbb{MS}^{ext}(v_{min}/2)$, there is an equal probability that the random basepoint p_n will end up in the ball with volume $v_{min}/2$ around any given point in the random subset S_n. Hence, ν_n assigns equal weight to all the possible basepoints for a complex. So, Bowen's Lemma 2.2 implies that the limit

$$\lim_{n\to\infty} \frac{E[b_k(K_n)]}{E[\mathrm{vol}(K_n)]} \tag{18}$$

exists for all k, where for each n, a pointed complex $(K_n, \star)$ is chosen ν_n-randomly. (Note that neither the Betti numbers nor the volume of a complex depend on the base point, though, so it is not necessary to write $\star$ in (18).)

Let's try to unpack (18) into an expression just involving the spaces $\mathfrak{M}_n = (M_n, \mathrm{vol}_n, E_n)$ and the associated σ_n-random subsets $S_n \subset M_n$. Now the expectations in (18) are taken with respect to the measure ν_n, which is constructed using σ_n, and tracing back through the construction, the key point is the following:

Claim 2.14. *If $f : \mathcal{K} \longrightarrow \mathbb{R}_{\geq 0}$ is a Borel function that is basepoint independent then*

$$E[f(K)] = \frac{v_{min}}{2vol_n(M_n) \cdot \lambda_n(\mathbb{MS}^{ext}(v_{min}/2))} \cdot E\big[f(N_{E_n}(S_n, \rho_n)) \cdot |S_n|\big],$$

where on the left $(K, \star)$ is chosen ν_n-randomly, and on the right $S_n \subset M_n$ is chosen σ_n-randomly and the values of $\rho_n : S_n \longrightarrow [r_2, r_3]$ are independent and Lebesgue distributed.

Proof. Since ν_n is the push forward of λ'_n under the random nerve complex map, we have

$$E[f(K)] = E[f(N_{E_n}(S_n, \rho_n))], \tag{19}$$

where on the right side $(M_n, \mathrm{vol}_n, p_n, E_n, S_n)$ is selected λ'_n*-randomly*, and then the values of ρ_n are selected independently and Lebesgue-randomly. But then (19) becomes

$$\begin{aligned}
&\int_{\mathbb{MS}^{ext}} \int f(N_{E_n}(S_n, \rho_n))\, d\rho_n\, d\lambda'_n \\
&= \frac{1}{\lambda_n(\mathbb{MS}^{ext}(v_{min}/2))} \int_{\mathbb{MS}^{ext}(v_{min}/2)} \int f(N_{E_n}(S_n, \rho_n))\, d\rho_n\, d\lambda_n \\
&= \frac{1}{\mathrm{vol}_n(M_n) \cdot \lambda_n(\mathbb{MS}^{ext}(v_{min}/2))} \int_{S_n \subset M_n} \int_{p \in S_n(v_{min}/2)} \int f(N_{E_n}(S_n, \rho_n))\, d\rho_n\, d\mathrm{vol}_n\, d\sigma_n,
\end{aligned}$$

where here the reader should recall that $S_n(v)$ is the union of all volume-v closed balls in M_n centered at points of S_n. Since the integrand does not depend on p, this becomes

$$\begin{aligned}
&= \frac{1}{\mathrm{vol}_n(M_n) \cdot \lambda_n(\mathbb{MS}^{ext}(v_{min}/2))} \int_{S_n \subset M_n} \mathrm{vol}_n(S_n(v_{min}/2)) \int f(N_{E_n}(S_n, \rho_n))\, d\rho_n\, d\sigma_n, \\
&= \frac{v_{min}/2}{\mathrm{vol}_n(M_n) \cdot \lambda_n(\mathbb{MS}^{ext}(v_{min}/2))} \int_{S_n \subset M_n} |S_n| \int f(N_{E_n}(S_n, \rho_n))\, d\rho_n\, d\sigma_n,
\end{aligned}$$

and the claim follows. ☐

Applying the claim to both the numerator and denominator of (18), we get

$$\lim_{n\to\infty} \frac{E[b_k(K)]}{E[\mathrm{vol}(K)]} = \lim_{n\to\infty} \frac{E\big[b_k(N_{E_n}(S_n, \rho_n)) \cdot |S_n|\big]}{E[|S_n|^2]}, \tag{20}$$

where for each n, $S_n \subset M_n$ is chosen σ_n-randomly, the values of $\rho_n : S_n \longrightarrow [r_2, r_3]$ are chosen independently and Lebesgue-randomly. Now, by Proposition 2.9 and the definition of j in the beginning of §2.4.3, the expectation of the number of points needed to extend S_n to a $(r_0, 2r_1)$-net $T_n \subset M_n$ is at most $\epsilon \cdot \mathrm{vol}_n(M_n)$. The nerves of S_n and T_n then differ by $O(\epsilon \cdot \mathrm{vol}_n(M_n))$ simplices, where here and below, all constants in big-O notation depend only on the constants in conditions (1) and (2) of the theorem. By condition (3) of the theorem, the Betti numbers of the nerve of T_n differ sublinearly in volume from B_n:

$$b_k(N_{E_n}(T_n, \rho_n)) = B_n + o(\mathrm{vol}_n(M_n)).$$

So, applying Mayer–Vietoris, we have

$$b_k(N_{E_n}(S_n,\rho_n)) = B_n + O(\epsilon \cdot \mathrm{vol}_n(M_n)).$$

Hence, the numerator in (20) becomes:

$$E\big[b_k(N_{E_n}(S_n,\rho_n)) \cdot |S_n|\big] = \big(B_n + O(\epsilon \cdot \mathrm{vol}_n(M_n)\big) \cdot E[|S_n|].$$

For the denominator, note that by Lemma 2.11, we have

$$|E[|S_n|^2] - E[|S_n|]^2| \le C \cdot \mathrm{vol}_n(M_n),$$

where here C is independent of n. By Claim 2.8, though, $E[|S_n|] \ge c \cdot \mathrm{vol}_n(M_n)$ for some uniform $c = c(r_0, r_1, v_{min}, v_{max})$, so as $\mathrm{vol}_n(M_n) \to \infty$ it follows that

$$\lim_{n\to\infty} \frac{E[|S_n|^2]}{E[|S_n|]^2} = 1.$$

Combining these estimates for the numerator and denominator of (20), we get that

$$\frac{E\big[b_k(N_{E_n}(S_n,\rho_n)) \cdot |S_n|\big]}{E[|S_n|^2]} = \left(\frac{B_n}{E[|S_n|]} + \frac{O(\epsilon \cdot \mathrm{vol}_n(M_n))}{E[|S_n|]}\right) \cdot \frac{E[|S_n|]^2}{E[|S_n|^2]},$$

where the limit of the last ratio is 1, as above. But since the left hand side converges and

$$\lim_{n\to\infty} \frac{E[|S_n|]}{\mathrm{vol}_n(M_n)} = \frac{v_{min}}{2} \lambda_\infty(\mathbb{MS}^{ext}(v_{min}/2)) \ge c = c(r_0, r_1, v_{min}, v_{max}),$$

the takeaway here is that

$$\limsup_{n\to\infty} \frac{B_n}{\mathrm{vol}_n(M_n)} - \liminf_{n\to\infty} \frac{B_n}{\mathrm{vol}_n(M_n)} \le C \cdot \epsilon$$

for some C depending only on the constants in the statement of the theorem. Of course, $\epsilon > 0$ was arbitrary, so this means that the sequence $B_n/\mathrm{vol}_n(M_n)$ converges.

3. Pinched negative curvature and Theorem 1.6

In this section, we consider only d-manifolds M with sectional curvature

$$-1 \le K \le -a^2 < 0,$$

and we let $\epsilon(d)$ be the corresponding d-dimensional Margulis constant. For any $\epsilon \le \epsilon(d)$, each component of the ϵ-thin part $(M_n)_{\le\epsilon}$ is either:

- a *Margulis tube*, which is (topologically) a tubular neighborhood of a closed geodesic, and so is homeomorphic to a ball bundle over the circle, or
- a *cusp neighborhood*, which is homeomorphic to $S \times [0, \infty)$ for some compact aspherical $(d-1)$-manifold S with virtually nilpotent fundamental group.

See for instance [7, §8] for a proof.

In the introduction, we explained how to produce BS-convergent sequences (M_n) of hyperbolic 3-manifolds where the normalized Betti numbers do not converge, using Dehn filling. In the example we gave, the volumes $\mathrm{vol}(M_n)$ were bounded, but one can construct similar examples with unbounded volumes by filling the complements of links with unboundedly many components, instead of a fixed knot complement. Instead of doing the details of this approach, though, we'll briefly describe a similar example in which the BS-limit is easier to understand.

Example 3.1. *Let M be the mapping torus of a homeomorphism $\phi : S \longrightarrow S$, where ϕ is a pseudo-Anosov homeomorphism of some closed surface S with genus at least 2. So, M comes with a fibration $M \longrightarrow S^1$. Identify S with a fiber of this fibration, and let γ be a simple closed curve on S. By Thurston's Hyperbolization Theorem [20], the manifold*

$$M(\infty) := M \setminus \gamma$$

admits a finite volume hyperbolic metric.

Let $M(k)$ be the closed 3-manifold obtained from $M(\infty)$ by $(1, k)$-Dehn filling[5]. For large k, Thurston's Dehn Filling Theorem [8] implies that $M(k)$ admits a hyperbolic metric; moreover, as $k \to \infty$ the manifolds $M(k) \to M(\infty)$ geometrically. Note that since we are doing $(1, k)$ filling, each $M(k)$ is also a genus g mapping torus. Indeed, if T_γ is a Dehn twist around γ, the monodromy map of $M(k)$ is $T_\gamma^k \circ \phi$.

For $k \in \mathbb{N} \cup \{\infty\}$, let $M_n(k)$ be the degree n cyclic cover of $M(k)$ corresponding to the subgroup of $\pi_1 M(k)$ that is the preimage of $n\mathbb{Z} \leq \mathbb{Z} \cong \pi_1(S^1)$ under the map induced by

$$M(k) \hookrightarrow M \longrightarrow S^1.$$

Then for every n and $k < \infty$, the manifold $M_n(k)$ is a mapping torus over a genus g surface, and hence

$$b_1(M_n(k)) \leq 2g + 1.$$

On the other hand, setting $k = \infty$ the manifold $M_n(\infty)$ has n cusps, so we have

$$b_1(M_n(\infty)) \geq n.$$

Now set $k = n$. As $n \to \infty$, the sequences $M_n(n)$ and $M_n(\infty)$ both BS-converge to the same limit measure μ on $\mathcal{M}$. This μ is supported on pointed manifolds isometric to the infinite cyclic cover $M_\infty(\infty)$ of $M(\infty)$ corresponding to the kernel of the map on fundamental groups induced by $M(\infty) \longrightarrow S^1$; more carefully, μ is the push forward of the normalized Riemannian measure on $M(\infty)$ under the map

$$M(\infty) \longrightarrow \mathcal{M}, \quad p \longmapsto [(M_\infty(\infty), p_\infty)],$$

where p_∞ is any point that projects to p under the covering map $M_\infty(\infty) \longrightarrow M(\infty)$. (This is a special case of the construction in Example 2.4 in [2].) However,

$$b_1(M_n(n))/vol(M_n(n)) \to 0, \quad b_1(M_n(\infty))/vol(M_n(\infty)) \not\to 0.$$

[5]Here, we use meridian-longitude coordinates to parametrize the boundary of a cusp neighborhoods, where the meridian is the curve that was homotopically trivial before we drilled out γ.

Essentially, the reason why Dehn filling is problematic is that from the perspective of most points in a manifold, a Margulis tube with very small core length can look nearly identical to a rank two cusp. (One can only see the difference if one is close enough to be able to distinguish the core geodesic of the tube, and when the core length is small, the set of points a bounded distance from the core has very small volume.) This coincidence is particularly three-dimensional, though. For instance, note that the boundary of a d-dimensional Margulis tube is a S^{d-2}-bundle over S^1, while the boundary of a cusp neighborhood is a Euclidean $(d-1)$-manifold. If $d = 3$, the torus T^2 satisfies both descriptions, but when $d \geq 4$, S^{d-2}-bundles over S^1 are not aspherical, so cannot be Euclidean.

The plan for the rest of §3 is as follows. In §3.1 we show that Margulis tubes with short cores have large volume in dimension at least 4. In §3.2 we adapt some of Gelander's work in [17], showing that one can approximate (shrinkings of) the ϵ-thick parts of manifolds with pinched negative curvature with certain nerve complexes. And then finally, in §3.3 we prove Theorem 1.6.

3.1. **Lower volume bounds for Margulis tubes.** As mentioned above, the basic idea in Theorem 1.6 is to show that the number of Margulis tubes with very short cores that appear in a manifold with pinched negative curvature is a very small fraction of its volume. To verify this, we will use the following proposition.

Proposition 3.1 (Short geodesics imply large volume)**.** *Let $d \geq 4$ and let M be a complete Riemannian d-manifold with sectional curvatures in the interval $[-1, -a^2]$, where $a > 0$. Suppose that $T \subset M_{\leq \epsilon}$ is a component of the ϵ-thin part of M whose core geodesic has length ℓ. Then $vol(T) \geq C := C(d, a, \epsilon, \ell)$, where $C \to \infty$ as $\ell \to 0$.*

By Wang's finiteness theorem [26], for $d \geq 4$ a finite volume hyperbolic d-manifold M can only have a very short geodesic if its volume is very large. So for hyperbolic manifolds, one can think of the above as a strengthening of this statement that says that the large volume has to come from the Margulis tube around the short geodesic.

One could probably prove (at least a version of) the proposition using a geometric limit argument informed by the above discussion on Dehn filling. We assume there is a sequence of manifolds and Margulis tubes $T_n \subset M_n$ where the core length $\ell_n \to 0$, but where $\sup \mathrm{vol}(T_n) < \infty$. Take base points $p_n \in \partial T_n$ and extract pointed Gromov-Hausdorff limits of everything, giving $T_\infty \subset M_\infty$ and $p_\infty \in \partial T_\infty$. Since $\ell_n \to 0$, this T_∞ is a cusp neighborhood, rather than a Margulis tube. And since $\sup \mathrm{vol}(T_n) < \infty$, one can argue that the diameter of ∂T_n is bounded, which means that ∂T_∞ should actually be homeomorphic to ∂T_n. But as mentioned above, this is impossible since the boundary of a cusp neighborhood is always aspherical, but the boundary of a Margulis tube is not if $d \geq 4$.

We chose not to use the geometric limit approach because pushing through the limiting arguments requires control over higher order derivatives of the metric tensors, which we do not necessarily want to include in the statement of Theorem 1.6. Also, the proof we give below is attractive in that one could use it to write down an explicit formula for C.

Before starting the proof of Proposition 3.1, we establish the following simple lemma.

Lemma 3.2. *Suppose that $n \geq 3$ and $A \leq O(n)$ is an abelian subgroup, which we consider as acting on the unit sphere S^{n-1} by isometries. Then diam$(G\backslash S^{n-1}) \geq \pi/2$.*

Here, the distance between two points in the quotient is the minimal distance in S^{n-1} between points in their preimages. Note that $G\backslash S^{n-1}$ is a path metric space.

Proof. The subgroup A is contained in a subgroup $\mathbb{T} \leq O(n)$ of the form

$$\mathbb{T} = \begin{pmatrix} O(2) & & \\ & \ddots & \\ & & O(2) \end{pmatrix} \text{ or } \mathbb{T} = \begin{pmatrix} O(2) & & & \\ & \ddots & & \\ & & O(2) & \\ & & & \pm 1 \end{pmatrix},$$

written in suitable orthonormal coordinates $(x_1, \ldots, x_n)$ for $\mathbb{R}^n$, depending on whether n is even or odd. But in these coordinates, the action of A preserves the intersection I of S^{n-1} with the x_1x_2-coordinate plane, and it also preserves the intersection J of S^{n-1} with either the x_3x_4-coordinate plane or the x_3-axis, depending on whether $n \geq 4$ or $n = 3$. The distance in S^{n-1} between I and J is $\pi/2$, so the lemma follows. □

Proof of Proposition 3.1. Pick a universal covering map $\tilde{M} \longrightarrow M$ and lift the core geodesic $\gamma \subset T$ to a complete geodesic $\tilde{\gamma} \subset \tilde{M}$. Let $\tilde{T}$ be the component of the preimage of T that contains $\tilde{\gamma}$, and let $g : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ be a nontrivial deck transformation stabilizing $\tilde{\gamma}$ that is primitive in the deck group. So, g is determined up to inversion, the cyclic group $\langle g \rangle$ is the stabilizer of $\tilde{T}$, and any deck transformation not in $\langle g \rangle$ moves $\tilde{T}$ completely off itself.

Pick a point $\tilde{p} \in \tilde{\gamma}$ and isometrically identify the fiber $N^1(\tilde{\gamma})_{\tilde{p}}$ of the unit normal bundle of $\tilde{\gamma}$ with S^{n-2}. Parallel transport then determines a global trivialization

$$\tilde{\gamma} \times S^{n-2} \longrightarrow N^1(\tilde{\gamma}),$$

and we can then write the action of g on $N^1(\tilde{\gamma})$ in these coordinates as

$$g = \tau \times r,$$

where τ is a translation by ℓ along $\tilde{\gamma}$ and $r \in O(d-1)$.

Since $O(d-1)$ is a compact manifold, there is some $c > 0$ such that if $S \subset O(d-1)$ is any set of $m^{\dim O(d-1)}$ points, there are $s, t \in S$ such that

$$d(s(\xi), t(\xi)) \leq c/m, \quad \forall \xi \in S^{d-2}.$$

Setting $m = \lfloor \ell^{\frac{-1}{\dim O(d-1)+1}} \rfloor$, we get that there is some power r^k, $k \leq m^{\dim O(d-1)}$ with

$$d(r^k(\xi), \xi) \leq c/m \leq c \cdot \ell^{\frac{1}{\dim O(d-1)+1}}, \quad \forall \xi \in S^{d-2}. \tag{21}$$

Note that we also have

$$d(\tau^k(\tilde{x}), \tilde{x}) \leq \ell \cdot m^{\dim O(d-1)} \leq \ell \cdot \ell^{\frac{-\dim O(d-1)}{\dim O(d-1)+1}} = \ell^{\frac{1}{\dim O(d-1)+1}}, \quad \forall \tilde{x} \in \tilde{\gamma}. \tag{22}$$

Since $\tilde{M}$ has sectional curvatures in $[-1, -a^2]$, it follows from the triangle comparison theorems that if two unit speed geodesic segments α, β in $\tilde{M}$ share an endpoint $\alpha(0) = \beta(0)$ at which they intersect with angle θ, then we have that

$$\theta \sinh(a \cdot t)/a \leq d(\alpha(t), \beta(t)) \leq \theta \sinh(t), \quad \forall t > 0. \tag{23}$$

Similarly, by an application of Berger's extension of Rauch's comparison theorem [12, Theorem 1.34], if α, β start out with $\alpha(0), \beta(0) \in \tilde{\gamma}$, and both are perpendicular to $\tilde{\gamma}$, then

$$d(\alpha(t), \beta(t)) \leq d(\alpha(0), \beta(0)) \cdot \cosh(t), \quad \forall t > 0. \tag{24}$$

Combining (21) and (22) with the upper bounds in (23) and (24), and using the decomposition $g = \tau \times r$, we get that for a point $\tilde{x} \in \tilde{M}$ that lies at distance t from $\tilde{\gamma}$,

$$d(g^k(\tilde{x}), \tilde{x}) \leq \ell^{\frac{1}{\dim O(d-1)+1}} \big(c \cdot \sinh(t) + \cosh(t)\big) \leq 2\ell^{\frac{1}{\dim O(d-1)+1}} \cdot c \cdot \cosh(t).$$

So if ℓ is small, we can set $L = \cosh^{-1}(\epsilon/(6c\ell^{\frac{1}{\dim O(d-1)+1}}))$, and then the L-neighborhood of $\tilde{\gamma}$ will be contained in the subset $\tilde{T}_{\epsilon/3} \subset \tilde{T}$ that is the lift of $T \cap M_{\leq \epsilon/3}$.

Since $d \geq 4$, Lemma 3.2 implies that

$$\operatorname{diam}(\langle r \rangle \backslash S^{n-2}) \geq \pi/2.$$

Since the quotient is a path metric space, we can then choose for any $\theta > 0$ a set S of $\lfloor \pi/(2\theta) \rfloor$ points in S^{n-2}, with the property that the $\langle r \rangle$-orbits of any two distinct points in S are at least a distance of θ from each other in S^{n-2}. Identify S^{n-2} with the fiber $N^1(\tilde{\gamma})_{\tilde{p}}$ of the unit normal bundle as above, let $\exp_{\tilde{p}}$ is the Riemannian exponential map, and let

$$\mathcal{S} = \{\exp_{\tilde{p}}(L \cdot \xi) \mid \xi \in S\}.$$

By the lower bound in (23), we get that the distance between the $\langle g \rangle$-orbits of any two distinct points in $\mathcal{S}$ is at least $\theta \sinh(a \cdot L)/a$. So taking

$$\theta = \frac{\epsilon \cdot a}{2 \sinh(a \cdot L)},$$

the $\langle g \rangle$-orbits of points in $\mathcal{S}$ are at least $\epsilon/3$ apart in $\tilde{M}$.

Now $\tilde{T}_{\epsilon/3}$ is star-shaped with respect to the geodesic $\tilde{\gamma}$, so we can now project $\mathcal{S} \subset \tilde{T}_{\epsilon/3}$ radially from $\tilde{\gamma}$ to a subset $\mathcal{S}' \subset \partial T_{\epsilon/3}$. Since $\tilde{M}$ has negative curvature, this radial projection cannot decrease the distance between any two points in $\tilde{T}_{\epsilon/3}$ that are the same distance from $\tilde{\gamma}$. Hence, the $\langle g \rangle$-orbits of points in $\mathcal{S}'$ are still at least $\epsilon/3$ apart in $\tilde{M}$. It follows that the covering map $\tilde{M} \longrightarrow M$ restricts to an embedding on the union of the $\epsilon/3$-balls in $\tilde{M}$ around the points of $\mathcal{S}'$. (It is an embedding on each individual ball by definition of $\tilde{T}_{\epsilon/3}$.) Each of these $\epsilon/3$-balls is contained in $\tilde{T}$, so volume of T is bounded below by the sum of the volumes of these balls. Each ball has volume at least some $V = V(\epsilon, d, a)$, by the usual comparison arguments, and there are $\lfloor \pi/(2\theta) \rfloor$ balls in total. By our definitions of θ and L, the number of balls goes to infinity with ℓ, and the proposition follows. $\square$

3.2. **Simplicial approximation of the thick part.** Suppose that M is a metric space and $A \subset M$. Following [17], we denote the metric ξ-neighborhood of A by $(A)_\xi$, and we define the ξ-*shrinking* of A to be the subset

$$)A(_\xi := M \setminus (M \setminus A)_\xi \subset A.$$

Fix now $\epsilon, \xi > 0$, with ϵ less than the Margulis constant $\epsilon(d)$, and let M be a Riemannian d-manifold with curvatures in $[-1, -a^2]$. The main result of this section is the following, which is an application of techniques of Gelander [17]. Informally, it says that the shrinking $)M_{\geq\epsilon}(_\xi$ of the ϵ-thick part of M can be simplicially modeled (up to homotopy) by the nerve complex associated to a certain open cover.

Proposition 3.3 $(([17])_\epsilon)$**.** *For any sufficiently small $\epsilon > \epsilon' > 0$ and any $c' \geq c \geq 1$, there is a constant $b = b(d, a, \epsilon)$ and some small $\delta_0 = \delta_0(d, a, c, \epsilon')$ such that the following holds for every d-manifold M with curvatures in $[-1, -a^2]$, and all $\delta < \delta_0$.*

Set $\xi = \epsilon/2 + \delta$ and let S be a $(\delta, c\delta)$-net in $)M_{\geq\epsilon}(_\xi$. Let

$$\rho : S \longrightarrow [(b + c)\delta, (b + c')\delta]$$

be any function and let $N(S, \rho)$ be the nerve of the collection of balls $B_M(x, \rho(x))$, where $x \in S$. Then $N(S, \rho)$ is homotopy equivalent to $M_{\geq\epsilon}$.

Here, recall from §2.3 that S is a $(\delta, c\delta)$-net if it is δ-separated and $c\delta$-covers. The key to the above is the following restatement of a result from [17].

Lemma 3.4 (essentially Lemma 4.1 in [17])**.** *Let M be a complete Riemannian d-manifold with sectional curvatures in the interval $[-1, 0]$, let $M' \subset M$ be a connected submanifold with boundary and let $\epsilon, b, c, c' > 0$, with $c < c'$, be fixed. Suppose that*

(1) M' is contained in the ϵ-thick part of M,
(2) M' is homotopy equivalent to $)M'(_{\epsilon/2}$,
(3) the preimage $\tilde{X}$ of $X = M \setminus M'$ under a universal covering map $\tilde{M} \longrightarrow M$ is a locally finite union $\tilde{X} = \cup_\gamma \tilde{X}_\gamma$ of convex open sets with smooth boundary,
(4) for any point $x \in \tilde{M} \setminus \tilde{X}$ with $d(x, \tilde{X}) \leq \epsilon$, there is a unit tangent vector $n(x) \in T_x(\tilde{M})$ such that for each i with $d(x, \tilde{X}) = d(x, \tilde{X}_\gamma)$, we have

$$n(x) \cdot \nabla d(\cdot, \tilde{X}_\gamma)|_x \geq 1/b.$$

Then there is some small $\delta_0 = \delta_0(\epsilon, b, d) > 0$ such that the following holds for all $\delta < \delta_0$. Let S be any δ-separated subset of $)M'(_{\epsilon/2+\delta}$ that $c\delta$-covers $)M'(_{\epsilon/2+\delta}$, and let

$$\rho : S \longrightarrow [(b + c)\delta, (b + c')\delta]$$

be a function. Then the nerve of the collection of balls

$$\mathcal{C} = \{B_M(x, \rho(x)) \mid x \in S\}$$

is homotopy equivalent to M'.

We should say that Lemma 4.1 in [17] is not quite stated as above. The biggest difference is that *maximal* δ-separated subsets of $)M'(_{\epsilon/2+\delta}$ are used in [17] instead of subsets that $c\delta$-cover, and the radii of the balls in the collection $\mathcal{C}$ are all chosen to be $(b+1)\delta$. However, the proof works just as well if all the 1's are replaced by numbers between c and c'. (So for instance, one should use c' instead of 1 in Proposition 4.7 of [17], and allow the radius to vary in Proposition 4.8.) A purely cosmetic difference is that Lemma 4.1 in [17] is stated for locally symmetric spaces, but local symmetry is not used in its proof. Finally, the conclusion of Lemma 4.1 in [17] is that M' is homotopy equivalent to an unnamed simplicial complex, but if one looks at his Proposition 4.8, one will see that this unnamed complex is just the nerve mentioned above. (The statement of Proposition 4.8 references the cover of $)M'(_{\epsilon/2}$ given by the collection of intersections $C\cap)M'(_{\epsilon/2}$, rather than the cover by $C \in \mathcal{C}$, and a priori the difference matters when constructing the nerve complex. However, the proof of Proposition 4.8 shows that when a finite subset of $\mathcal{C}$ has a nonempty intersection, this intersection intersects $)M'(_{\epsilon/2}$, so one gets the same nerve whether one considers the collection $\mathcal{C}$ referenced in our statement of Lemma 3.4, or the collection consisting of the intersections of its elements with $)M'(_{\epsilon/2}$, as in [17].)

Proof of Proposition 3.3. Set $M' = M_{\geq\epsilon}$. First, note that M' is homotopy equivalent to its $\epsilon/2$-shrinking, since its components are star-shaped neighborhoods of either a closed geodesic or a point at infinity, so we can deformation retract M' to its shrinking by flowing outwards. See the proof of Claim 8.5 of [17] for more details. So, by the Nerve Theorem it suffices to show that M' satisfies the conditions of the lemma above.

Conditions (1) and (2) are immediate from the definition of M', where

$$\tilde{X}_\gamma = \{x \in \tilde{X} \mid d(x,\gamma(x)) < \epsilon\}, \quad \gamma \in \pi_1 M.$$

For condition (3), we define the vector $n(x)$ in two cases. As long as ϵ is small, we can assume that any x in condition (3) is contained in the preimage of the $\epsilon(d)$-thin part of M, where $\epsilon(d)$ is the Margulis constant. If x lies in a component of this preimage that covers a Margulis tube, we define $n(x)$ exactly as in the proof of Lemma 7.4 of [17], i.e. by using Lemma 7.3 to some $b = b(d)$ and a unit vector $n(x)$ whose inner products with the gradients $\nabla d(\cdot, \tilde{X}_\gamma)$ are all at least $1/b$. If x lies in a component that covers a cusp neighborhood, we let $n(x)$ point away from the point at infinity to which the lifted cusp neighborhood accumulates, just as in Section 6 of [17]. Now Gelander uses that his manifolds are locally symmetric to control the associated constant b. Instead, we use the following:

Claim 3.5 (Moving away from the cusp)**.** *Suppose $\tilde{M}$ is a simply connected Riemannian manifold with curvatures in $[-1,-a^2]$ and let $\xi \in \partial_\infty \tilde{M}$. Let γ be a parabolic isometry of $\tilde{M}$ with $\gamma(\xi) = \xi$, let $x \in \tilde{M}$ be a point with $d(x,\gamma(x)) \geq \epsilon$ and let $c : \mathbb{R} \longrightarrow \tilde{M}$ be a unit speed geodesic with $c(-\infty) = \xi$ and $c(0) = x$. Then we have*

$$\frac{d}{dt} d(c(t), \tilde{X}_\gamma)|_{t=0} \geq \epsilon \cdot a/2.$$

Proof. Since the geodesics $c(t)$ and $\gamma \circ c(t)$ are asymptotic to ξ as $t \to -\infty$ and always lie on the same horospheres, [19, Proposition 4.1] says that for any fixed s,

$$d(c(t), \gamma \circ c(t)) \leq d(c(s), \gamma \circ c(s)) \cdot e^{a(t-s)}, \quad \forall t \leq s.$$

Since the two sides are equal at $t = s$ and we are saying that $t \leq s$, it follows that

$$\frac{d}{dt} d(c(t), \gamma \circ c(t))|_{t=s} \geq \frac{d}{dt} d(c(s), \gamma \circ c(s)) \cdot e^{at}|_{t=s}.$$

Apply this to the (unique) value $s \leq 0$ such that $c(s) \in \partial \tilde{X}_\gamma$. Then

$$\frac{d}{dt} d(c(t), \gamma \circ c(t))|_{t=s} \geq \epsilon \cdot a. \tag{25}$$

On $\partial \tilde{X}_\gamma$, the gradients of d_γ and $d(\cdot, \tilde{X}_\gamma)$ are parallel. As

$$d_\gamma(y) \leq d_\gamma(x) + 2d(x, y) \quad \forall x, y,$$

we have $|\nabla d_\gamma| \leq 2$, while $\nabla d(\cdot, \tilde{X}_\gamma)$ is a unit vector. So, this and (25) imply:

$$\frac{d}{dt} d(c(t), \tilde{X}_\gamma)|_{t=s} = \nabla d(\cdot, \tilde{X}_\gamma) \cdot c'(0) \geq \frac{1}{2} \nabla d_\gamma \cdot c'(0) \geq \frac{1}{2} \epsilon \cdot a.$$

Finally, as curvature is nonpositive and X_γ is a convex set, $d(c(t), \tilde{X}_\gamma)$ is a convex function and hence has increasing derivative. As $s \leq 0$, the claim follows. ☐

So, to finish the proof of Proposition 3.3, we just take b to be at least the constant $b = b(d)$ from the Margulis tube case, and at least $\epsilon \cdot a/2$. With this b and n, the conditions of Lemma 3.4 are satisfied, so the proposition follows. ☐

Finally, we prove the following estimate on the volumes of balls in the shrunk thick parts $)M_{\geq \epsilon}(_\delta$, which is necessary if we want to invoke Theorem 2.5.

Lemma 3.6. *Suppose that M is a complete Riemannian d-manifold with sectional curvatures in $[-1, -a^2]$. Fix $\epsilon < \epsilon(d)$, let $\delta, r < \max\{\epsilon, \epsilon(d) - \epsilon\}/4$ and set $N :=)M_{\geq \epsilon}(_\delta$. Then there is some $c = c(d, \epsilon, a) > 0$ such that*

$$vol(B_N(p, r)) \geq cr^d, \ \forall p \in N.$$

Note that since M is non-positively curved, the volume of any embedded metric ball $B \subset M$ is at least the volume of a ball with the same radius in $\mathbb{R}^d$, see e.g. [16, Theorem 3.101]. So, as long as we choose $\rho < \epsilon$, the lemma is trivial for balls $B_N(p, r)$ that do not intersect ∂N. The point of the lemma, then, is that the boundary of N is convex enough that balls centered near ∂N still have a definite amount of volume that is contained in N.

Proof. As described in the paragraph above, it suffices to consider only points $p \in N$ that are within r of ∂N. The fact that $\delta < (\epsilon(d) - \epsilon)/4$ ensures that the radius r ball $B_M(p, r)$ in M around p will be an embedded ball contained the $\epsilon(d)$-thin part $M_{<\epsilon(d)}$. Choose a universal cover

$$\pi : \tilde{M} \longrightarrow M,$$

components $\tilde{T}_{<\epsilon} \subset \tilde{T}_{<\epsilon(d)} \subset \tilde{M}$ of the preimages of $M_{<\epsilon}, M_{<\epsilon(d)}$, and a point

$$\tilde{p} \in \tilde{T}_{<\epsilon(d)} \setminus (\tilde{T}_{<\epsilon})_\delta, \quad \pi(\tilde{p}) = p.$$

Then we can write $\tilde{T}_{<\epsilon}$ as the union

$$\tilde{T}_{<\epsilon} = \cup_\gamma \tilde{X}_\gamma,$$

where γ ranges over the group of deck transformations stabilizing $\tilde{T}_{<\epsilon}$, and

$$\tilde{X}_\gamma := \{\tilde{x} \in \tilde{M} \mid d(\gamma(\tilde{x}), \tilde{x}) < \epsilon\}.$$

We claim that there is a unit vector $n \in T\tilde{M}_{\tilde{p}}$ and some $b = b(d, a, \epsilon)$ such that

$$n \cdot \nabla d(\cdot, \tilde{X}_\gamma)|_{\tilde{p}} \geq 1/b > 0, \quad \forall \gamma. \tag{26}$$

Now if $\tilde{T}_{<\epsilon}$ covers a Margulis tube, any two γ, γ' as above commute, so we have

$$\nabla d(\cdot, \tilde{X}_\gamma) \cdot \nabla d(\cdot, \tilde{X}'_\gamma) \geq 0$$

by the argument of [17, Lemmas 7.1 and 7.2] (see also [5, Lemma 3.5]), and then one can construct n as in [17, Lemma 7.3] (or [5, Lemma 3.12]). If $\tilde{T}_{<\epsilon}$ covers a cusp neighborhood, we can just let n be the unit vector that points away from the point $\xi \in \partial_\infty \tilde{M}$ to which $\tilde{T}_{<\epsilon}$ accumulates (i.e. let $n = c'(0)$ where c is a unit speed geodesic with $c(-\infty) = \xi$ and $c(0) = \tilde{p}$) and then the claim follows from Claim 3.5 above, after setting $b = \epsilon a/2$.

It follows from (26) that for every $v \in T\tilde{M}_{\tilde{p}}$ with $|v - n| < 1/b$ we have

$$\begin{aligned} v \cdot \nabla d(\cdot, \tilde{X}_\gamma) &= n \cdot \nabla d(\cdot, \tilde{X}_\gamma) + (v - n) \cdot \nabla d(\cdot, \tilde{X}_\gamma) \\ &= 1/b - |v - n| \\ &> 0, \end{aligned}$$

so v points out of the convex subset $(\tilde{X}_\gamma)_{d(\tilde{p}, \tilde{X}_\gamma)} \subset \tilde{M}$ on whose boundary $\tilde{p}$ lies. And since

$$\tilde{p} \notin (\tilde{T}_{<\epsilon})_\delta \implies (\tilde{T}_{<\epsilon})_\delta \subset \cup_\gamma (\tilde{X}_\gamma)_{d(\tilde{p}, \tilde{X}_\gamma)},$$

we then have that for all $v \in T\tilde{M}_{\tilde{p}}$ with $|\, v/|v| - n \,| < 1/b$, the Riemannian exponential

$$\exp_{\tilde{p}}(v) \notin (\tilde{T}_{<\epsilon})_\delta.$$

Now as explained in the beginning of the proof, r is small enough so that $B_M(p, r)$ is an embedded ball in $M_{<\epsilon(d)}$. So, if we let

$$V = \{v \in T\tilde{M}_{\tilde{p}} \;\mid\; |v| < r, \; |\, v/|v| - n \,| < 1/b\},$$

the composition $\pi \circ \exp_{\tilde{p}}$ of the universal covering map and the Riemannian exponential map embeds V as a subset of N, where $N =)M_{\geq \epsilon}(_\delta$. The ratio of the Euclidean volume of V to r^d is certainly bounded below by some constant depending only on $b = b(d, a, \epsilon)$, so nonpositive curvature implies that the same is true of $B_N(p, r)$, [21, Corollary 11.4]. □

3.3. **The proof of Theorem 1.6.** We will assume everywhere below that $d \geq 4$, since the theorem follows trivially from Gauss–Bonnet when $d = 2$ and we have assumed that $d \neq 3$. Fix a sequence of finite volume d-manifolds (M_n) with curvatures in $[-1, -a^2]$ and for each n, let μ_n be the measure on $\mathcal{M}_a^d$ obtained by pushing forward the Riemannian measure on M_n under $p \mapsto (M_n, p)$. By assumption, the sequence $(\mu_n/\mathrm{vol}(M_n))$ converges weakly to some probability measure μ on $\mathcal{M}_a^d$.

Claim 3.7. *For some $\epsilon_{max} > 0$, we have that $\mu(\mathcal{E}_\epsilon) = 0$ for all but countably many $\epsilon \in (0, \epsilon_{max})$, where here $\mathcal{E}_\epsilon$ is the set of all $(M, p) \in \mathcal{M}_a^d$ such that M has a primitive closed geodesic with length exactly 2ϵ.*

Proof. Take ϵ_{max} less than the Margulis constant $\epsilon(d)$. Using Proposition 3.1, we may assume that ϵ_{max} is small enough so that if $\epsilon \in (0, \epsilon_{max})$, then any $\epsilon(d)$-Margulis tube with core length 2ϵ has volume at least 1. For each $\epsilon \in (0, \epsilon_{max})$ and $R > 0$, consider the set $\mathcal{E}_{\epsilon,R}$ of all $(M, p) \in \mathcal{M}_a^d$ such that there is an $\epsilon(d)$-Margulis tube with core length 2ϵ that is completely contained in the radius R ball around p. In any manifold M with sectional curvatures at least -1, the radius R ball around any point has volume at most some constant $V(d, R)$, see [16, Theorem 3.101]. So, it follows that for fixed R, any $(M, p) \in \mathcal{M}_a^d$ can be contained in $\mathcal{E}_{\epsilon,R}$ for at most $V(d, R)$-many choices of ϵ. Hence, we have

$$\sum_\epsilon \mu(\mathcal{E}_{\epsilon,R}) \leq V(d, R),$$

implying that $\mu(\mathcal{E}_{\epsilon,R}) \neq 0$ for at most countably many ϵ. But letting $R \in \mathbb{N}$, there are then only countably many *pairs* (ϵ, R) such that $\mu(\mathcal{E}_{\epsilon,R}) \neq 0$, and hence only countably many ϵ such that $\mu(\mathcal{E}_{\epsilon,R}) \neq 0$ for *some* $R \in \mathbb{N}$. Since $\mathcal{E}_\epsilon = \cup_{R \in \mathbb{N}} \mathcal{E}_{\epsilon,R}$, the claim follows. ☐

Fix now some small $\epsilon, \xi > 0$, to be determined later, such that $\mu(\mathcal{E}_\epsilon) = 0$. Using the notation and terminology of §2.2, consider the extended mm-space

$$\mathfrak{M}_n := (\ )(M_n)_{\geq\epsilon}(_\xi, M_n).$$

and let $\mu_{\mathfrak{M}_n}$ be the associated measure on $\mathbb{M}^{ext}$. Then if

$$\mathcal{T} = \{(M, p) \in \mathcal{M}_a^d \mid d(p, M_{<\epsilon}) > \xi\},$$

the measure $\mu_{\mathfrak{M}_n}$ is just the push forward of the restriction $\mu_n|_\mathcal{T}$ under the map

$$\mathcal{T} \longrightarrow \mathbb{M}^{ext}, \quad (M, p) \longmapsto (\ )M_{\geq\epsilon}(_\xi, p, M).$$

Lemma 3.8. *The measures $\mu_{\mathfrak{M}_n}/vol(M_n)$ weakly converge.*

Note that these are not probability measures.

Proof. Let $f : \mathbb{M}^{ext} \longrightarrow \mathbb{R}$ be a bounded, continuous function and define

$$F : \mathcal{M}_a^d \longrightarrow \mathbb{R}, \quad F(M, p) = \begin{cases} f(\ )M_{\geq\epsilon}(_\xi, p, M) & (M, p) \in \mathcal{T} \\ 0 \text{ otherwise.} \end{cases}$$

We have $\int f\, d\mu_{\mathfrak{M}_n} = \int F\, d\mu_n$, so it suffices to show that the limit

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(M_n)} \int F\, d\mu_n$$

exists. Recall that the measures $\mu_n/\mathrm{vol}(M_n) \to \mu$ weakly. So by the Portmanteau theorem, it suffices to show that F is continuous on a subset of $\mathcal{M}_a^d$ that has full μ-measure.

Claim 3.9. *The map F is continuous on the difference $\mathcal{M}_a^d \setminus (\mathcal{E}_\epsilon \cup \mathcal{D})$, where*

$$\mathcal{D} := \{(M,p) \in \mathcal{M}_a^d \mid d(p, M_{<\epsilon}) = \xi\}$$

and $\mathcal{E}_\epsilon$ is as in Claim 3.7.

Proof of Claim 3.9. Suppose that we have a convergent sequence

$$(N_n, p_n) \to (N,p) \in \mathcal{M}_a^d \setminus (\mathcal{E}_\epsilon \cup \mathcal{D}).$$

Assume first that $d(p, N_{<\epsilon}) < \xi$. By a result of Ehrlich [14], injectivity radius is continuous under smooth convergence, so it follows that $d(p_n, (N_n)_{<\epsilon}) < \xi$ as well for large n. In this case, the continuity of F along our sequence is obvious, since for large n,

$$0 = F(N_n, p_n) \to F(N,p) = 0.$$

So, assume that $d(p, N_{<\epsilon}) > \xi$, i.e. that $(N,p) \in \mathcal{T}$. First, we claim that $(N_n, p_n) \in \mathcal{T}$ for large n. If not, then after passing to a subsequence there would be points $q_n \in N_n$ with $d(p_n, q_n) \le \xi$ and $\mathrm{inj}_{N_n}(q_n) \le \epsilon$. Again by continuity of injectivity radius, we can take a subsequential limit of the q_n to produce some $q \in N$ with $d(p,q) \le \xi$ and $\mathrm{inj}_N(q) \le \epsilon$. If $\mathrm{inj}_N(q)$ is *less than* ϵ, then this contradicts that $d(p, N_{<\epsilon}) > \xi$. So assume $\mathrm{inj}_N(q) = \epsilon$. Since $(N,p) \notin \mathcal{E}_\epsilon$, the point q cannot lie on a closed geodesic of length exactly 2ϵ, so q can be perturbed to a point q' with $\mathrm{inj}_N(q') < \epsilon$. Taking the perturbation small enough so that $d(p,q') < d(p, N_{<\epsilon})$, we have a contradiction.

In order to avoid a debauch of parentheses, set $T_n =)(N_n)_{\ge\epsilon}(_\xi$ and define $T \subset N$ similarly. To prove that F is continuous along $(N_n, p_n) \to (N,p)$, it suffices to show that

$$(T_n, p_n, N_n) \;\to\; (T, p, N) \in \mathbb{M}^{ext}. \tag{27}$$

Fixing some large $R > 0$, choose a sequence of embeddings

$$\phi_n : B_N(p,R) \longrightarrow N_n, \quad \phi_n(p) = p_n,$$

such that the pullback metrics $\phi_n^*(g_n) \to g$ in the smooth topology, as described in §2.1. To prove (27), we would like to apply Lemma 2.3 to say that for a given $\alpha > 0$, the triples in (27) are (α, R)-related for large n. This requires proving that for an arbitrary $\delta > 0$, conditions (1)–(3) in Lemma 2.3 hold for large n.

Condition (1) in Lemma 2.3 is immediate, since the maps ϕ_n are nearly isometries when n is large. The proof of condition (2) in Lemma 2.3 is similar to the first two paragraphs of the current claim. Namely, suppose that the first part of condition (2) fails for infinitely many n. Then for infinitely many n, there are points

$$q_n \in T_n \cap \phi(B_N(p,R)), \quad d(\phi_n^{-1}(q_n), T) > \delta.$$

Passing to a subsequence, we can assume that $\phi_n^{-1}(q_n) \to q \in B_N(p,R)$, and by continuity of injectivity radius we have $q \in T$, a contradiction. The second part of condition (2) is similar, although as we did above one has to use that there are no closed geodesics of length exactly 2ϵ in N. So, it remains to prove condition (3) of Lemma 2.3, i.e. that

$$\mathrm{vol}(\phi_n^{-1}(T_n) \triangle T) < \delta$$

for large n. Pick a neighborhood $U \supset \partial T \cap B_N(p,R)$ with volume less than δ. If n is large, then the same arguments as above show that $\phi_n^{-1}(T_n) \triangle T \subset U$, so we are done. □

By our choice of ϵ, we have $\mu(\mathcal{E}_\epsilon) = 0$. So, to prove Lemma 3.8 it suffices to show that $\mu(\mathcal{D}) = 0$. Essentially, the point is that $d(p, M_{<\epsilon}) = \xi$ is a measure zero condition within each fixed M, and as a weak limit of measures constructed using Riemannian measures on finite volume manifolds, μ is distributed on each 'leaf'

$$\mathcal{L}_M = \{(M,p) \mid p \in M\} \subset \mathcal{M}_a^d$$

according to the Riemannian measure of M. (This is not quite precise, the leaves may be highly singular, but one can make this argument work in the foliated 'desingularization' of $\mathcal{M}$ constructed in [2, Theorem 1.6]). However, an easier approach is to use that μ satisfies the *mass transport principle*, see [2, (1)]. Namely, define a Borel function

$$\varphi : (\mathcal{M}_a^d)_2 \longrightarrow \{0,1\}, \quad \varphi(M,p,q) = \begin{cases} 1 & d(p, M_{<\epsilon}) = \xi \text{ and } d(p,q) \le \epsilon \\ 0 & \text{otherwise} \end{cases},$$

where $(\mathcal{M}_a^d)_2$ is the space of *doubly pointed* d-manifolds with curvature in $[-1, -a^2]$, endowed with the natural version of smooth convergence, see [2]. Note that

$$d(p, M_{<\epsilon}) = \xi \implies \int_{q \in M} \varphi(M,p,q)\, d\mathrm{vol} \ge \mathrm{vol}\, B_{\mathbb{R}^d}(0,\epsilon),$$

since embedded ϵ-balls in a d-manifold of nonpositive curvature have volume at least that of an ϵ-ball in $\mathbb{R}^d$, c.f. [16, Theorem 3.101]. By the mass transport principle [2, (1)],

$$\begin{aligned} \mu(\mathcal{D}) &\le 1/B_{\mathbb{R}^d}(0,\epsilon) \cdot \int_{(M,p) \in \mathcal{M}_a^d} \int_{q \in M} \varphi(M,p,q)\, d\mathrm{vol}_M\, d\mu \\ &= 1/B_{\mathbb{R}^d}(0,\epsilon) \cdot \int_{(M,p) \in \mathcal{M}_a^d} \int_{q \in M} \varphi(M,q,p)\, d\mathrm{vol}_M\, d\mu \\ &= 0, \end{aligned}$$

where the last equality is because for small ϵ, the set of points exactly at distance ξ from the ϵ-thin part has measure zero in any manifold with negative curvature. □

We now know that the sequence of measures $\mu_{\mathfrak{M}_n}/\mathrm{vol}(M_n)$ weakly converges, and we would like to apply Theorem 2.5, or really Corollary 2.7. Lemma 3.6 will give the lower bound on ball volumes needed in Theorem 2.5 (1), and the upper bound needed in (2) comes from the uniform lower sectional curvature bound, see e.g. [16, Theorem 3.101 on p. 169]. The key, though, is to use our work in §2 to define the appropriate r_0, r_1, r_2, r_3.

Namely, take $\epsilon, \delta > 0$ small enough so that Proposition 3.3 applies, set $\xi = \epsilon/2 + \delta$, and let b be as given in Proposition 3.3 for $c = 3, c' = 4$, say. If

$$r_0 = \delta,\ r_1 = 3\delta,\ r_2 = (b+6)\delta,\ r_3 = (b+7)\delta,$$

then Proposition 3.3 says that the nerve $N_{M_n}(S_n, \rho_n)$ in M_n associated to any $[r_2, r_3]$-weighted (r_0, r_1)-net (S_n, ρ_n) in $)(M_n)_{\geq\epsilon}(_\xi$ is homotopy equivalent to $(M_n)_{\geq\epsilon}$. So, applying Corollary 2.7 to the sequence of extended mm-spaces $\mathfrak{M}_n$, with $B_n = b_k((M_n)_{\geq\epsilon})$, $V_n = \mathrm{vol}(M_n)$ and r_0, r_1, r_2, r_3 as above, we get that the limit

$$\lim_{n\to\infty} \frac{b_k((M_n)_{\geq\epsilon})}{\mathrm{vol}(M_n)} = L \in [0, \infty). \tag{28}$$

But Proposition 3.1 says that the number of components of the ϵ-thin part of M_n is at most $\mathrm{vol}(M_n)/C$, where $C = C(\epsilon, d, a) \to \infty$ as $\epsilon \to 0$. Removing a cusp neighborhood from M_n does not change the homotopy type, and by Mayer–Vietoris removing a Margulis tube can only change Betti numbers by 1. So, we get that for each n and k,

$$|b_k((M_n)_{\geq\epsilon}) - b_k(M_n)| \leq \mathrm{vol}(M_n)/C.$$

Combining this with (28), we get that

$$L - 1/C \leq \liminf_{n\to\infty} \frac{b_k(M_n)}{\mathrm{vol} M_n} \leq \limsup_{n\to\infty} \frac{b_k(M_n)}{\mathrm{vol} M_n} \leq L + 1/C,$$

so sending $\epsilon \to 0$, and hence $C \to \infty$, proves the theorem.

4. Manifolds of nonpositive curvature and Theorem 1.7

In this section we prove Theorem 1.7, i.e. the convergence of normalized Betti numbers for BS-convergent sequences of analytic d-manifolds of nonpositive curvature without Euclidean factors, when the limit is thick. For that purpose it will be more convenient to work not with the standard thick thin decomposition but a close variant of it, introduced in [7], which we call a 'stable' thick thin decomposition:

4.1. **A stable thick thin decomposition.** Suppose that M is a finite volume, real analytic d-manifold with sectional curvatures in the interval $[-1, 0]$ and that the universal cover X of M has no Euclidean deRham factors. Write $M = \Gamma\backslash X$. Then Γ operates freely and the displacement functions d_γ $(\gamma \in \Gamma)$ are analytic. In particular the convex sets

$$\mathrm{Min}(\gamma) = \{x \in X \mid d_\gamma(x) = \min(d_\gamma)\}$$

are complete submanifolds. An element $\gamma \in \Gamma$ is called *J-stable* if we have

$$\mathrm{Min}(\gamma^i) = \mathrm{Min}(\gamma), \quad \forall i = 1, \dots, J.$$

Let ϵ be less than the Margulis constant, and fix also the constants δ, I_δ, J defined at the beginning of [7, §13.4], *but using ϵ instead of the actual Margulis constant.* The interested reader can refer to [7] if necessary, but it is not necessary to know what these constants are to read our proof below. As in [7], let

$$\Delta_0 := \{\gamma \in \Gamma \setminus \{1\} \mid \gamma \text{ is } J\text{-stable and } \inf_{x\in X} d_\gamma(x) \le \delta\}, \text{ and}$$

$$\Delta := \{\gamma^1, \dots, \gamma^{I_\delta} \mid \gamma \in \Delta_0\}.$$

Define the *stable ϵ-thick part* M_+ and the *stable ϵ-thin part* M_- by

$$M_+ := \{x \in M \mid d_\gamma(x) \ge \epsilon \ \ \forall \gamma \in \Delta\}, \quad M_- = \overline{M \setminus M_+}.$$

Then M_+, M_- are codimension zero submanifolds of M with $\partial M_+ = \partial M_-$. By [7, Corollary 12.5], there is some integer $m = m(J)$ such that for any $\gamma \in \Gamma \setminus \{1\}$, there is some $j \le m$ such that γ^j is J-stable. So, if $\epsilon' = \epsilon/m$ we have

$$M_{<\epsilon'} \subset M_- \subset M_{<\epsilon}, \quad M_{\ge\epsilon} \subset M_+ \subset M_{\ge'\epsilon}. \tag{29}$$

The following proposition is a modification of [7, Theorem 13.1].

Proposition 4.1. *Suppose that M is a finite volume, real analytic d-manifold with sectional curvatures in the interval $[-1,0]$ and that the universal cover X of M has no Euclidean de Rham factors. Then there is some $C = C(d)$ such that for all $k \in \mathbb{N}$, both $b_k(M_-)$ and $b_k(\partial M_-)$ are less than or equal to $C\mathrm{vol}(M_{<\epsilon})$.*

It is necessary to assume here that M is analytic and that X has no Euclidean de Rham factors. If $M = N \times S^1$ for some $(d-1)$-manifold N, we can scale the S^1-factor so that $M = M_{\le\epsilon} = M_-$ and $\mathrm{vol}(M) \approx 0$. And unless we assume analyticity (or some weaker alternative, see [7, §A2]) there are finite volume manifolds with sectional curvatures in $[-1,0]$ where the thin parts have infinite Betti numbers, see [7, §11.1].

Proof. First, it suffices to show that $b_k(M_-) \le C\mathrm{vol}(M_-)$. For by Poincaré duality,

$$b_k(M_-) = \dim H_k(M_-;\mathbb{R}) = \dim H^{d-k}(M_-, \partial M_-;\mathbb{R}) = \dim H_{d-k}(M_-, \partial M_-;\mathbb{R}),$$

so all the relative homology groups $H_*(M_-, \partial M_-;\mathbb{R})$ have rank at most $C\mathrm{vol}(M_-)$. But then from the long exact sequence for the pair $(M_-, \partial M_-)$ we quickly see that the Betti numbers of ∂M_- are bounded above by a constant, namely $2C$, times $\mathrm{vol}(M_-)$.

The rest of the argument follows closely the proof of [7, Theorem 13.1]. Let

$$g : (0,\infty) \longrightarrow [0,\infty)$$

be a C^∞ function with

- $g(t) > 0, g'(t) < 0$ for $t \in (0,\epsilon)$,
- $g(t) = 0$ for $t \ge \epsilon$,
- $g(t) \to \infty$ as $t \to 0$,
- $g(\delta) = 1$.

and with Δ as above, consider the smooth function

$$F : X \longrightarrow [0,\infty), \quad F(x) = \sum_{\gamma\in\Delta} g \circ d_\gamma(x).$$

Since Δ is conjugation invariant in Γ, this F descends to a smooth function $f : M \longrightarrow \mathbb{R}$. On [7, pg 145], it is shown that f and F have finitely many critical values

$$0 = r_0 < r_1 < \ldots < r_s.$$

Note that the 0-critical set $f^{-1}(0)$ is exactly M_+. Moreover, they show that the Betti numbers of M can be estimated via the following summation:

$$b_*(M) = \text{rank } H_*(M) \leq \sum_{i,j} \text{rank } H_*(\{f_{x_{ij}} < r_i + \rho\}). \tag{30}$$

Here, for each i the indices j correspond to different pieces of the critical set $f^{-1}(r_i)$; each such piece is contained in a complete immersed submanifold $V_{x_{ij}} \looparrowright M$, the restriction of f to $V_{x_{ij}}$ is called $f_{x_{ij}}$, and $\rho > 0$ is very small, so $\{f_{x_{ij}} < r_i + \rho\}$ is just a small neighborhood of that piece in $V_{x_{ij}}$. The proof of (30) is essentially via Morse theory, applied to the function F: one considers the homology of the sublevel set $\{f < r\}$, starting with $r < 0$ when the set is empty, and one shows that passing through the critical point $r = r_i$ contributes at most the corresponding index-i terms of the summation to the rank of the homology. In particular, we can run the exact same argument on $int(M_-) = M \setminus f^{-1}(0)$, just by starting with some small positive r instead, and the only difference is that it is not necessary to include the terms with index $i = 0$ in the summation.

Now on [7, pg 145], the authors show not only that there are finitely many critical values, but that the number of terms in the summation (30) is bounded above by some constant depending only on d. They then show in [7, pg 148, (16)] that the rank of each $H_*(\{f_{x_{ij}} < r_i + \rho\})$ is bounded above by a dimensional constant times the 'essential volume' of the immersed submanifold $V_{x_{ij}}$ mentioned above[6]. Here, essential volume is an integer that estimates volume up to some fixed multiplicative constant, but in a way that disregards small volume Euclidean factors, see [7, §12.8].

In [7, Theorem 12.11], they then show[7] that the sum N of all these essential volumes is at most a dimensional constant times $\text{vol}(M)$. However, the proof of [7, Theorem 12.11] is stronger: the authors construct a collection of N injectively embedded r-balls centered in points of $M_{<\epsilon/2}$[8] that overlap with uniformly bounded multiplicity (see [7, (3) p. 133]), where here $r > 0$ depends only on d. Hence, this shows N is at most a dimensional constant times $\text{vol}(M_{<\epsilon})$, so tracing through the three paragraphs above, we have

$$b_*(M_-) \leq C(d)\text{vol}(M_{<\epsilon}). \quad \square$$

4.2. **Proof of Theorem 1.7.** Let $\epsilon_0 > 0$ and let (M_n) be a sequence of real analytic, finite volume Riemannian d-manifolds with sectional curvatures in the interval $[-1, 0]$, and assume the universal covers of the M_n do not have Euclidean de Rham-factors. Assume

[6] In [7], they set $V_x = Y_x/\Gamma_x$ at some point, but mostly use the latter notation.

[7] For M, essential volume agrees with volume up to a dimensional constant, since M has no Euclidean factors.

[8] In their construction of the centers z of these balls there exists an element $\beta_z \in \Gamma$ such that $d_{\beta_z}(z) = \epsilon/2$, see top of page 132.

(M_n) BS-converges to a measure μ on $\mathcal{M}$ that is supported on ϵ_0-thick manifolds. Here, recall that BS-convergence means that if μ_n are the associated measures on $\mathcal{M}^d$, then

$$\mu_n/\text{vol}(M_n) \to \mu$$

weakly. We want to show that the following limit exists for all k:

$$\lim_{n\to\infty} b_k(M_n)/\text{vol}(M_n).$$

First, here is the reason we assume that μ is supported on ϵ_0-thick manifolds.

Claim 4.2. *For all $R > 0$ and $0 < \epsilon < \epsilon_0$, we have*

$$vol(\{x \in M_n \mid d(x, M_{\leq\epsilon}) \leq R\})/vol(M_n) \to 0.$$

Proof. By the continuity of injectivity radius with respect to smooth convergence [14],

$$D := \{(M,p) \in \mathcal{M}^d \mid d(p, M_{\leq\epsilon}) \leq R\} \subset \mathcal{M}^d$$

is closed, so by the Portmanteau theorem,

$$\limsup_n \frac{\text{vol}(\{x \in M_n \mid d(x, M_{\leq\epsilon}) \leq R\})}{\text{vol}(M_n)} \leq \limsup_n \mu_n(D) \leq \mu(D) = 0,$$

so the limit of the sequence on the left is zero. □

Pick some $\epsilon > 0$ that is less than ϵ_0 and also less than the Margulis constant, and fix some $\xi < \epsilon/20$. With the notation $(\ )_+$ of the last section, with input ϵ, let

$$N_n := \overline{((M_n)_+)_\xi} = \{x \in M_n \mid d(x, (M_n)_+) \leq \xi\},$$

and using the notation and terminology of §2.2, consider the extended mm-spaces

$$\mathfrak{M}_n := (M_n, M_n),\ \mathfrak{N}_n := (N_n, M_n)$$

and their associated measures $\mu_{\mathfrak{M}_n}, \mu_{\mathfrak{N}_n}$ on $\mathbb{M}^{ext}$. (Note that the space N_n may be disconnected, but since it has finitely many components, it is special and hence our work in §2.3 still applies to $\mathfrak{N}_n$.) Here, if

$$\iota : \mathcal{M}^d \longrightarrow \mathbb{M}^{ext},\ (M,p) \longmapsto (M,p,M),$$

is the natural continuous map (see Corollary 2.2), then $\mu_{\mathfrak{M}_n} = \iota_*(\mu_n)$, so

$$\mu_{\mathfrak{M}_n}/\text{vol}(M_n) = \iota_*\big(\mu_n/\text{vol}(M_n)\big) \to \iota_*(\mu).$$

Claim 4.3. *We have $\mu_{\mathfrak{N}_n}/vol(M_n) \to \iota_*(\mu)$ as well.*

Here, note that by Claim 4.2, we have that $\text{vol}(N_n)/\text{vol}(M_n) \to 1$, so one could replace the normalizing factor by $\text{vol}(N_n)$ if desired.

Proof. Let $f : \mathbb{M}^{ext} \longrightarrow [0, M]$ be a continuous function, and fix $\alpha > 0$. Given δ, R, let

$$C_{\delta,R} = \{\mathfrak{M} \in \mathbb{M}^{ext} \mid \mathfrak{M} \text{ is } (\delta, R)\text{-related to } \mathfrak{N} \in \mathbb{M}^{ext} \implies |f(\mathfrak{M}) - f(\mathfrak{N})| < \alpha\}.$$

Since the sets $C_{\delta,R}$ are open, are nested when δ is decreased and R is increased, and union to all of $\mathbb{M}^{ext}$, we can choose δ, R such that

$$\iota_*(\mu)(C_{\delta,R}) > 1 - \alpha.$$

By the Portmanteau theorem, $\liminf_n \mu_{\mathfrak{M}_n}(C_{\delta,R}) > 1 - \alpha$, so there is some N such that

$$\mu_{\mathfrak{M}_n}(C_{\delta,R}) > 1 - \alpha, \quad \forall n \geq N.$$

Furthermore, in light of Claim 4.2 and (29), we can also assume that

$$\frac{\operatorname{vol}(\{x \in M_n \mid d(x, (M_n)_+) \leq R\})}{\operatorname{vol}(M_n)} < \alpha, \quad \forall n \geq N.$$

Combining the above two estimates, we see that the $\operatorname{vol}/\operatorname{vol}(M_n)$-measure of the set of points $p \in M_n$ such that *both* $d(x, (M_n)_+) > R$ and $(M, p, M) \in C_{\delta,R}$ is at least $(1 - 2\alpha)$. Now at any such point p, we have $p \in N_n$ as well, and the pointed extended mm-spaces (N_n, p, M_n) and (M_n, p, M_n) are obviously (δ, R)-related. Hence, at any such p, we have

$$|f(N_n, p, M_n) - f(M_n, p, M_n)| < \alpha. \tag{31}$$

Breaking the domains of the following integrals in two, and using the upper bound $M \geq f$ on the piece where (31) is not helpful, we see that

$$\left| \int f \, d\mu_{\mathfrak{N}_n} - \int f \, d\mu_{\mathfrak{M}_n} \right| \leq (1 - 2\alpha) \cdot \alpha + \alpha \cdot 2M, \quad \forall n \geq N.$$

So, since $\alpha > 0$ was arbitrary and $\int f \, d\mu_{\mathfrak{M}_n} \to \int f \, d\iota_*(\mu)$, we have that $\int f \, d\mu_{\mathfrak{N}_n} \to \int f \, d\iota_*(\mu)$ as well, and the claim follows. □

We now want to apply Corollary 2.7 to the sequence $\mu_{\mathfrak{N}_n}/\operatorname{vol}(M_n)$, in order to say something about normalized Betti numbers. We'll apply it with $r_0 = 4\xi, r_1 = 5\xi, r_2 = 10\xi$ and $r_3 = 11\xi$, with $B_n = b_k(M_n)$ and $V_n = \operatorname{vol}(M_n)$. So, let's verify its hypotheses.

For condition (1) of Corollary 2.7, just note that any point $p \in N_n$ is within ξ of a point q in $(M_n)_+$, so $B_{N_n}(p, 2\xi)$ contains an embedded ξ-ball around q, which by nonpositive curvature has volume at least that of a ξ-ball in $\mathbb{R}^d$, see e.g. [16, Theorem 3.101]. Similarly, for condition (2) the lower curvature bound implies that any r-ball in N_n has volume at most that of an r-ball in $\mathbb{H}^d$, again see [16, Theorem 3.101].

For condition (3), we need to prove the following.

Lemma 4.4. *Whenever (S_n, ρ_n) is a sequence of $[10\xi, 11\xi]$-weighted $(4\xi, 10\xi)$-nets in N_n, then*

$$\frac{\left|b_k(N_{M_n}(S_n, \rho_n)) - b_k(M_n)\right|}{vol(M_n)} \to 0.$$

Assuming the lemma, the hypotheses of Corollary 2.7 are satisfied, so

$$B_n/V_n = b_k(M_n)/\mathrm{vol}(M_n)$$

converges, proving Theorem 1.7. So, it remains to prove the lemma.

Proof of Lemma 4.4. Since $\mathrm{inj} : M_n \longrightarrow \mathbb{R}$ is 2-lipschitz, we have

$$\forall x \in S_n \subset N_n := \overline{((M_n)_+)_\xi}, \quad \mathrm{inj}(x) \geq \epsilon - 2\xi > \frac{9\epsilon}{10} > \frac{11\epsilon}{20} > 11\xi \geq \rho_n(x).$$

Nonpositive curvature then implies that the balls $B_{\rho(x)}(x)$ are convex, so the Nerve Lemma (c.f. [18, Corollary 4G.3]) says that $N_{S_n} := N_{M_n}(S_n, \rho_n)$ is homotopy equivalent to

$$U_n := \cup_{x \in S_n} B_{\rho_n(x)}(x).$$

So to prove the claim, it suffices to show the following:

(a) If $D_{k,n}$ is the dimension of the image of the map $H_k(U_n, \mathbb{R}) \longrightarrow H_k(M_n, \mathbb{R})$ induced by inclusion, then $b_k(M_n) = D_{n,k} + o(\mathrm{vol}(M_n))$.

(b) If $K_{k,n}$ is the dimension of the kernel of the map $H_k(U_n, \mathbb{R}) \longrightarrow H_k(M_n, \mathbb{R})$ induced by inclusion, then $K_{k,n} = o(\mathrm{vol}(M_n))$.

For (a), apply Mayer–Vietoris to $M_n = \overline{(M_n)_-} \cup (M_n)_+$, giving the long exact sequence

$$\cdots \longrightarrow H_k(\overline{(M_n)_-}; \mathbb{R}) \oplus H_k((M_n)_+; \mathbb{R}) \longrightarrow H_k(M_n; \mathbb{R}) \longrightarrow H_{k-1}(\partial (M_n)_-, \mathbb{R}) \longrightarrow \cdots.$$

By Proposition 4.1 and Claim 4.2, $b_k(\overline{(M_n)_-})$ and $b_k(\partial (M_n)_-)$ are $o(\mathrm{vol}(M_n))$, so

$$b_k(M_n) = \dim \mathrm{Im}\Big(H_k((M_n)_+; \mathbb{R}) \longrightarrow H_k(M_n; \mathbb{R}) \Big) + o(\mathrm{vol}(M_n)).$$

But the inclusion map $(M_n)_+ \longrightarrow M_n$ factors through $U \longrightarrow M_n$, so we have

$$b_k(M_n) \geq D_{n,k} \geq b_k(M_n) - o(\mathrm{vol}(M_n))$$

as well, proving (a).

For (b), let $T_n \subset M_n \setminus N_n$ be a maximal collection of points such that

$$d(s,t) \geq \frac{1}{3} \min\{\mathrm{inj}(s), \mathrm{inj}(t)\}, \quad \forall s, t \in T_n.$$

Since inj is continuous, T_n is locally finite. Moreover, suppose $x \in M_n \setminus N_n$ and $x \notin T_n$. By maximality, there must be some $t \in T_n$ with

$$d(x,t) \leq \frac{1}{3} \min\{\mathrm{inj}(x), \mathrm{inj}(t)\} \leq \frac{1}{3}\mathrm{inj}(t),$$

so the open balls of radius $\rho_n(t) := \frac{1}{2}\mathrm{inj}(t)$ around all $t \in T_n$ cover $M_n \setminus N_n$. Let $N_{S_n \cup T_n}$ be the nerve complex associated to the cover of M_n by the collection of all such balls $B_{\rho_n(t)}(t)$, $t \in T_n$, *together with the balls* $B_{\rho_n(x)}(x)$, $x \in S_n$. As all these balls are convex, $N_{S_n \cup T_n}$ is homotopy equivalent to M_n. In fact, more is true:

Claim 4.5. *There is a diagram of maps*

$$\begin{array}{ccc} U_n & \xrightarrow{\quad \Phi \quad} & N_{S_n} \\ \big\downarrow & & \big\downarrow \\ M_n & \xleftarrow[\quad F \quad]{} & N_{S_n \cup T_n} \end{array}$$

that is commutative up to homotopy, where the vertical maps are the natural inclusions and the horizontal maps Φ, F are homotopy equivalences.

The claim does not assert that the pairs (M_n, U_n) and $(N_{S_n \cup T_n}, N_{S_n})$ are homotopy equivalent, although it is certainly a result along those lines. We should note that there is at least one 'Relative Nerve Lemma' for pairs in the literature, see e.g. [5, Lemma 2.9], but this does not apply in our situation since $U_n \hookrightarrow M_n$ is not a cofibration. One can get around this, but the fix is not particularly pretty, and it is much more direct just to prove the claim above without referencing any citations.

Before proving the claim, let us quickly indicate how to finish the proof of (b). Any point $x \in S_n$ such that $B_{\rho_n(x)}(x)$ intersects a ball $B_{\rho_n(y)}(y)$, where $y \in T_n$, must lie close to the ϵ-thin part of M_n. By Claim 4.2 the volume of any fixed R-neighborhood of $(M_n)_-$ is $o(\mathrm{vol}(M_n))$, so this means that there are only $o(\mathrm{vol}(M_n))$-many vertices of N_{S_n} that are adjacent to vertices of $N_{S_n \cup T_n} \setminus N_{S_n}$. So, Mayer–Vietoris implies that the kernel of the map

$$H_k(N_{S_n}; \mathbb{R}) \longrightarrow H_k(N_{S_n \cup T_n}; \mathbb{R})$$

induced by inclusion has rank $o(\mathrm{vol} M_n)$. Therefore, Claim 4.5 implies that the same is true for the kernel of the map on homology induced by $U_n \hookrightarrow M_n$.

Proof of Claim 4.5. Let's review the proof of the Nerve Lemma. For a much more general proof that essentially specializes to the one below, see Hatcher [18, 4G].

We start with a Riemannian manifold X and an open cover $\mathcal{O}$ by small convex balls. If N is the nerve complex of the cover $\mathcal{O}$, we can define homotopy inverses

$$\alpha : X \longrightarrow N, \quad \beta : N \longrightarrow X$$

as follows. Pick a partition of unity $\{\phi_O \mid O \in \mathcal{O}\}$ subordinate to $\mathcal{O}$, and define

$$\alpha : X \longrightarrow N, \quad \alpha(p) = \sum_{O \in \mathcal{O},\ p \in O} \phi_O(p) \cdot O \ \in N.$$

Here, the values $\phi_O(p)$ are the barycentric coordinates of $\alpha(p)$, within the simplex of N spanned by those O containing p. The map β is defined inductively on the i-skeleta BN^i of the first barycentric subdivision BN of N. Before starting the construction, note that every vertex v of BN is the barycenter of a simplex of N, which corresponds to some finite

$$\mathcal{F}_v \subset \mathcal{O}, \ \ \cap_{O \in \mathcal{F}_v} O \neq \emptyset,$$

and if Δ is a simplex of BN, there is one vertex $v(\Delta)$ of Δ such that $\mathcal{F}_{v(\Delta)}$ is contained in $\mathcal{F}_w$ for every other vertex w of Δ. (This $v(\Delta)$ is just the vertex that is the barycenter of the simplex of N with minimal dimension.) For $i = 0, 1, 2, \ldots$, we now construct the map β on BN^i in such a way that for any i-simplex Δ,

$$\beta(\Delta) \subset \cap_{O \in \mathcal{F}_{v(\Delta)}} O. \tag{32}$$

If v is a vertex of BN, just pick $\beta(v) \in \cap_{O \in F_v} O$ arbitrarily. In general, assuming β has been defined on $\partial\Delta$, it follows from the definition of $v(\Delta)$ and (32) that

$$\beta(\partial\Delta) \subset \cap_{O \in \mathcal{F}_{v(\Delta)}} O.$$

This intersection is contractible, so there is some extension of β to Δ satisfying (32). The homotopy $\alpha \circ \beta \simeq 1$ is constructed inductively on the skeleta of BN, using the homotopy extension principle at each step. To see that $\beta \circ \alpha \simeq 1$, one just notes that if $p \in X$, then $\alpha(p)$ is in some simplex Δ of BN that has as a vertex some $O \ni p$, so by (32), $\beta \circ \alpha(p) \in O$. In other words, p and $\beta \circ \alpha(p)$ are both contained in one of the small convex balls in our cover, so we can just take a straight line homotopy from $\beta \circ \alpha$ to 1.

With the above presentation of the proof of the Nerve Lemma (which we could not find a reference for) the claim becomes trivial. Namely, let $\Phi : U_n \longrightarrow N_{S_n}$ be the map called α above, where the manifold is U_n and the cover is by the $\rho_n(x)$-balls around $x \in S_n$. Let $F : N_{S_n \cup T_n} \longrightarrow M_n$ be the map called β above, where the manifold is M_n and the cover is by the $\rho_n(x)$-balls around $x \in S_n \cup T_n$. These are both homotopy equivalences, and just as above the straight-line homotopy connects $F \circ \Phi$ to the inclusion $U_n \longrightarrow M_n$. □

Now that we have proved the claim, the lemma follows. □

And so does the theorem.

References

1. Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet, *On the growth of L^2-invariants for sequences of lattices in Lie groups*, Ann. of Math. (2) **185** (2017), no. 3, 711–790. MR 3664810
2. Miklos Abert and Ian Biringer, *Unimodular measures on the space of all Riemannian manifolds*, https://arxiv.org/abs/1606.03360.
3. David Aldous and Russell Lyons, *Processes on unimodular random networks*, Electron. J. Probab. **12** (2007), no. 54, 1454–1508. MR 2354165 (2008m:60012)
4. Jesús A. Álvarez López, Ramón Barral Lijó, and Alberto Candel, *A universal Riemannian foliated space*, Topology Appl. **198** (2016), 47–85. MR 3433188
5. Uri Bader, Tsachik Gelander, and Roman Sauer, *Homology and homotopy complexity in negative curvature*, arXiv preprint arXiv:1612.04871 (2016).
6. Werner Ballmann, *Lectures on Kähler manifolds*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2006. MR 2243012
7. Werner Ballmann, Mikhael Gromov, and Viktor Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics, vol. 61, Birkhäuser Boston Inc., Boston, MA, 1985. MR 823981 (87h:53050)
8. Riccardo Benedetti and Carlo Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992. MR 1219310 (94e:57015)
9. Itai Benjamini and Oded Schramm, *Recurrence of distributional limits of finite planar graphs*, Selected Works of Oded Schramm, Springer, 2011, pp. 533–545.

10. Ian Biringer and Jean Raimbault, *Ends of unimodular random manifolds*, Proceedings of the American Mathematical Society **145** (2017), no. 9, 4021–4029.
11. Lewis Bowen, *Cheeger constants and L^2-Betti numbers*, Duke Math. J. **164** (2015), no. 3, 569–615. MR 3314481
12. Jeff Cheeger and David G. Ebin, *Comparison theorems in Riemannian geometry*, AMS Chelsea Publishing, Providence, RI, 2008, Revised reprint of the 1975 original. MR 2394158
13. David L. de George and Nolan R. Wallach, *Limit formulas for multiplicities in $L^2(\Gamma\backslash G)$*, Ann. of Math. (2) **107** (1978), no. 1, 133–150. MR 0492077 (58 #11231)
14. Paul E Ehrlich, *Continuity properties of the injectivity radius function*, Compositio Math **29** (1974), 151–178.
15. Gábor Elek, *Betti numbers are testable*, Fete of Combinatorics and Computer Science, Springer, 2010, pp. 139–149.
16. Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, vol. 3, Springer, 1990.
17. Tsachik Gelander, *Homotopy type and volume of locally symmetric manifolds*, Duke Math. J. **124** (2004), no. 3, 459–515. MR 2084613 (2005i:53050)
18. Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR MR1867354 (2002k:55001)
19. Ernst Heintze, Hans-Christoph Im Hof, et al., *Geometry of horospheres*, Journal of Differential geometry **12** (1977), no. 4, 481–491.
20. Michael Kapovich, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics, vol. 183, Birkhäuser Boston Inc., Boston, MA, 2001. MR 1792613 (2002m:57018)
21. John M Lee, *Riemannian manifolds: an introduction to curvature*, vol. 176, Springer Science & Business Media, 2006.
22. W. Lück, *Approximating L^2-invariants and homology growth*, Geom. Funct. Anal. **23** (2013), no. 2, 622–663. MR 3053758
23. AI Mal'cev, *On homomorphisms onto finite groups*, Trans. Amer. Math. Soc **119** (1983), 67–79.
24. Hossein Namazi, Pekka Pankka, and Juan Souto, *Distributional limits of Riemannian manifolds and graphs with sublinear genus growth*, Geom. Funct. Anal. **24** (2014), no. 1, 322–359. MR 3177385
25. M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. MR 0507234 (58 #22394a)
26. Hsien Chung Wang, *Topics on totally discontinuous groups*, Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970), Dekker, New York, 1972, pp. 459–487. Pure and Appl. Math., Vol. 8. MR 0414787 (54 #2879)

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