1 Introduction

Let $\phi : S \to S$ be a homeomorphism of a closed, orientable surface $S$ of genus $g$. The mapping torus of $\phi$ is the 3-manifold $M_\phi$ defined by

$$M_\phi = S \times [0,1] / \sim, \ (x,0) \sim (\phi(x),1).$$

Our interest is in the minimal number of elements needed to generate the fundamental group of $M_\phi$; this is called the rank of $\pi_1 M_\phi$. As we have

$$1 \longrightarrow \pi_1 S \longrightarrow \pi_1 M_\phi \longrightarrow \mathbb{Z} \longrightarrow 1,$$

a generating set for $\pi_1 M_\phi$ can be constructed by adding to a generating set for $\pi_1 S$ any element of $\pi_1 M_\phi$ the projects to $1 \in \mathbb{Z}$. Therefore,

$$\text{rank}(\pi_1 M_\phi) \leq \text{rank}(\pi_1 S) + 1 = 2g + 1.$$

The mapping torus of the identity map is $S \times S^1$, in which case the above is an equality. However, there are also examples where the inequality is strict. For instance, if $g = 1$ then $\text{rank}(\pi_1 M_\phi) = 2 < 3$ when $\phi$ is the map

$$\phi : T^2 \to T^2, \quad \phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For although $\pi_1 M_\phi = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ is generated by $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, the second is the conjugate of the first by the third. Similar examples can also be constructed by hand in higher genus.

For a more dramatic example, suppose $M_\phi$ is hyperbolic and $b_1(M_\phi) \geq 2$. By work of Thurston [21], there are surfaces $S'$ with arbitrarily large genus and homeomorphisms $\phi' : S' \to S'$ such that $M_\phi \cong M_{\phi'}$. So, if the genus $g'$ of $S'$ is large, $\text{rank}(\pi_1 M_{\phi'}) = \text{rank}(\pi_1 M_{\phi})$ will be much less than $2g' + 1$.

**Theorem 1.1.** When $g \geq 2$, there is some $L = L(g)$ such that if $\phi : S \to S$ has translation length at least $L$ in the curve complex $C(S)$, then

$$\text{rank}(\pi_1 M_\phi) = 2g + 1.$$
The curve complex $C(S)$ is the graph whose vertices are isotopy classes of essential simple closed curves on $S$ and whose edges connect vertices with minimally intersecting representatives. When $g \geq 2$, ‘minimally intersecting’ means ‘disjoint’, while on the torus it means ‘intersecting once’. $C(S)$ has a path metric in which all edges have length 1, and the translation length of $\phi$ is

$$\tau(\phi) = \inf_{v \in C(S)} d(v, \phi(v)).$$

Theorem 1.1 is a strengthening of previous results of the authors [3], [19]. The proof outline is similar to the argument given in [3], except that more machinery is required to deal with the thin parts of the mapping tori, which were excluded in the previous work. We would like to stress, however, that this paper is the first of [3, 4, 19] to harness the power of carrier graphs when the injectivity radius is not bounded away from zero.

The new result also has application to random fibered hyperbolic 3-manifolds. Specifically, Maher [12] has shown that certain random walks in the mapping class group of a surface $S$ make linear progress in the complex of curves. Combining his full statement with our theorem gives:

Corollary 1.2. Suppose that $S$ is a closed, orientable surface with genus $g \geq 2$ and let $\mu$ be a probability distribution on $\text{Mod}(S)$ with finite first moment, whose support is bounded in the relative metric and generates a non-elementary subgroup. If $\omega_n$ is a $\mu$-random walk on $\text{Mod}(S)$, then the probability that $\text{rank}(\pi_1 M_{\omega_n}) = 2g + 1$ converges to 1 as $n \to \infty$.

The ‘relative metric’ on $\text{Mod}(S)$ is a metric under which the mapping class group is quasi-isometric to the complex curves, and the condition of bounded support is equivalent to requiring that all elements of the support translate a given base point in the curve complex by a bounded amount. We refer the reader to Maher’s paper [12] for more details. In particular, though, a simple random walk on a Cayley graph for $\text{Mod}(S)$ satisfies the assumptions above.

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2 The proof of Theorem 1.1

Let $S$ be a closed, orientable surface with genus $g \geq 2$ and let $\phi : S \to S$ be a homeomorphism such that the rank of $\pi_1 M_\phi$ is less than $2g + 1$. We want to show that $\phi$ has translation length $\tau(\phi)$ at most some $L = L(g)$.

We may assume that $\tau(\phi) > 0$, so that $\phi$ is pseudo-Anosov. By Thurston’s hyperbolization theorem [22], $M_\phi$ admits a hyperbolic metric. Fix some small
\(\epsilon > 0\) and let \(d_\epsilon\) be the path pseudo-metric on \(M_\phi\) that agrees with the hyperbolic metric on the \(\epsilon\)-thick part of \(M_\phi\) and vanishes on the \(\epsilon\)-thin part (see §4). We say that \(d_\epsilon\) is constructed by \textit{electricifying} the \(\epsilon\)-thin part of \(M_\phi\). We call the \(d_\epsilon\)-diameter of \(M_\phi\) its \(\epsilon\)-\textit{electric diameter} and write \(\text{diam}_\epsilon(M_\phi)\).

\textbf{Lemma 2.1} \((\text{Electric diameter} \approx \tau(\phi)).\) For small \(\epsilon = \epsilon(g) > 0\), there is some constant \(k = k(g, \epsilon)\) such that \(\frac{1}{k} \cdot \tau(\phi) \leq \text{diam}_\epsilon(M_\phi) \leq k \cdot \tau(\phi)\).

In some sense, \(\tau(\phi)\) is more closely related to something like the ‘\(\epsilon\)-electric circumference of \(M_\phi\)’, which one could define as the smallest \(d_\epsilon\)-length of a loop in \(M_\phi\) that projects to an essential loop under \(M_\phi \to S^1\). However, this is the same as electric diameter up to a multiplicative error, so as such an error is built into Lemma 2.1, it does not matter which we use.

Lemma 2.1 is well-known, and is a moral starting point for Minsky’s program \([14, 15, 16]\) for building bilipschitz models for hyperbolic 3-manifolds, which led to a proof of the Ending Lamination Conjecture by Brock-Canary-Minsky \([7]\). However, to our knowledge it is not written, so we give a proof in Section 4.

The goal now is to show that when \(\text{rank}(\pi_1 M_\phi) < 2g + 1\), the \(\epsilon\)-electric circumference of \(M_\phi\) is bounded by some constant depending on \(g\). To do this, we need a tool that allows us to view the rank of \(\pi_1 M\) geometrically.

A \textit{carrier graph} in a hyperbolic manifold \(M\) is a map \(f : X \to M\), where \(X\) is a connected finite graph, that induces a surjection on \(\pi_1\). Note that any generating set for \(\pi_1 M\) gives, for instance, a carrier graph whose domain is a wedge of circles. As long as \(f\) is rectifiable, the hyperbolic metric on \(M\) pulls back to a path (pseudo)-metric on \(X\), which we use to measure the lengths of the edges of \(X\). We’ll say that \(f\) has \textit{minimal length} if its total edge length is that most that of any other carrier graph \(\gamma : Y \to M\), say with \(\text{rank}(\pi_1 X) = \text{rank}(\pi_1 Y)\). (Everything we will now say, however, also works for a weaker notion of minimality, in which there are no length-reducing ‘edge moves’. This is the perspective we take later in the paper, see §5.3.)

White \([23]\) has shown that the geometry of a minimal length carrier graph is well-behaved: for instance, \(X\) is trivalent and the edges of \(X\) map to geodesic segments in \(M\) that connect at \(2\pi/3\)-angles. Using these features, he showed that any minimal length carrier graph has a cycle with total edge length less than some constant depending on \(\text{rank}(\pi_1 X)\). An even stronger statement was proved in \([3]\): any minimal length carrier graph \(f : X \to M\) is filtered by subgraphs each of which has bounded length ‘relative to’ the previous subgraph.

\textbf{Proposition 2.2} \((\text{Chains of Bounded Length}, \text{see } \S 5.4).\) Fixing \(\epsilon > 0\), there is an \(L = L(\epsilon, n)\) such that if \(M\) is a closed hyperbolic \(n\)-manifold and \(f : X \to M\) is a minimal length carrier graph, there is a sequence of (possibly disconnected) subgraphs \(\emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X\) such that \(\text{length}_{Y_i, \epsilon}(Y_{i+1}) < L\) for all \(i\).

Intuitively, \(\text{length}_{Y_i, \epsilon}(Y_{i+1})\) is the hyperbolic length of the part of \(f(Y_{i+1})\) that lies outside of the union of the convex hulls of the components of \(f(Y_i)\) with any \(\epsilon\)-thin parts of \(M\) where a curve from \(f_\ast(\pi_1 Y_i)\) is short, although to make this completely accurate one must work in the universal cover – see Section 5.4.
Returning to Theorem 1.1, we want to show that if \( \text{rank}(\pi_1 M_\phi) < 2g + 1 \), then the \( \epsilon \)-electric circumference of \( M_\phi \) is bounded above by a constant depending only on \( g \). Choose a minimal length carrier graph

\[
\phi : X \to M_\phi
\]

with \( \text{rank}(\pi_1 X) < 2g + 1 \), and let \( \emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X \) be the filtration from Proposition 2.2. The proof relies on the following lemma.

**Lemma 2.3.** Fixing \( i \), suppose that for each connected component \( Y_i^j \subset Y_i \), the image \( f_\ast(\pi_1 Y_i^j) \) is an infinite index subgroup of \( \pi_1 S \subset \pi_1 M_\phi \).

Then for every component \( Y_{i+1}^k \) of \( Y_{i+1} \), the \( d_e \)-diameter of \( f(Y_{i+1}^k) \) in \( M_\phi \) is bounded above by a constant depending only on \( g, \epsilon \) and the \( d_e \)-diameters of \( f(Y_i^j) \), where \( Y_i^j \) is a component of \( Y_i \).

We will prove the lemma in §6, but the point is the following. Since we know that \( \text{length}_{Y_i^j}(Y_{i+1}) \) is bounded, the \( d_\epsilon \)-diameter of the part of each \( f(Y_{i+1}^k) \) that lies outside of the convex hulls of the \( f(Y_i^j) \subset M_\phi \) is also bounded. So, the goal is to bound the \( d_e \)-diameter of these convex hulls.

Applying Lemma 2.3 iteratively, we get a bound in terms of \( g, \epsilon \) for the \( \epsilon \)-electric diameter of the first subgraph \( Y_i \) with a component \( Y_i^j \) such that \( f_\ast(\pi_1 Y_i^j) \) is not an infinite index subgroup of \( \pi_1 S \subset \pi_1 M_\phi \).

If \( f_\ast(\pi_1 Y_i^j) \) is a finite index subgroup of \( \pi_1 S \), then it has rank at least \( 2g \). However, then \( Y_i^j \) is a proper subgraph of \( X \) with \( \text{rank}(\pi_1 Y_i^j) \geq 2g \), which contradicts the fact that \( X \) is trivalent and \( \text{rank}(\pi_1 X) \leq 2g \).

Thus, \( f_\ast(\pi_1 Y_i^j) \) is not contained in \( \pi_1 S \). If \( p \in M_\phi \), Corollary 3.4 gives an embedded surface \( S' \subset M_\phi \) in the homotopy class of the fiber that has bounded \( \epsilon \)-electric diameter and lies at bounded distance from \( p \). Since \( M_\phi \backslash S' \cong S \times (0,1) \) and \( f_\ast(\pi_1 Y_i^j) \not\subset \pi_1 S \), the surface \( S' \) must intersect \( Y_i^j \). Therefore, the \( d_e \)-diameter bound for \( Y_i^j \) translates into a \( d_e \)-diameter bound for \( M_\phi \).

### 3 Preliminaries

#### 3.1 Convex cores and ends of hyperbolic 3-manifolds

Let \( M \) be a hyperbolic 3-manifold with finitely generated fundamental group, and assume that \( M \) has no cusps. By the Tameness theorem [1, 8], \( M \) is homeomorphic to the interior of a compact 3-manifold. If \( M \) is noncompact, each of its ends then has a neighborhood homeomorphic to \( \Sigma \times (0, \infty) \) for some closed surface \( \Sigma \). In the following, we recall a geometric classification of these ends. As all the stated results are extremely well-known, we eschew pinpoint references and refer the reader to [5, 9, 13, 20], for instance, for details.

The **convex core** of \( M \), written \( \text{core}(M) \), is the smallest convex submanifold of \( M \) whose inclusion is a homotopy equivalence. If \( M = \mathbb{H}^3 / \Gamma \), where \( \Gamma \) is a discrete group of isometries of \( \mathbb{H}^3 \), then \( \text{core}(M) \) is the quotient by \( \Gamma \) of the convex hull in \( \mathbb{H}^3 \) of the limit set \( \Lambda(\Gamma) \subset \partial_\infty \mathbb{H}^3 \). The ends of \( M \) are classified...
according to their intersections with $\text{core}(M)$. Namely, an end $E$ of $M$ either has a neighborhood disjoint from $\text{core}(M)$, in which case $E$ called \textit{convex-compact}, or has a neighborhood contained in $\text{core}(M)$, in which case it is called \textit{degenerate}.

Both types of ends can occur in hyperbolic 3-manifolds $M \cong S \times \mathbb{R}$. For example, orthogonally extending a discrete, cocompact action of $\Gamma = \pi_1 S$ on some totally geodesic plane $\mathbb{H}^2 \subset \mathbb{H}^3$ gives an action $\Gamma \circlearrowleft \mathbb{H}^3$. The quotient $M = \mathbb{H}^3 / \Gamma$ is homeomorphic to $S \times \mathbb{R}$ and is called \textit{Fuchsian}. In this case, the convex core is $\mathbb{H}^2 / \Gamma$, which is compact, so both ends are convex cocompact.

Now suppose $\phi : S \rightarrow S$ is pseudo-Anosov. By Thurston’s hyperbolization theorem [22], the mapping torus $M_\phi$ admits a hyperbolic metric. Let $\tilde{M}_\phi$ be the infinite cyclic cover of $M_\phi$ corresponding to the subgroup $\pi_1 S \subset \pi_1 M_\phi$. Then $\tilde{M}_\phi \cong S \times \mathbb{R}$, and the hyperbolic metric on $M_\phi$ lifts to a metric on $\tilde{M}_\phi$. As $\tilde{M}_\phi$ regularly covers a closed manifold, it is its own convex core; for instance, one can see this by noting that a cocompact group $\Gamma$ of isometries of $\mathbb{H}^3$ has full limit set and a nontrivial normal subgroup of $\Gamma$ has the same limit set as $\Gamma$. It follows that both ends of $\tilde{M}_\phi$ are degenerate.

### 3.2 Simplicial hyperbolic surfaces

Suppose that $M$ is a hyperbolic 3-manifold. We describe here a useful class of maps of surfaces into $M$; more information can be found in [8, 10, 18].

**Definition 3.1.** A simplicial hyperbolic surface is a map $f : S \rightarrow M$ such that

1. each face of a triangulation $T$ of $S$ maps to a totally geodesic triangle,

2. for each vertex $v \in T$ the angles between the images of the edges adjacent to $v$ sum to at least $2\pi$.

A simplicial hyperbolic surface pulls back the hyperbolic metric on $M$ to a path-metric on $S$ that is smooth and hyperbolic away from the vertices of $T$, at which there are possible excesses of angle. By the Gauss-Bonnet Theorem,

$$\text{vol}(S) \leq 2\pi(2g - 2).$$

Therefore, the injectivity radius of $S$ at every point is at most some $C = C(g)$, and the $\epsilon$-electric diameter of $S$ is at most some $C = C(\epsilon, g)$. Accordingly,

**Fact 3.2.** If $f : S \rightarrow M$ is a simplicial hyperbolic surface, there is a simple closed curve on $S$ whose image has length at most some $C = C(g)$. Moreover, if $f$ is $\pi_1$-injective, the $\epsilon$-electric diameter of $f(S) \subset M$ is at most some $C = C(g, \epsilon)$.

Here, the $\pi_1$-injectivity ensures that the $\epsilon$-thin parts of $S$ map into $\epsilon$-thin parts of $M$, so that electric distance on $S$ bounds electric distance in $M$.

Most existence results about simplicial hyperbolic surfaces stem from the following, which is essentially due to Thurston.
Lemma 3.3 (compare with Example 8.7.3, [20]). Let \( g : S \to M \) be a \( \pi_1 \)-injective map into a hyperbolic 3-manifold \( M \) without cusps and \( \alpha \) a multi-curve on \( S \). Then \( g \) is homotopic to a simplicial hyperbolic surface \( f : S \to M \) that realizes \( \alpha \), i.e. such that \( f \) maps each component of \( \alpha \) to a closed geodesic.

Canary showed in [10] how to interpolate between simplicial hyperbolic surfaces using simplicial hyperbolic surfaces, which gives more powerful existence results. The following is a version of the much stronger theorem of [10] with the same name; it can be proved using the arguments of [10, Theorem 6.2].

**Canary's Filling Theorem.** There is some constant \( C = C(g) \) such that if \( M \) is a hyperbolic 3-manifold homeomorphic to \( S \times \mathbb{R} \), then every point in the convex core of \( M \) lies at most at distance \( C \) from the image of a simplicial hyperbolic surface \( f : S \to M \) in the homotopy class of the fiber.

When \( \phi : S \to S \) is pseudo-Anosov, the infinite cyclic cover \( \hat{M}_\phi \) of the mapping torus \( M_\phi \) is its own convex core (see §3.1), so Canary’s Filling Theorem gives simplicial hyperbolic surfaces that coarsely exhaust \( \hat{M}_\phi \). Projecting these surfaces to \( M_\phi \), we have that every point of \( M_\phi \) is within \( C \) of the image of a simplicial hyperbolic surface in the homotopy class of the fiber.

Simplicial hyperbolic surfaces \( f : S \to M_\phi \) are in general not embedded. However, work of Freedman-Hass-Scott (see [11, Theorem 2.5]), implies that any \( f \) in the homotopy class of the fiber is homotopic to an embedding, which maybe is not simplicial hyperbolic, but lies near to the image of \( f \) and still has an electric diameter bound:

**Corollary 3.4.** If \( \phi : S \to S \) is pseudo-Anosov, then for every point \( p \in M_\phi \), there is an embedding \( S \to M \) in the homotopy class of the fiber such that

1. the hyperbolic distance from \( p \) to \( S \) is at most some \( C = C(g) \),

2. the \( \epsilon \)-electric diameter of \( S \subset M \) is at most some \( C = C(g, \epsilon) \).

## 4 Electric distance and the curve complex

Let \( M \) be a hyperbolic 3-manifold without cusps. The \emph{injectivity radius} at a point \( p \in M \), written \( \text{inj}_M(p) \), is half the length of a shortest homotopically essential loop through \( p \). The \emph{\( \epsilon \)-thin part} of \( M \) is the subset

\[
M_{\leq \epsilon} = \{ p \in M \mid \text{inj}_M(p) \leq \epsilon \}.
\]

The complement of \( M_{\leq \epsilon} \) is the \emph{\( \epsilon \)-thick part} of \( M \), written \( M_{>\epsilon} \). When \( \epsilon \) is less than the Margulis constant (see [2]), every component of \( M_{\leq \epsilon} \) is a neighborhood of a closed geodesic in \( M \) and is homeomorphic to a solid torus.

We construct a path pseudometric \( d_\epsilon \) on \( M \) by declaring that the length of any path contained in \( M_{\leq \epsilon} \) is zero, while path lengths outside \( M_{\leq \epsilon} \) are measured hyperbolically. We say that \( d_\epsilon \) is constructed from the hyperbolic metric on \( M \).
by electrifying the $\epsilon$-thin part of $M$. Metric quantities such as length and diameter have their $\epsilon$-electric counterparts, written e.g. $\text{length}_\epsilon$ and $\text{diam}_\epsilon$.

As stated in Section 2, the $\epsilon$-electric diameter of a hyperbolic mapping torus is related to the curve complex translation distance $\tau(\phi)$ of its monodromy:

**Lemma 2.1** (Electric diameter and $\tau(\phi)$). There are $\epsilon = \epsilon(g)$ and $k = k(g, \epsilon)$ such that if $M_\phi$ is the mapping torus of a pseudo-Anosov map $\phi : S \to S$,

$$\text{diam}_\epsilon(M_\phi) \approx_k \tau(\phi).$$

For convenience, here and below we write $A \leq_k B$ if $A \leq k \cdot B + k$ and $A \approx_k B$ if $A \leq k \cdot B$ and $B \leq k \cdot A$. Lemma 2.1 is a consequence of the following:

**Theorem 4.1** (Curve complex models electric distance). For every $L > 0$ and small $\epsilon > 0$, there are constants $k_1 = k_1(g, L)$ and $k_2 = k_2(g, \epsilon)$ such that if $M \cong S \times \mathbb{R}$ is a hyperbolic 3-manifold without cusps and $\alpha, \beta$ are simple closed curves on $S$ with geodesic representatives $\hat{\alpha}, \hat{\beta} \subset M$ of length at most $L$, then

$$d_{C(S)}(\alpha, \beta) \leq_{k_1} d_{\epsilon}(\hat{\alpha}, \hat{\beta}) \leq_{k_2} d_{C(S)}(\alpha, \beta).$$

We feel that some sort of credit for this theorem should be given to Yair Minsky, since in some sense it is implicit in the development of the model manifolds of [14]. However, as we find ourselves without a citation, we give credit only to Brock-Bromberg [6] for the key fact used in the lower bound and apologize profusely. In the proofs below, unless we say otherwise we use ‘bounded’ to mean bounded above by a constant depending only on $\epsilon$ and on $g$.

**Proof of Theorem 4.1.** For the upper bound, suppose that

$$\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$$

is a path in $C(S)$. For each $i$, the multi-curve $\alpha_i \cup \alpha_{i+1}$ can be realized geodesically by a simplicial hyperbolic surface $f_i : S_i \to M$ in the homotopy class of the fiber, by Lemma 3.3. By Fact 3.2, the images $f_i(S_i)$ have bounded electric diameter. Since each image $f(S_i)$ intersects $f(S_{i+1})$ along the geodesic $\hat{\alpha}_{i+1}$, the electric distance from $\hat{\alpha}$ to $\hat{\beta}$ is bounded above by a linear function of $n$, and therefore a linear function of $\tau(\phi)$.

The lower bound is a bit harder. Let $\gamma$ be a path realizing the electric distance from $\hat{\alpha}$ to $\hat{\beta}$ and subdivide $\gamma$ into a concatenation

$$\gamma = \gamma_1 \cdots \gamma_n,$$

such that for odd $i$, the path $\gamma_i$ lies in the $\epsilon$-thick part of $M$ and for even $i$, in the $\epsilon$-thin part of $M$. (Start the index at $i = 0$ if necessary.) We then have

$$\sum_{i \text{ odd}} \text{length}(\gamma_i) = \text{length}_\epsilon(\gamma),$$

where on the left length is absolute and on the right it is electric.
By Canary’s Filling Theorem (§3) and Fact 3.2, for each $i$ there is a simple closed curve $\alpha_i$ on $S$ whose geodesic representative $\hat{\alpha}_i$ has bounded length and lies at bounded (absolute) distance from the left endpoint of $\gamma_i$. Here and in the rest of this proof, ‘bounded’ means by a constant depending on $g$. As long as $\epsilon$ is much smaller than the Margulis constant $\epsilon_0$, for even $i$ the geodesics $\hat{\alpha}_i$ and $\hat{\alpha}_{i+1}$ must lie in the same component of the $\epsilon_0$-thin part of $M$, which can only happen if the simple closed curves $\alpha_i$ and $\alpha_{i+1}$ are homotopic. For odd $i$, it follows from a theorem of Brock-Bromberg [6, Theorem 7.16] that

$$d_{C(S)}(\alpha_i, \alpha_{i+1}) \leq k \text{ length}(\gamma_i),$$

where $k = k(g)$. Therefore, we conclude from Equation (1) that

$$d_{C(S)}(\alpha_1, \alpha_n) \leq k \text{ length}(\epsilon)(\gamma),$$

for some slightly larger $k = k(g)$. Applying Brock-Bromberg again to bound the distance between $\alpha, \beta$ and $\alpha_1, \alpha_n$, we can replace this with the inequality

$$d_{C(S)}(\alpha, \beta) \leq k \text{ length}(\epsilon)(\gamma),$$

except that now $k = k(g, L)$ depends on the length bound $L$ for $\hat{\alpha}$ and $\hat{\beta}$.

Proof of Lemma 2.1. To set up some notation, let $\hat{M}_\phi \cong S \times \mathbb{R}$ be the infinite cyclic cover corresponding to the subgroup $\pi_1 S \subset \pi_1 M_\phi$, see §3.1. We also let

$$\hat{\phi} : \hat{M}_\phi \rightarrow M_\phi$$

be the isometry that generates the deck group of $\hat{M}_\phi \rightarrow M_\phi$.

We first show that $\text{diam}_\epsilon(M_\phi) \leq k \tau(\phi)$. Pick a simple closed curve $\alpha$ on $S$ such that $d_{C(S)}(\alpha, \phi(\alpha)) = \tau(\phi)$. Then if $\hat{\alpha}$ is the geodesic representative of $\alpha$ in $\hat{M}_\phi$, the geodesic representative of $\phi(\alpha)$ is $\hat{\phi}(\alpha)$. By Theorem 4.1, there is a path $\hat{\gamma}$ from $\hat{\alpha}$ to $\hat{\phi}(\alpha)$ in $\hat{M}_\phi$ with

$$\text{length}(\hat{\gamma}) \leq k \tau(\phi).$$

By Lemma 3.3 and Fact 3.2, $\hat{\phi}(\alpha)$ has bounded $\epsilon$-electric diameter in $\hat{M}_\phi$. So, after increasing the length of $\hat{\gamma}$ by a bounded $\epsilon$-electric amount, we may assume that the endpoints of $\hat{\gamma}$ differ by $\hat{\phi}$. The image of $\hat{\gamma}$ in $\hat{M}_\phi$ is then a loop $\gamma$ that has $\epsilon$-electric length at most some multiple of $\tau(\phi)$, and which projects to an essential loop under the fibration $M_\phi \rightarrow S^1$.

If $p \in M$, Corollary 3.4 gives an embedded surface $S \rightarrow M_\phi$ in the homotopy class of the fiber that has bounded $\epsilon$-electric diameter and lies at bounded distance from $p$. As $\gamma$ projects essentially under $M_\phi \rightarrow S^1$, this surface intersects $\gamma$. So, the $\epsilon$-electric diameter of $M_\phi$ is at most a bounded constant plus the $\epsilon$-electric length of $\gamma$, and therefore is at most a bounded multiple of $\tau(\phi)$.

We now claim that $\tau(\phi) \leq k \text{ diam}_\epsilon(M_\phi)$. By Fact 3.2 and Lemma 3.3, there is a simple closed curve $\alpha$ on $S$ such that geodesic $\hat{\alpha}$ in $\hat{M}_\phi$ has bounded length.
We claim that the \(d_c(\hat{\alpha}, \hat{\phi}(\alpha))\) is at most a bounded multiple of \(\text{diam}_c(M_\phi)\). Theorem 4.1 will then imply that the curve complex distance between \(\alpha\) and \(\phi(\alpha)\) is similarly bounded, finishing the lemma.

Using Corollary 3.4, pick an embedding \(f : S \to M_\phi\) such that \(f(S)\) has bounded \(\epsilon\)-electric diameter and lies at bounded distance from the projection of \(\hat{\alpha}\) to \(M_\phi\). Then \(f\) lifts to \(Z\)-many disjoint embeddings \(S \to \hat{M}_\phi\). If \(\hat{f}\) is such a lift that intersects \(\hat{\alpha}\), the rest of the lifts are of the form \(\psi^n \circ \hat{f}\), where \(n \in Z\).

As \(\pi_1M_\phi\) is generated by loops with \(\epsilon\)-electric length at most twice the \(\epsilon\)-electric diameter of \(M_\phi\), there must be a loop \(\gamma\) in \(M_\phi\) with

\[
\text{length}_\epsilon(\gamma) \leq 2\text{diam}_\epsilon(M_\phi)
\]

that does not lie in \(\pi_1S \subset \pi_1M_\phi\). It follows that \(\gamma\) must intersect \(f(S)\). Cutting \(\gamma\) at some such intersection point, lift it to an arc \(\hat{\gamma}\) in \(\hat{M}_\phi\) that starts on \(\hat{f}(S)\) and ends on \(\hat{\phi}^n \circ \hat{f}(S)\) for some \(n \neq 0\). Up to reversing the direction we traverse \(\gamma\), we may assume \(n > 0\). For \(n > 1\), the surface \(\hat{\phi} \circ \hat{f}(S)\) separates \(\hat{f}(S)\) from \(\hat{\phi}^n \circ \hat{f}(S)\) in \(\hat{M}_\phi\). Therefore, in any case \(\hat{\gamma}\) must intersect \(\hat{\phi} \circ \hat{f}(S)\).

To recap, \(\hat{\alpha}\) lies at bounded distance from \(\hat{f}(S)\), which has bounded electric diameter. An arc of \(\hat{\gamma}\) connects \(\hat{f}(S)\) to \(\hat{\phi} \circ \hat{f}(S)\), which has bounded electric diameter and intersects \(\hat{\phi}(\alpha)\). So, we would be done if we knew that the \(\epsilon\)-electric length of \(\hat{\gamma}\) was bounded by a multiple of \(\text{diam}_\epsilon(M_\phi)\).

The \(\epsilon\)-electric length of \(\gamma\) in \(M_\phi\) is at most twice \(\text{diam}_\epsilon(M_\phi)\). Unfortunately, the electric length of \(\gamma\) may be larger, since in \(\hat{M}_\phi\) we electrify the \(\epsilon\)-thin parts of \(\hat{M}_\phi\), which may not be the entire preimage of the \(\epsilon\)-thin part of \(M_\phi\). However, this is only a problem if there is an \(\epsilon\)-short loop in \(M_\phi\) that does not lift to \(\hat{M}_\phi\). In this case, we could take that short loop as our \(\gamma\), which then has short absolute length, implying that the same is true for \(\hat{\gamma}\) in \(\hat{M}_\phi\).

\[
\square
\]

5 Convexity, carrier graphs and relative length

In this section, we give some background on convexity and carrier graphs, and introduce the notion of relative length referenced in Proposition 2.2.

5.1 Convex hulls in \(\mathbb{H}^n\)

A subset \(A \subset \mathbb{H}^n\) is called convex if every geodesic segment with endpoints in \(A\) is entirely contained in \(A\). The convex hull of a subset \(A \subset \mathbb{H}^n\) is the smallest convex subset \(\text{hull}(A) \subset \mathbb{H}^n\) containing \(A\). Additionally, if \(D \geq 0\) then a subset \(A \subset \mathbb{H}^n\) is called \(D\)-quasiconvex if every geodesic segment with endpoints in \(A\) is contained in the radius-\(D\) neighborhood \(N_D(A)\) of \(A\).

Every geodesic with endpoints in a set \(A \subset \mathbb{H}^n\) lies inside of \(\text{hull}(A)\), so any set \(A\) with a \(D\)-neighborhood that contains \(\text{hull}(A)\) is \(D\)-quasiconvex. The converse is also true, up to an increase in the constant:

**Fact 5.1.** If \(A \subset \mathbb{H}^n\) is \(D\)-quasiconvex, then \(\text{hull}(A) \subset N_{nD}(A)\).
Proof. Let \( \text{hull}^1(A) \) be the union of all geodesic segments with endpoints in \( A \) and define \( \text{hull}^i(A) = \text{hull}^1(\text{hull}^{i-1}(A)) \) inductively. As \( A \) is \( D \)-quasiconvex, \( \text{hull}^1(A) \subset N_D(A) \). An inductive argument using the convexity of the distance function shows that \( \text{hull}^i(A) \subset N_i D(A) \) for all \( i \). But by [17, Corollary 2.8], for instance, in hyperbolic or Euclidean \( n \)-space we have \( \text{hull}^n(A) = \text{hull}(A) \). \( \square \)

In \( \mathbb{H}^n \), quasiconvexity is somewhat robust.

**Fact 5.2.** For every \( D \geq 0 \), there is some \( D' = D'(D,n) \) such that if \( \{ A_i, \ i \in I \} \) is a collection of \( D \)-quasiconvex subsets of \( \mathbb{H}^n \) and every \( A_i \) intersects some fixed \( A_{i_0} \), then the union \( \bigcup_i A_i \) is \( D' \)-quasiconvex.

**Proof.** Every geodesic with endpoints in \( \bigcup_i A_i \) is one side of a geodesic quadrilateral whose other three sides lie in a \( D \)-neighborhood of \( \bigcup_i A_i \). So, the fact follows from the \( \delta \)-hyperbolicity of \( \mathbb{H}^n \). \( \square \)

Suppose that \( \Gamma \) is a discrete subgroup of \( \text{Isom}(\mathbb{H}^n) \). The smallest nonempty \( \Gamma \)-invariant convex subset of \( \mathbb{H}^n \) is the convex hull of the limit set of \( \Gamma \), which we briefly introduced in \( \S 3 \) and now refer to as \( \text{hull}(\Gamma) \). We will also need an enlargement of \( \text{hull}(\Gamma) \) that contains all points on which \( \Gamma \) acts with small displacement: if \( \epsilon > 0 \) is smaller than the Margulis constant, define

\[
\text{Thin}_\epsilon(\Gamma) = \{ x \in \mathbb{H}^n | \exists \gamma \in \Gamma \text{ with } d(x, \gamma(x)) < \epsilon \}
\]

and define the \( \epsilon \)-thickened convex hull of \( \Gamma \) to be the smallest nonempty \( \Gamma \)-invariant convex subset \( \text{hull}_\epsilon(\Gamma) \subset \mathbb{H}^n \) that contains \( \text{Thin}_\epsilon(\Gamma) \).

By \( \Gamma \)-invariance, the \( \epsilon \)-thickened convex hull contains the convex hull of \( \Lambda(\Gamma) \); it also obviously contains \( \text{Thin}_\epsilon(\Gamma) \). Here is a coarse converse:

**Lemma 5.3.** There is some \( D = D(n) \) such that \( \text{hull}_\epsilon(\Gamma) \) is contained in the \( D \)-neighborhood of the union of \( \text{hull}(\Gamma) \) and \( \text{Thin}_\epsilon(\Gamma) \).

By the Margulis lemma, the components of \( \text{Thin}_\epsilon(\Gamma) \) are all convex and all intersect \( \text{hull}(\Gamma) \) along their core axes. So, Lemma 5.3 follows from Fact 5.2.

### 5.2 The geometry of (equivariant) carrier graphs

We introduced carrier graphs in \( \S 2 \) as \( \pi_1 \)-surjective maps \( f : X \to M \), where \( X \) is a finite graph and \( M \) is a hyperbolic \( n \)-(orbifold). Many arguments using carrier graphs are best performed in the universal cover, so we now introduce an equivariant version of carrier graphs in \( \mathbb{H}^n \).

Suppose that \( \Gamma \) is a discrete subgroup of \( \text{Isom}(\mathbb{H}^n) \). A \( (\Delta, \Gamma) \)-equivariant carrier graph is a \( (\Delta, \Gamma) \)-equivariant map

\[
\tilde{f} : \tilde{X} \to \mathbb{H}^n,
\]

where \( \tilde{X} \) is a connected, locally finite graph and \( \Delta < \text{Aut}(\tilde{X}) \) acts freely and cocompactly on \( \tilde{X} \). For example, any carrier graph \( X \to M \) for a hyperbolic
Lemma 5.4. Suppose that \( \hat{f} : \hat{X} \rightarrow \mathbb{H}^n \) is a \((\Delta, \Gamma)\)-equivariant carrier graph that maps each edge of \( \hat{X} \) to a geodesic segment in \( \mathbb{H}^n \), that \( \alpha \) is an edge-path in \( \hat{X} \) from \( x \) to \( y \), and that \( \gamma(h(x)) = h(y) \) for some nontrivial \( \gamma \in \Gamma \).

Then there is a point on \( h(x) \) whose distance to \( h_l(\Gamma) \) is at most some constant depending only on \( n, \epsilon \) and the number of edges in \( \alpha \).

Proof. Let \( x', y' \) be the nearest point projections of \( h(x), h(y) \) to \( h_l(\Gamma) \). Form a geodesic polygon in \( \mathbb{H}^n \) by concatenating \( h(\alpha) \) with the segments

\[
\{h(\gamma)\}, \{y', x' \}, \{x', h(x)\}.
\]

If \( k \) is the number of edges in \( \alpha \), then by \( \delta \)-hyperbolicity, \( [x', h(x)] \) is contained in a \( C \)-neighborhood of the other sides, for some \( C = C(k, n) \).

The angle between \( [x', h(x)] \) and \( [x', y'] \) is at least \( \pi/2 \). So, \( N_C([x', y']) \) can only contain a segment of \( [x', h(x)] \) with length at most some \( C' = C'(k, n) \).

If \( h(x) \in h_l(\Gamma) \), we’re done. Otherwise, \( x' \in \partial h_l(\Gamma) \). As \( y' = \gamma(x') \) for some nontrivial \( \gamma \in \Gamma \), the segment \( [y', x'] \) has length at least \( \epsilon \). So, as both \( [h(x), x'] \) and \( [h(y), y'] \) intersect \( [y', x'] \) at an angle at least \( \pi/2 \), \( N_C([h(y), y']) \) can only contain a segment of \( [x', h(x)] \) of length at most \( C'' = C''(k, n, \epsilon) \).

Therefore, there is a point on \( [x', h(x)] \) at most \( \epsilon \) away from \( x' \) that is within \( C \) of \( h(Y_i) \), which bounds the distance between \( h(Y_i) \) and \( h_l(\Gamma) \) by a constant depending only on \( n, k, \epsilon \). \( \square \)

The \( \epsilon \)-thickened convex hull of \( h(\hat{X}) \) with respect to \( \Gamma \), written

\[
h_h(\hat{f}(\hat{X}), \Gamma),
\]

is the convex hull of the union of \( h(\hat{X}) \) and \( h_l(\Gamma) \). Actually, the union is quasiconvex, so taking the convex hull does not change it dramatically:

Proposition 5.5. Suppose that \( \hat{f} : \hat{X} \rightarrow \mathbb{H}^n \) is a \((\Delta, \Gamma)\)-equivariant carrier graph that maps each edge of \( \hat{X} \) to a geodesic segment in \( \mathbb{H}^n \). Then there is some \( D \) depending only on \( n, \epsilon \) and the number \( k \) of edges in \( \hat{X}/\Delta \) such that

\[
h_h(\Gamma) \cup \hat{f}(\hat{X})
\]

is \( D \)-quasiconvex. Consequently,

\[
h_h(\Gamma) \cup \hat{f}(\hat{X}) \subset h_h(\hat{f}(\hat{X}), \Gamma) \subset N_D\left(h_h(\Gamma) \cup \hat{f}(\hat{X})\right).
\]

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Proof. The point is to show that $\text{hull}_c(\Gamma) \cup \tilde{f}(\tilde{X})$ is $D$-quasiconvex, for then the second statement follows from Fact 5.1.

Let $Y_i \subset \tilde{X}$, where $i = 1, 2, \ldots$, be all the translates of a connected fundamental domain $Y_0$ for the cocompact action $\Delta \circlearrowright \tilde{X}$. Each $\tilde{f}(Y_i)$ is a connected union of $k$ geodesic segments in $\mathbb{H}^n$, and therefore is $C$-quasiconvex for some $C = C(k, n)$, by the $\delta$-hyperbolicity of $\mathbb{H}^n$. Applying Lemma 5.4 to a simple path connecting (extremal) vertices of $Y_i$ that are identified by the $\Delta$-action, we see that a $C'$-neighborhood of each $\tilde{f}(Y_i)$ intersects $\text{hull}_c(\Gamma)$, where $C' = C'(n, k, \epsilon)$.

Applying Fact 5.2 to $\text{hull}_c(\Gamma)$ and these $C'$-neighborhoods then shows that $\text{hull}_c(\Gamma) \cup \tilde{f}(\tilde{X})$ is $D$-quasiconvex for some $D = D(k, n, \epsilon)$. \hfill $\square$

5.3 Edge moves and carrier graphs of minimal length

Let $f : X \longrightarrow M$ be a carrier graph in a hyperbolic $n$-manifold $M$. As in Figure 1, an edge move on $f$ creates a new carrier graph $g : Y \longrightarrow M$ as follows:

1. Pick an edge $e \subset X$ with vertices $v_0, v_1$ and a path $\alpha$ in $X \setminus e$ from $v_0$ to some point $w \in X$.

2. Create a new graph $Y$: sever the attachment of $e$ to $v_1$, create an additional vertex at $w$ and reattach the free endpoint of $e$ to $w$.

3. Define $g : Y \longrightarrow M$: set $g = f$ on $Y \setminus e$, and send $e$ to a path from $f(v_0)$ to $f(w)$ in the homotopy class of $f(e) \cdot f(\alpha)$.

4. Homotope $g$ while fixing all the vertices of $Y$, except possibly $w$.

Edge moves have a natural interpretation in the equivariant setting as well: if $\tilde{f} : \tilde{X} \longrightarrow M$ is a $(\Delta, \Gamma)$-equivariant carrier graph, an edge move on $\tilde{f}$ slides one endpoint of an edge $e \subset \tilde{X}$ along a path in $\tilde{X} \setminus \Delta e$ and then repeats this equivariantly for every edge in the $\Delta$-orbit $\Delta e$.

Definition 5.6. We say that a carrier graph (equivariant or not) has minimal length rel vertices if it does not admit a length-decreasing edge move.
This version of minimality is strictly weaker than that used in [3, 4], in which two carrier graphs \( f_i : X_i \to M \), \( i = 1, 2 \) were said to be equivalent if there is a homotopy equivalence \( h : X_1 \to X_2 \) such that \( f_1 \simeq f_2 \circ h \), and minimal length meant minimal in an equivalence class. The advantage of the new definition is that it is inherited by subgraphs, since any length-reducing edge move on a subgraph clearly extends to such a move on the supergraph. Note that in contrast to 'equivalence', edge moves are not always reversible.

Most of the key geometric properties of the minimal length carrier graphs of [3, 4, 23] still apply to carrier graphs with minimal length rel vertices. In particular, we have the following adaptation of a result of White:

**Proposition 5.7** (see [23]). Let \( f : X \to M \) be a carrier graph with minimal length rel vertices in a hyperbolic \( n \)-manifold \( M \). Then \( X \) is at most trivalent and each edge maps to a geodesic segment in \( M \). The angle between any two adjacent edges is at least \( \frac{2\pi}{3} \), and the image of any simple closed path in \( X \) is either a point or an essential loop in \( M \).

In particular, if \( X \) has no valence 1 or 2 vertices, it is trivalent with all angles equal to \( \frac{2\pi}{3} \) and has \( 3(\text{rank}(\pi_1 X) - 1) \) edges, by an Euler characteristic count.

![Figure 2: As long as the angle \( \theta < \frac{2\pi}{3} \), folding part of the two edges together is a length-decreasing edge move.](image)

To prove the proposition, if two adjacent edges meet at an angle \( \theta < \frac{2\pi}{3} \), then the edge move shown in Figure 2 decreases length. (Since it suffices to perform the move arbitrarily close to the vertex, this can be checked with a quick Euclidean computation.) It follows that \( X \) is at most trivalent, and at every trivalent vertex the angles are all equal to \( \frac{2\pi}{3} \). Finally, if a simple closed path \( \alpha \) in \( X \) maps homotopically trivially to \( M \), then either \( \alpha \) maps to a point or there is a length-decreasing edge move that replaces some edge \( e \subset \alpha \) with a loop based at one of its vertices that maps to a point in \( M \).

All the topology of a hyperbolic \( n \)-manifold is contained in its convex core, outside of which its geometry exponentially expands. Therefore, it should be most length-efficient for a carrier graph to stay near the convex core:

**Proposition 5.8** (Minimal length graphs orbit the convex core). Suppose that \( f : X \to M \) is a reduced carrier graph with minimal length rel vertices in a hyperbolic \( n \)-manifold \( M \). If \( \text{core}(X) \) is the Stallings core of \( X \), then we have

\[
\text{f(core}(X)) \subset N_D(\text{core}_e(M)),
\]

where \( D \) is a constant depending on \( n, e \) and the number of edges in \( \text{core}(X) \).
A carrier graph \( f : X \to M \) is reduced if no simple closed curve on \( X \) is mapped to a point in \( M \). Any carrier graph can be reduced by collapsing the problematic loops in \( X \), so the above still has content for nonreduced graphs.

The Stallings core of \( X \) is the smallest subgraph of \( X \) whose inclusion is a homotopy equivalence, which is obtained by removing each edge from \( X \) whose interior divides \( X \) into two components, one of which is a tree.

Finally, \( \text{core}_\epsilon(M) \) is the \( \epsilon \)-thickened convex core of \( M \), which is defined as the smallest convex subset of \( M \) that contains the \( \epsilon \)-thin part of \( M \) and whose inclusion into \( M \) is a homotopy equivalence. If \( M = \mathbb{H}^n/\Gamma \), then \( \text{core}_\epsilon(M) \) is exactly the projection to \( M \) of the \( \epsilon \)-thickened convex hull of \( \Gamma \).

**Proof of Proposition 5.8.** In order to make the argument more readable, we’ll omit reference to specific constants and understand that ‘bounded’ means in terms of \( n, \epsilon \) and the number of edges in \( \text{core}(X) \), and that ‘large’ and ‘far’ mean much larger and further than all the bounded constants appearing below.

Suppose that \( f(\text{core}(X)) \) strays very far from \( \text{core}_\epsilon(M) \). As there are only so many edges, some subsegment \( e' \) of an edge \( e \subset X \) must have endpoints with radically different distances to \( \text{core}_\epsilon(M) \), such that both distances are very large.

Write \( M = \mathbb{H}^n/\Gamma \), lift \( f|_{\text{core}(X)} \) to a \((\Delta, \Gamma)\)-equivariant carrier graph
\[
\tilde{f} : \text{core}(X) \to \mathbb{H}^n,
\]
choose lifts \( \tilde{e}' \subset \tilde{e} \subset \tilde{X} \) of \( e' \subset e \subset X \), let \( \tilde{v}_0 \) and \( \tilde{v}_1 \) be the endpoints of \( \tilde{e}' \), and assume that \( \tilde{v}_1 \) lies further from \( \text{hull}_\epsilon(\Gamma) \) than \( \tilde{v}_0 \).

We claim there is a path \( \tilde{a} : [0, 1] \to \tilde{X} \) with \( \tilde{a}(0) = \tilde{v}_1 \) such that

1. \( \tilde{a}(1) \) lies at bounded distance from \( \text{hull}_\epsilon(M) \),
2. the projection of \( \tilde{e}' \cdot \tilde{a} \) to \( X \) is a simple path.

By Lemma 5.4, any simple closed curve in \( X \) must come within a bounded distance of \( \text{core}_\epsilon(M) \). If \( e \) does not separate \( X \), we can take \( \tilde{a} \) to be some segment of a lift of a simple closed curve in \( X \) that contains \( e \). In the other case, \( e \) separates \( X \) into two components; let \( Y \) be the component adjacent to \( v_1 \), the projection of \( \tilde{v}_1 \). As \( e \) is in \( \text{core}(X) \), there is a simple closed curve \( \beta \) in \( Y \), and we can take \( \tilde{a} \) to be the lift of a simple path in \( Y \) from \( v_1 \) to a point on \( \beta \) that lies at bounded distance from \( \text{core}_\epsilon(M) \).

Form a geodesic polygon in \( \mathbb{H}^n \) by concatenating \( \tilde{e}' \), \( \tilde{a} \) and the geodesic segment \([\tilde{a}(1), \tilde{v}_0] \). Divide \( \tilde{e}' \) in half, and let \( \tilde{e}'' \) be the half adjacent to \( \tilde{v}_1 \). By \( \delta \)-hyperbolicity, there is some subsegment \( \tilde{f}'' \) of an edge of \( \tilde{a} \cup [\tilde{a}(1), \tilde{v}_0] \) that boundedly fellow travels a very long segment of \( \tilde{e}'' \). By convexity,

\[
d_{\mathbb{H}^n}(x, \text{hull}_\epsilon(\Gamma)) \leq d_{\mathbb{H}^n}(\tilde{v}_0, \text{hull}_\epsilon(\Gamma)), \quad \forall x \in [\tilde{a}(1), \tilde{v}_0].
\]

So, as \( \tilde{e}'' \) lies much further from \( \text{hull}_\epsilon(\Gamma) \) than \( \tilde{v}_0 \), it cannot be that \( \tilde{f}'' \subset [a, b] \). Therefore, \( \tilde{f}'' \) is part of an edge of \( \tilde{X} \). As illustrated in Figure 3, there is then an edge move on \( \tilde{X} \) that dramatically decreases length. 

\[\square\]
5.4 Relative length

We now introduce the notion of relative length referenced in §2, first in the equivariant setting and then for carrier graphs in 3-manifolds.

Suppose that \( \tilde{f} : \tilde{X} \rightarrow \mathbb{H}^n \) is a \((\Delta, \Gamma)\)-equivariant carrier graph with geodesic edges, where \( \Gamma < \text{Isom}(\mathbb{H}^n) \). If \( Y \) is a subgraph of \( \tilde{X} \) that is stabilized cocompactly by some subgroup of \( \Delta \), then \( \tilde{f} \) restricts to an equivariant carrier graph \( \tilde{Y} \rightarrow \mathbb{H}^n \) for the subgroup \( \Gamma_{\tilde{Y}} < \Gamma \) that stabilizes \( \tilde{f}(\tilde{Y}) \).

**Definition 5.9 (Equivariant relative edge length).** If a vertex of an edge \( \tilde{e} \) of \( \tilde{X} \) lies in \( \tilde{Y} \), we define the length of \( \tilde{e} \) relative to \( \tilde{Y} \) and \( \epsilon \) to be the hyperbolic length of the part of \( \tilde{e} \) that lies outside of a radius-1 neighborhood of \( \text{hull}_\epsilon(\tilde{f}(\tilde{Y}), \Gamma_{\tilde{Y}}) \).

Moreover, if \( \tilde{Y}_0 \) and \( \tilde{Y}_1 \) are both subgraphs as above and the two vertices of \( \tilde{e} \) lie in \( \tilde{Y}_0 \) and \( \tilde{Y}_1 \), respectively, we define the length of \( \tilde{e} \) relative to \( \tilde{Y}_0, \tilde{Y}_1 \) and \( \epsilon \) as the hyperbolic length of the part of \( \tilde{e} \) that lies outside the radius-1 neighborhoods of both of the associate \( \epsilon \)-thickened convex hulls.

Note that the length of an edge of \( \tilde{Y} \) relative to \( \tilde{Y} \) is zero. However, we are really interested in the case that \( \tilde{e} \) is an edge of \( \tilde{X} \setminus \tilde{Y} \) but has a vertex in \( \tilde{Y} \).

Let’s extend this definition to the non-equivariant case. Suppose that

\[ f : X \rightarrow M \]

is a carrier graph with geodesic edges in a hyperbolic \( n \)-manifold \( M \) and that \( Y \subset X \) is some subgraph. If \( e \) is an edge of \( X \) that is adjacent to \( Y \), lift \( e \) to an edge \( \tilde{e} \) in an equivariant carrier graph \( \tilde{f} : \tilde{X} \rightarrow \mathbb{H}^n \), where \( \pi : \tilde{X} \rightarrow X \) is the universal cover. We define the length of \( e \) relative to \( Y \) and \( \epsilon \), written

\[ \text{length}_{Y,\epsilon}(e) \]
as the length of \( \tilde{e} \) relative to \( \epsilon \) and any components of \( \pi^{-1}(Y) \) that \( \tilde{e} \) touches. The relative length of a subgraph \( Z \subset X \), written \( \text{length}_{Y, \epsilon}(Z) \), is then defined to be the sum of the relative lengths of its edges.

This definition is a slight extension of that given in [3], where it was always assumed that \( \epsilon = 1 \). This restriction does make a real difference in relative length, not just a difference that is bounded by a function of \( \epsilon \). However, as you might expect\(^1\), the choice \( \epsilon = 1 \) was arbitrary and the proof from [3] of the following proposition, previously stated in §2, goes through unchanged.

**Proposition 2.2** (Chains of Bounded Length, [3]). Fixing \( \epsilon > 0 \), if \( M \) is a closed hyperbolic \( n \)-manifold and \( f : X \rightarrow M \) is a carrier graph with minimal length rel vertices, there is a sequence of (possibly disconnected) subgraphs

\[ \emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X \]

such that \( \text{length}_{Y_i, \epsilon}(Y_{i+1}) < L \) for all \( i \), where \( L = L(\epsilon, n) \).

Note the assumption in Proposition 2.2 that \( M \) is closed. Without this, \( M \) could be the quotient of \( \mathbb{H}^n \) by a Schottky group generated by two hyperbolic-type isometries that both have large translation distance and whose axes are far apart. In this case, a minimal length, rank 2 carrier graph in \( M \) might look like the union of the two closed geodesics in \( M \) corresponding to the generators and a geodesic segment connecting them, modified so that all angles are \( 2\pi/3 \). In particular, there may not be any short edges at all. The basic idea of the proof is to show inductively that if there are no edges in \( X \setminus Y_i \) with short relative length, then \( \pi_1 M \) splits as a free product of the images of the fundamental groups of the components of \( Y_i \), which is impossible if \( M \) is closed.

### 6 The proof of Lemma 2.3

To remove a level of notational complexity, we assume in this section that our carrier graphs are embedded. This does not simplify any of the proofs, but allows us to omit reference to \( f \) and \( \tilde{f} \), which makes the arguments more readable.

We also encourage the reader to read through §5 before continuing, since we will make heavy use of notation introduced there.

Let’s recall the setup from Section 2. Fix a homeomorphism \( \phi : S \rightarrow S \) of a closed surface of genus \( g \), and suppose that \( X \hookrightarrow M_\phi \) is a carrier graph that has minimal length among all carrier graphs with \( \text{rank}(\pi_1 X) \leq 2g \). From Proposition 2.2, we have a sequence of subgraphs

\[ \emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X \]

such that \( \text{length}_{Y_i, \epsilon}(Y_{i+1}) < L \) for all \( i \), where \( L = L(\epsilon, n) \) is some constant. Let \( d_\epsilon \) be the pseudometric on \( M_\phi \) obtained by electrifying its \( \epsilon \)-thin parts.

The remaining task is to prove Lemma 2.3, rephrased with \( X \) embedded.

\(^1\)It is a well known meta-theorem in geometry that whenever ‘1’ appears in a theorem statement, a similar statement arises if 1 is replaced by any other positive real number.
Lemma 2.3. Fixing $i$, suppose that for each connected component $Y^i_j$ of $Y_i$, the image of $\pi_1Y^i_j$ is an infinite index subgroup of $\pi_1S \subset \pi_1M_\phi$.

Then the $d_\epsilon$-diameter of every component of $Y_{i+1}$ in $M_\phi$ is bounded above by a constant depending on $g, \epsilon$ and the $d_\epsilon$-diameters of the components of $Y_i$.

Proof. We may assume that $X$ is ‘reduced’ (no edge maps to a point) and does not have any valence 1 or 2 vertices. By Proposition 5.7, $X$ is trivalent. As $\text{rank}(\pi_1X) \leq 2g$, $X$ has at most $3g - 3$ edges. So, it suffices to bound the $d_\epsilon$-diameter of each edge $e$ of $Y_{i+1}$ individually.

We know that the length of $e$ is bounded relative to $Y_i$ and $\epsilon$. By definition, the parts of $e$ that do not count towards relative length lie in the projections to $M_\phi$ of the $\epsilon$-thickened convex hulls of the components of the preimage of $Y_i$ in universal cover of $M_\phi$, which we identify with $\mathbb{H}^3$. So, to bound the $d_\epsilon$-diameter of $e$, it suffices to bound the $d_\epsilon$-diameters of these projections.

Let $\tilde{Z}$ be a component of the preimage of $Y_i$ in $\mathbb{H}^3$ and let $\Gamma$ be the group of deck transformations of $\mathbb{H}^3 \rightarrow M_\phi$ that stabilize $\tilde{Z}$, so that $\tilde{Z}/\Gamma$ is identified with a component $Z \subset Y_i$. By Proposition 5.5 and Lemma 5.3,

$$ (*) \quad \text{hull}_\epsilon(\tilde{Z}, \Gamma) \subset N_D(\text{hull}(\Gamma) \cup \text{Thin}_\epsilon(\Gamma) \cup \tilde{Z}), $$

for some constant $D = D(\epsilon, g)$, where the dependence on $g$ is since Proposition 5.5 requires a bound on the number of edges of $Z$, which is at most $3g - 3$.

$\text{Thin}_\epsilon(\Gamma)$ projects into the $\epsilon$-thin part of $M_\phi$, so its $d_\epsilon$-diameter is zero. The constant in the statement of the lemma is supposed to depend on the $d_\epsilon$-diameter of the components of $Y_i$, one of which is the projection $Z$ of $\tilde{Z}$ to $M_\phi$. So, as the union in $(*)$ is coarsely connected (Proposition 5.5) we have reduced the lemma to bounding the $d_\epsilon$-diameter of the projection of $\text{hull}(\Gamma)$ to $M_\phi$.

Consider the cover $N = \mathbb{H}^3/\Gamma$ of $M_\phi$. As the group $\Gamma$ is isomorphic to the image of $\pi_1(Z)$ in $\pi_1M_\phi$, it is an infinite index subgroup of a surface group, and therefore is free. By the Tameness Theorem [1, 8], $N$ is homeomorphic to the interior of a handlebody. Canary’s covering theorem [10] implies that a one ended 3-manifold whose end is degenerate cannot cover a closed manifold. So, the convex core $\text{core}(N)$, which is the projection of $\text{hull}(\Gamma)$ to $N$, is compact.

Let $\pi : N \rightarrow M_\phi$ be the covering map. It suffices to show that $\pi(\text{core}(N))$ has $d_\epsilon$-diameter bounded by a constant depending on $g, \epsilon$ and $\text{diam}_\epsilon(Z)$.

We first show that the projection $\pi(\partial \text{core}(N))$ has bounded $d_\epsilon$-diameter. By work of Thurston [20], the intrinsic metric on $\partial \text{core}(N)$ is hyperbolic. The components of the $\epsilon$-thick part of $\partial \text{core}(N)$ then have bounded diameter, so the same is true of their projections in $M_\phi$. The $\epsilon$-thin parts of $\partial \text{core}(N)$ come in two flavors: either the core curve is essential in $N$ or it is not. $\pi$ sends thin parts of the first type into the $\epsilon$-thin part of $M_\phi$, so their projections have zero $d_\epsilon$-diameter. Therefore, to bound the $d_\epsilon$-diameter of $\pi(\partial \text{core}(N))$ it suffices to deal with the compressible $\epsilon$-thin parts.

The graph $Z$ lifts homeomorphically to a reduced carrier graph $\tilde{Z} \hookrightarrow N$ with minimal length rel vertices. By Proposition 5.8, we have

$$ \text{core}(\tilde{Z}) \subset N_C(\text{core}(N)), $$

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where \( \text{core}(\hat{Z}) \) is the Stallings core of \( \hat{Z} \) and \( C = C(\epsilon, g) \). Let \( A \subset \partial \text{core}(N) \) be a compressible \( \epsilon \)-thin part and let \( p \in A \). There is a properly embedded, bounded diameter disk \( D \subset N_\epsilon(\text{core}(N)) \) with \( p \in D \): the isoperimetric inequality gives a properly embedded disk in \( \text{core}(N) \) whose boundary passes through \( p \), and this can be extended by an annulus from \( \partial \text{core}(N) \) to \( \partial N_\epsilon(\text{core}(N)) \). As the inclusion \( \hat{Z} \hookrightarrow N_\epsilon(\text{core}(N)) \) is \( \pi_1 \)-surjective, \( \hat{Z} \) must intersect \( D \), and therefore passes within bounded distance of \( p \) (see Figure 4). Thus, the \( d_\epsilon \)-diameter of the projection \( \pi(A) \) is only boundedly bigger than the \( d_\epsilon \)-diameter of \( Z \).

We now know that \( \pi(\partial \text{core}(N)) \) has bounded \( d_\epsilon \)-diameter, and we would like to say the same about \( \pi(\text{core}(N)) \). If \( p \in \pi(\text{core}(N)) \), then by Corollary 3.4 there is an embedding \( f : S \rightarrow M_\phi \) in the homotopy class of the fiber whose image has bounded \( d_\epsilon \)-diameter. We claim that the image \( f(S) \) intersects \( \pi(\partial \text{core}(N)) \); as \( p \) is arbitrary, this will show that \( \pi(\text{core}(N)) \) lies in a bounded \( d_\epsilon \)-neighborhood of \( \pi(\partial \text{core}(N)) \), finishing the proof.

Let \( \hat{M}_\phi \) be the cyclic cover of \( M_\phi \) corresponding to the fundamental group of the fiber and let \( \hat{\pi} : N \rightarrow \hat{M}_\phi \) be the covering map. We may identify

\[
\hat{M}_\phi \cong S \times \mathbb{R}
\]

such that the preimage of \( f(S) \) is \( \bigcup_{i \in \mathbb{Z}} S \times \{i\} \). As \( f(S) \) intersects \( \pi(\text{core}(N)) \), for some \( i \in \mathbb{Z} \) the surface \( S \times \{i\} \) intersect \( \hat{\pi}(\text{core}(N)) \). By the extreme value theorem, the frontier of \( \hat{\pi}(\text{core}(N)) \) intersects both \( S \times (-\infty, i] \) and \( S \times [i, \infty) \). However, this frontier is \( \hat{\pi}(\partial \text{core}(N)) \), which is connected, so \( \hat{\pi}(\partial \text{core}(N)) \) must intersect \( S \times \{i\} \). Projecting to \( M_\phi \), \( f(S) \) intersects \( \pi(\partial \text{core}(N)) \) as desired. \( \square \)

**References**


[22] ______, Hyperbolic structures on 3-manifolds. II. Surface groups and 3-manifolds fibering over the circle, preprint (1998).