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THE UNIVERSITY OF CHICAGO

ALGEBRA VERSUS GEOMETRY IN HYPERBOLIC 3-MANIFOLDS

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CHAPTER 1
INTRODUCTION

Mostow's rigidity theorem states that a closed hyperbolic n-manifold M is determined up to isometry by the isomorphism class of its fundamental group, as long as $n \geq 3$. This means that in some sense it is equivalent to know M through its algebra, its topology or its geometry. For this to be effectively true, though, one should be able to translate concrete information about M given in the language of one of these categories into the language of any other. This problem has attracted a fair amount of attention recently, particularly in dimension 3 where the theory of hyperbolic manifolds is richest ([19], [48], [49], [29], [62]).

The primary goal of this thesis is investigate how the rank of the fundamental group of a closed hyperbolic 3-manifold is reflected in its geometry. Answers will usually come as limits on the size and complexity of the manifold and will sometimes be phrased as finiteness results about the number of manifolds with fixed rank and certain geometric constraints.

1.1 Rank and fibered hyperbolic 3-manifolds

Let Σ_g be the closed orientable surface of genus g and $\phi : \Sigma_g \to \Sigma_g$ a homeomorphism. We can construct a 3-manifold M_ϕ, the *mapping torus* of ϕ, as the quotient space

$$M_\phi = \Sigma_g \times [0,1]/\sim, \quad (x,0) \sim (\phi(x),1).$$

The manifold M_ϕ fibers over the circle with fiber Σ_g; conversely, any closed 3-manifold fibering over the circle is the mapping torus of some surface homeomorphism. Thurston [60] has proven that if the map $\phi : \Sigma_g \to \Sigma_g$ is pseudo-anosov then M_ϕ can be given a hyperbolic metric.

The fundamental group of M_ϕ is given by a semi-direct product

$$1 \to \pi_1(\Sigma_g) \to \pi_1(M_\phi) \to \mathbb{Z} \to 1.$$

Since $\operatorname{rank}(\pi_1(\Sigma_g)) = 2g$ it follows that $\operatorname{rank}(\pi_1(M_\phi)) \leq 2g+1$. It is not hard to construct examples where this inequality is strict, but if the geometry of the manifold is sufficiently complicated then this is not the case.

Theorem 1.1.1. *Given $\epsilon > 0$ and a closed orientable surface Σ_g, there is some $D > 0$ with the following property. Let M be an ϵ-thick hyperbolic 3-manifold fibering over S^1 with fiber Σ_g. If the diameter of M is at least D, then* $\operatorname{rank}(\pi_1(M)) = 2g+1$.

This extends a previous result of Souto [57], which guarantees that $\operatorname{rank}(\pi_1(M)) = 2g+1$ whenever the monodromy ϕ is a sufficiently high power of a pseudo-Anosov map. Recall that ϵ-thick simply means that the injectivity radius of M is at least ϵ. It is worth noting that if ϵ is small then there are many hyperbolic 3-manifolds fibering over the circle that are ϵ-thick. In fact, if $\phi_1, \ldots, \phi_n$ are pseudo-anosov homeomorphisms of Σ_g, then for large m the maps $\phi_1^m, \ldots, \phi_n^m$ freely generate a subgroup of $\operatorname{Mod}(\Sigma_g)$ all of whose elements are monodromies of mapping tori with a common lower bound on injectivity radius. This follows from combining work of Farb-Mosher [31, Theorem 1.4], Kent-Leininger [38, Theorem 1.2] and Rafi [54, Theorem 1.6]. It also demonstrates that Theorem 1.1.1 is strictly stronger than Souto's earlier result.

1.2 Carrier graphs and 3-manifolds with bounded rank

Theorem 1.1.1 relies heavily on a tool introduced by White [62] to gain geometric control over generating sets for the fundamental groups of hyperbolic manifolds. Namely, a *carrier graph* is a π_1-surjective immersed graph $X \looparrowright M$; usually we assume that $\operatorname{rank}(\pi_1 X) = \operatorname{rank}(\pi_1 M)$ as well. A carrier graph is said to have *minimal length* if its total edge length, with respect to the hyperbolic metric on M, is smaller than that of any other such graph.

The following decomposition of minimal length carrier graphs is the key technical element in theorems above. It is originally due to Souto [57], in a different form, but his proof was somewhat incomplete. We will present a full proof of a more general statement in Chapter 3.

Proposition 1.2.1 (Chains of Bounded Length)**.** *Let M be a closed hyperbolic 3-manifold with $X \looparrowright M$ a minimal length carrier graph. Then we have a sequence of (possibly disconnected) subgraphs*

$$\emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X$$

such that the length of any edge in $Y_{i+1} \setminus Y_i$ is bounded above by some constant depending only on $\mathrm{inj}(M)$*,* $\mathrm{rank}(\pi_1 M)$*,* $\mathrm{length}(Y_i)$ *and the diameters of the convex cores of the covers of M corresponding to $\pi_1(Y_i^j)$, where $Y_i^1, \ldots, Y_i^n$ are the connected components of Y_i.*

This strengthens a result of White [62], which says that a minimal length carrier graph must have a cycle with length bounded by some function of $\mathrm{rank}(\pi_1 M)$. The proof is a slightly complicated argument in elementary hyperbolic geometry.

Ultimately, one would like to produce from the decomposition above some sort of global combinatorial description of the geometry of closed hyperbolic 3-manifolds of a certain rank. For instance, one of the most basic consequences of Proposition 1.2.1 is that a minimal length carrier graph contains some number of short subgraphs that are connected together with long edges. At least if M has injectivity radius bounded below, we expect that the short subgraphs and long edges should reflect the geometry of underlying parts of M.

Conjecture 1. *Assume that M is a closed hyperbolic* 3*-manifold with injectivity radius* $\mathrm{inj}(M) \geq \epsilon$ *and* $\mathrm{rank}(\pi_1 M) \leq k$*. Then there are disjoint, compact 3-dimensional submanifolds $N_1, \ldots, N_l \subset M$ with the following properties.*

- *The number l of these submanifolds is bounded above by some constant depending on k. Each N_i has diameter bounded above in terms of k, ϵ, and is homeomorphic to some manifold on a finite list that also depends only on k, ϵ.*

- *Every component of $M \setminus \bigcup_i N_i$ is homeomorphic to a product $\Sigma_g \times (0,1)$ for some g. The homeomorphism can be chosen so that the level surfaces $\Sigma_g \times \{t\}$ have both diameter and area bounded above by a constant depending on ϵ.*

One should visualize the submanifolds N_i as small *building blocks* and the complementary components as *long product regions* that connect them. Here, 'long' refers to the distance between the two components of the frontier of a product region. Each short subgraph of a minimal length carrier graph in M should lie inside of one of these building blocks and each long edge should run through a product region.

In the special case $k = 2$, Conjecture 1 follows from unpublished work of Agol. The case $k = 3$ was completed by Souto using similar arguments, but the general case is significantly more complicated. The following result, which is joint work with J. Souto, is a step in that direction.

Proposition 1.2.2. *Assume that (M_i) is a sequence of pairwise distinct hyperbolic 3-manifolds with $\mathrm{inj}(M_i) \geq \epsilon$ and $\mathrm{rank}(\pi_1(M_i)) \leq k$. Then there are base points $x_i \in M_i$ such that, up to passing to a subsequence, the based manifolds (M_i, x_i) converge in the based Gromov-Hausdorff topology to a based hyperbolic 3-manifold (M_∞, x_∞) that has a degenerate end.*

The relation to Conjecture 1 is that a degenerate end of an ϵ-thick hyperbolic 3-manifold has a neighborhood homeomorphic to $\Sigma_g \times [0, \infty)$ in such a way that the level surfaces have bounded diameter and area; in other words, it is an infinitely long product region. So if M_i converges to a manifold with a degenerate end, then for large i one can see long product regions inside of M_i. The Proposition can therefore be interpreted as saying that once M_i becomes too big to be considered a building block on its own, it must contain a product region.

1.3 The Laplacian and arithmetic manifolds

Proposition 1.2.2 has an interesting consequence concerning the first eigenvalue of the Laplacian operator $\Delta(M) : H_{1,2}(M) \to H_{1,2}(M)$. Namely, we will see in

Chapter 5 that if a sequence of hyperbolic 3-manifolds (M_i) converges to a manifold with a degenerate end, then the associated sequence of Cheeger constants must converge to zero. Work of Buser [22] and Cheeger [27] then implies that the same must be true for the first eigenvalues $\lambda_1(M_i)$ of the Laplacian operators $\Delta(M_i)$. As a result,

Theorem 1.3.1. *For every $\epsilon, \delta, k > 0$, there are only finitely many isometry classes of hyperbolic 3-manifolds M with $\mathrm{inj}(M) \geq \epsilon$, $\mathrm{rank}(\pi_1 M) \leq k$ and first eigenvalue of the Laplacian $\lambda_1(M) \geq \delta$.*

A deep result of Vigneras, combined with a lemma of Long-Reid (see Agol [1]), asserts that if a closed hyperbolic 3-manifold M is arithmetic, then it covers some hyperbolic orbifold $\mathcal{O}_i$ with $\lambda_1(\mathcal{O}_i) \geq \frac{3}{4}$. In the arithmetic setting, this allows us to remove the assumption on the Laplacian from the above theorem if we weaken the conclusion to a finiteness of commensurability classes.

Corollary 1.3.2. *For all ϵ and k positive, there are only finitely many commensurability classes of closed arithmetic hyperbolic 3-manifolds M with $\mathrm{inj}(M) \geq \epsilon$ and $\mathrm{rank}(\pi_1(M)) \leq k$.*

Similar methods, combined with Theorem 1.1.1, also prove the following.

Corollary 1.3.3. *Given $\epsilon, k > 0$, there are only finitely many arithmetic closed hyperbolic 3-manifolds with $\mathrm{rank}(\pi_1 M) = k$ and $\mathrm{inj}(M) \geq \epsilon$ that do not fiber over the circle with fiber $\Sigma_{\frac{k-1}{2}}$ (k odd), or fiber over the orbifold $\mathbb{S}^1/(z \mapsto -z)$ with non-singular fiber Σ_{k-2} (k even).*

Recall that the geometric version of Lehmer's Conjecture states that there is some universal lower bound for the length of any closed geodesic in an arithmetic hyperbolic 3-manifold. If this were true, the above Corollaries would become finiteness statements for arithmetic 3-manifolds of bounded rank.

1.4 Algebraic and geometric limits

In the last part of this thesis, we present some key technical results about algebraic and geometric limits, all joint with J. Souto, that have applications to Conjecture 1.

Let Γ is a finitely generated group and define $\mathcal{D}(\Gamma)$ to be the set of all representations $\rho : \Gamma \to \mathrm{PSL}_2\,\mathbb{C}$ with discrete, torsion free and non-abelian image. We say that a sequence (ρ_i) in $\mathcal{D}(\Gamma)$ converges *algebraically* to a representation $\rho : \Gamma \to \mathrm{PSL}_2\,\mathbb{C}$ if it does so pointwise. The goal of this section is to investigate the relationship between the algebraic convergence of such a sequence and the *geometric convergence* of the images $\rho_i(\Gamma) \subset \mathrm{PSL}_2\,\mathbb{C}$. Recall that a sequence of closed subgroups (G_i) of $\mathrm{PSL}_2\,\mathbb{C}$ converges to a subgroup $G \subset \mathrm{PSL}_2\,\mathbb{C}$ geometrically if it does in the Chabauty topology.

Assume from now on that (ρ_i) is a sequence in $\mathcal{D}(\Gamma)$ converging algebraically to a representation $\rho \in \mathcal{D}(\Gamma)$, and that the groups $\rho_i(\Gamma)$ converge geometrically to a subgroup $G < \mathrm{PSL}_2\,\mathbb{C}$. For convenience, we will often say that ρ_i converges geometrically to G. It is not hard to see that $\rho(\Gamma) \subset G$, and in fact the inclusion may be strict. The first examples of this are due to Jorgensen [35]. Later, Thurston [60] constructed an algebraically convergent sequence in $\mathcal{D}(\pi_1(\Sigma_g))$ that converges geometrically to a subgroup of $\mathrm{PSL}_2\,\mathbb{C}$ that is not even finitely generated. Other examples, each one dramatic in its own way, were constructed by Kerckhoff-Thurston [39], Anderson-Canary [4] and Brock [20].

All these examples are related to the appearance of new parabolic elements in the algebraic limit; in fact, the following holds:

Theorem 1.4.1 (Anderson-Canary [5]). *Let Γ be a finitely generated group and assume that (ρ_i) is a sequence of faithful representations in $\mathcal{D}(\Gamma)$ converging algebraically to some $\rho \in \mathcal{D}(\Gamma)$. If $\rho(\Gamma)$ does not contain parabolic elements, then the groups $\rho_i(\Gamma)$ converge geometrically to $\rho(\Gamma)$.*

In [6], Anderson and Canary extended this result to the case where ρ and ρ_i map the same elements to parabolics for all i. Evans proved in [30] that the same conclusion holds under the weaker assumption that if an element of Γ is sent to a parabolic by ρ then it is also parabolic in ρ_i for all i. All these results were obtained in the presence of certain technical assumptions rendered unnecessary by work of Brock and Souto [16], and later by the resolution of the tameness conjecture by Agol [2] and Calegari-Gabai [23].

While attempting to prove Conjecture 1, we revisited Theorem 1.4.1 convinced that it would remain true after dropping the assumption that the representations ρ_i are faithful. To our surprise, we found the following examples showing that Theorem 1.4.1 fails dramatically in this more general setting.

Example 1.4.1. Let Γ be the fundamental group of a closed surface of genus 3. There is a sequence of representations (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to a faithful representation ρ and geometrically to a subgroup $G \subset \mathrm{PSL}_2\,\mathbb{C}$, such that

- G does not contain any parabolic elements.
- $\rho(\Gamma)$ has index 2 in G.

Example 1.4.2. Let Γ be the fundamental group of a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3. There is a sequence of representations (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to a faithful representation ρ and geometrically to a group G, such that

- G does not contain any parabolic elements.
- $\rho(\Gamma)$ has infinite index in G.

Example 1.4.3. Let Γ be the fundamental group of a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3. There is a sequence of representations (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to a faithful representation ρ and geometrically to a group G, such that

- $\rho(\Gamma)$ does not contain any parabolic elements.
- G is not finitely generated.

The reader may find it surprising that we mention Example 1.4.1 at all; the other two seem to be much more dramatic. However, Example 1.4.1 shows that the following theorem of Anderson fails if one considers unfaithful representations:

Theorem 1.4.2 (Anderson)**.** *Assume that* Γ *is a finitely generated group, and that* (ρ_i) *is a sequence of faithful representations in* $\mathcal{D}(\Gamma)$ *converging algebraically to some representation* ρ *and geometrically to a subgroup* G *of* $\mathrm{PSL}_2\,\mathbb{C}$*. Then no element of* $G \backslash \rho(\Gamma)$ *has a nonzero power that lies in* $\rho(\Gamma)$*. In particular,* $\rho(\Gamma)$ *cannot have finite index inside of* G *unless the two groups coincide.*

Apart from discussing the examples above, our goal is to understand the extent of Theorem 1.4.1's failure for unfaithful sequences and to provide some useful substitutes. Our first result in this direction is the following.

Theorem 1.4.3. *Let* Γ *be a finitely generated group and* (ρ_i) *a sequence in* $\mathcal{D}(\Gamma)$*. Assume that* (ρ_i) *converges algebraically to a representation* ρ *and geometrically to a subgroup* G *of* $\mathrm{PSL}_2\,\mathbb{C}$*. If*

- $\rho(\Gamma)$ *does not contain parabolic elements, and*
- *no element of* $G \setminus \rho(\Gamma)$ *has a nonzero power in* $\rho(\Gamma)$*,*

then $G = \rho(\Gamma)$*.*

Together, Theorems 1.4.2 and 1.4.3 imply Anderson-Canary's theorem above. In fact, our proof of Theorem 1.4.3 is quite different from and considerably simpler than Anderson-Canary's argument. One reason for this is that they use some fairly involved arguments to bypass the question of tameness, while we use here the resolution of the tameness conjecture by Agol [2] and Calegari-Gabai [23] in a crucial way.

Our last result is key to the study of Conjecture 1. Essentially, the Conjecture asserts a uniformity in the geometry of closed ϵ-thick hyperbolic 3-manifolds with bounded rank. To prove that a certain class of manifolds has uniform geometry, one commonly takes a sequence of such manifolds and shows that any geometric limit has controlled geometry. The following result, combined with the Tameness Theorem of Agol [2] and Calegari-Gabai [23], supplies this control in many situations. Note that the assumption that G does not contain parabolics is always satisfied for a sequence of ϵ-thick manifolds.

Theorem 1.4.4. *Let Γ be a finitely generated group and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$. Assume that (ρ_i) is algebraically convergent and converges geometrically to a subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$. If G does not contain parabolic elements, then G is finitely generated.*

Note that Example 1.4.3 shows that for G to be finitely generated it does not suffice to assume that $\rho(\Gamma)$ has no parabolics. Also, Example 1.4.2 shows that under the assumptions of Theorem 1.4.4 the algebraic limit $\rho(\Gamma)$ can have infinite index in the geometric limit G.

1.5 Organization

The remainder of this thesis is organized into five chapters. The first, Chapter 2, serves mainly to summarize some well-known results about the geometry of hyperbolic 3-manifolds. Chapter 3 develops the machinery of carrier graphs introduced above. Next, in Chapter 4 we study hyperbolic 3-manifolds fibering over the circle and prove Theorem 1.1.1. Chapter 5 combines a proof of Proposition 1.2.2 with its consequences concerning the first eigenvalue of the Laplacian and arithmetic 3-manifolds. We end with Chapter 6, in which we study the relationship between algebraic and geometric limits of representations into $\mathrm{PSL}_2\,\mathbb{C}$.

CHAPTER 2
PRELIMINARIES

In this chapter we recall some well-known facts about hyperbolic 3-manifolds.

2.1 Hyperbolic manifolds

A (complete) hyperbolic 3-manifold is a Riemannian 3-manifold isometric to $\mathbb{H}^3/\Gamma$, where Γ is a discrete, torsion-free group of isometries of hyperbolic 3-space. We will usually assume that the elements of Γ are orientation preserving, or equivalently that $\mathbb{H}^3/\Gamma$ is orientable. The full group of orientation preserving isometries of hyperbolic 3-space is written $\mathrm{Isom}_+(\mathbb{H}^3)$, and is often identified with $\mathrm{PSL}_2\,\mathbb{C}$ through its action on the boundary of $\mathbb{H}^3$.

Conjugate subgroups of $\mathrm{PSL}_2\,\mathbb{C}$ give isometric quotients of $\mathbb{H}^3$; in order to remove this indeterminacy we consider *pointed* hyperbolic 3-manifolds, i.e. pairs (M, ω) where ω is an orthonormal frame of some tangent space of M. Choosing once and for ever a fixed frame $\omega_{\mathbb{H}^3}$ of some tangent space of $\mathbb{H}^3$, every quotient manifold $\mathbb{H}^3/\Gamma$ has an induced framing $\omega_{\mathbb{H}^3/\Gamma}$ given by the projection of $\omega_{\mathbb{H}^3}$. Now, if (M, ω) is a pointed hyperbolic 3-manifold then there is a unique $\Gamma \subset \mathrm{PSL}_2\,\mathbb{C}$ such that the manifolds (M, ω) and $(\mathbb{H}^3/\Gamma, \omega_{\mathbb{H}^3/\Gamma})$ are isometric as pointed manifolds.

Remark. It would be more natural to speak of *framed* hyperbolic 3-manifolds instead of pointed; however, it is customary to use the given terminology.

2.2 Simplicial Hyperbolic Surfaces

We recall here some facts about negatively curved surfaces in hyperbolic 3-manifolds.

Definition 2.2.1. Let M be a hyperbolic 3-manifold. A *simplicial hyperbolic surface* in M is a map $f : S \to M$, where

- S is a closed surface equipped with a triangulation T
- f maps each face of T to a totally geodesic triangle in M
- for each vertex $v \in T$ the angles between the f-images of the edges adjacent to v sum to at least 2π.

If $f : S \to M$ is a simplicial hyperbolic surface then we get a path-metric on S by requiring that f preserves path lengths. The metric is smooth and hyperbolic away from the vertices of T, at which there are possible excesses of angle. By the Gauss-Bonnet Theorem, we have $\mathrm{vol}(S) \leq 2\pi|\chi(S)|$. Bounding the diameter of S by its volume and injectivity radius, we obtain:

Bounded Diameter Lemma (Thurston)**.** Assume $f : S \to M$ is an ϵ-thick simplicial hyperbolic surface of genus g. Then $\mathrm{diam}(S) \leq \frac{4}{\epsilon}(2g-2)$.

Mahler's Compactness Theorem ([9], E.1) states that the moduli space of ϵ-thick (smooth) hyperbolic surfaces of fixed genus is compact. Together with the following Proposition, this provides a number of upper bounds on the geometry of ϵ-thick simplicial hyperbolic surfaces, albeit without explicit constants.

Proposition 2.2.1 (Smooth Dominates Simplicial)**.** *Let S be a closed surface and d a metric on S that is the pullback metric for some simplicial hyperbolic surface. Then there exists a smooth hyperbolic metric d_{hyp} on S such that for all $x, y \in S$*

$$\frac{1}{C} d(x,y) \leq d_{hyp}(x,y),$$

where $C > 0$ depends only on the topological type of S. Note that if d is ϵ-thick then d_{hyp} is $\frac{\epsilon}{C}$-thick.

Proof of Proposition 2.2.1. Working in polar coordinates in small neighborhoods of the singular points of d, we can explicitly deform d to obtain a smooth metric d'

with Gaussian curvature $K \leq -1$ that is bilipschitz to d with bilipschitz constant depending only on the angles d has around the points in its singular locus. The argument is very similar to the proof of the 2π-Theorem of Gromov and Thurston [12], so we will omit it here. Since the Gauss-Bonnet Theorem gives an upper bound for the sum of these singular angles, d and d' are in fact C-bilipschitz for some C depending only on the topological type of S. Define d_{hyp} to be the hyperbolic metric in the conformal class of d'. The Ahlfors-Schwartz Lemma [3] states that distances measured in d' are less than or equal to distances in d_{hyp}; this proves the desired inequality. □

As an application, we can use Proposition 2.2.1 and a based version of Mahler's Compactness Theorem to show:

Corollary 2.2.2 (Short Markings)**.** *Set $\Gamma = \pi_1(\Sigma_g)$ and fix a generating set $X \subset \Gamma$. Then given $\epsilon, g > 0$ there is a constant L such that whenever $f : S \to M$ is an ϵ-thick simplicial hyperbolic surface of genus g and $p \in S$, there is an isomorphism $\Phi : \Gamma \to \pi_1(S, p)$ such that the image of each element of X can be represented by a loop based at p of length less than L.*

Observe that Corollary 2.2.2 is similar to Lemma 7.1 in [24], but it is slightly stronger and more easily applied in our work in Section 4.1.

2.3 Tameness

We will be mainly interested in hyperbolic 3-manifolds M with finitely generated fundamental group. Any such manifold is *tame* by the work of Agol [2] and Calegari-Gabai [23]:

Tameness Theorem (Agol, Calegari-Gabai)**.** Let M be a hyperbolic 3-manifold with finitely generated fundamental group. Then M is homeomorphic to the interior of a compact 3-manifold.

A *standard core* of a hyperbolic 3-manifold M with finitely generated fundamental group is a compact submanifold $C \subset M$ with $M \setminus C$ homeomorphic to $\partial C \times \mathbb{R}$. Observe that M is homeomorphic to the interior of every such compact core. It follows from the Tameness Theorem that every such M admits an exhaustion by nested compact cores. If $C \subset M$ is a standard compact core, then the ends of M correspond naturally to components of $M \setminus C$. The component $U_{\mathcal{E}}$ of $M \setminus C$ corresponding to an end $\mathcal{E}$ is said to be a *standard neighborhood* of $\mathcal{E}$ and the component of ∂C contained in the closure of $U_{\mathcal{E}}$ is said to *face* $\mathcal{E}$. We will often denote the component of ∂C facing $\mathcal{E}$ by $\partial\mathcal{E}$. Observe that $U_{\mathcal{E}}$ is homeomorphic to $\partial\mathcal{E} \times \mathbb{R}$.

2.4 Geometry of ends in the absence of cusps

Let M be a hyperbolic 3-manifold with finitely generated fundamental group and without cusps. An end $\mathcal{E}$ of M is *convex cocompact* if it has a neighborhood in M disjoint from the convex-core $\mathrm{CC}(M)$ of M. Recall that the convex core $\mathrm{CC}(M)$ is the smallest convex submanifold of M whose inclusion is a homotopy equivalence. A manifold with compact $\mathrm{CC}(M)$ is said to be *convex cocompact*; equivalently, all ends of M are convex cocompact.

For every $d > 0$, the set of points in M within distance d of $\mathrm{CC}(M)$ is homeomorphic to M and has strictly convex C^1-boundary. Smoothing the boundary, we obtain the following well-known fact:

Lemma 2.4.1. *Let M be a hyperbolic 3-manifold with finitely generated fundamental group. There is an exhaustion of M by a nested sequence of submanifolds K_i such that:*

1. *The boundary ∂K_i is smooth and strictly convex. Here, strictly convex means that $\langle \nabla_X \nu, X\rangle > 0$, where ∇ is the Levi-Civita connection, X is a vector tangent to ∂K_i and ν is the outer normal field along ∂K_i.*

2. *Every convex cocompact end of M has a neighborhood disjoint of K_i.*

3. The inclusion of K_i into M is a homotopy equivalence.

Continuing with the same notation as in Lemma 2.4.1, convexity implies that there is a well-defined map $\kappa_{K_i} : M \to K_i$ that takes a point in M to the point in K_i closest to it. Strict convexity implies that the preimage of a point $x \in \partial K_i$ under this projection is a geodesic ray. It follows that the map

$$M \setminus K_i \to \partial K_i \times (0, \infty), \quad x \mapsto (\kappa_{K_i}(x), d(x, \kappa_{K_i}(x))$$

is a diffeomorphism; in fact, its inverse is the radial coordinate map

$$\partial K_i \times (0, \infty) \to M \setminus K_i, \quad (x, t) \mapsto \exp_x(t\nu(x)),$$

where ν is the outer unit normal vector-field along ∂K_i.

An end $\mathcal{E}$ which is not convex cocompact is said to be *degenerate*. It follows from the Tameness Theorem and earlier work of Bonahon [13] and Canary [24] that degenerate ends have very well-behaved geometry. For instance, every degenerate end $\mathcal{E}$ has a neighborhood which is completely contained in the convex core $\mathrm{CC}(M)$. The prototypical examples of such ends come from unwrapping 3-manifolds fibering over the circle:

Example 2.4.1. Let M_ϕ be the mapping torus of a pseudo-anosov map $\phi : \Sigma_g \to \Sigma_g$. As mentioned in the introduction, $\pi_1(M_\phi)$ decomposes as

$$1 \to \pi_1(\Sigma_g) \to \pi_1(M_\phi) \to \mathbb{Z} \to 1.$$

Let N be the cyclic cover of M_ϕ corresponding to the subgroup $\pi_1(\Sigma_g)$. Then N is homeomorphic to $\Sigma_g \times \mathbb{R}$, and since it regularly covers a closed manifold we have $\mathrm{CC}(N) = N$, implying that both ends of N are degenerate. Note that unwrapping a fiber bundle structure for M_ϕ gives a product structure $N \cong \Sigma_g \times \mathbb{R}$ with fibers of bounded diameter, contrasting with the exponential growth of level surfaces in a convex-cocompact end.

From our point of view, the most important fact about degenerate ends is the Thurston-Canary [25] covering theorem, of which we state the following weaker version:

Covering theorem (Thurston-Canary)**.** Let M and N be non-compact hyperbolic 3-manifolds, assume that M has finitely generated fundamental group and no cusps, let $\pi : M \to N$ be a Riemannian cover and let $\mathcal{E}$ be a degenerate end of M. Then $\mathcal{E}$ has a standard neighborhood $U_{\mathcal{E}}$ such that the restriction

$$\pi|_{U_{\mathcal{E}}} : U_{\mathcal{E}} \to \pi(U_{\mathcal{E}})$$

of the covering π to $U_{\mathcal{E}}$ is a covering map onto a standard neighborhood of a degenerate end $\mathcal{E}'$ of N. More precisely, there is a finite covering $\sigma : \partial\mathcal{E} \to \partial\mathcal{E}'$ and homeomorphisms

$$\phi : \partial\mathcal{E} \times \mathbb{R} \to U_{\mathcal{E}}, \quad \psi : \partial\mathcal{E}' \times \mathbb{R} \to \pi(U_{\mathcal{E}})$$

with $(\psi^{-1} \circ \pi|_{U_{\mathcal{E}}} \circ \phi)(x,t) = (\sigma(x), t)$. In particular, the covering $\pi|_{U_{\mathcal{E}}}$ has finite degree.

Combining the Tameness theorem, Lemma 2.4.1 and the Covering theorem we obtain:

Proposition 2.4.2. *Let M and N be hyperbolic 3-manifolds with infinite volume, assume that M has finitely generated fundamental group and no cusps, and let $\pi : M \to N$ be a Riemannian cover. Then M admits an exhaustion by nested standard compact cores $C_i \subset C_{i+1}$ such that the following holds:*

1. *If a component S of ∂C_i faces a convex cocompact end of M then S is smooth and strictly convex.*

2. *If a component S of ∂C_i faces a degenerate end of M then the restriction $\pi|_S : S \to \pi(S)$ is a finite covering onto an embedded surface in N.*

2.5 Conformal boundaries and Ahlfors-Bers theory

Continuing with the same notation as in the previous section, we have a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ with finitely generated fundamental group and no cusps.

Recall that the group Γ acts on $\mathbb{H}^3$ by isometries, and that its *limit set*, written $\Lambda(\Gamma)$, is the closure of the set of fixed points in $\mathbb{S}^2_\infty = \partial\mathbb{H}^3$ of hyperbolic elements of Γ. The complement of $\Lambda(\Gamma)$ is the *domain of discontinuity* $\Omega(\Gamma) = \mathbb{S}^2_\infty \setminus \Lambda(\Gamma)$, which is the largest open subset of $\mathbb{S}^2_\infty$ on which Γ acts properly discontinuously. In fact, Γ acts properly discontinuously on $\mathbb{H}^3 \cup \Omega(\Gamma)$, and the quotient is a manifold with boundary $\overline{M}$ having interior M.

The action of Γ on $\mathbb{S}^2_\infty$ is by Mobius transformations, so $\partial\overline{M}$ inherits a natural conformal structure and is accordingly called the *conformal boundary* of M. Its structure is closely tied with the geometry of M: for instance, the unique hyperbolic metric on $\partial\overline{M}$ compatible with this conformal structure, called the *Poincaré metric*, is similar to the intrinsic metric on $\partial\,\mathrm{CC}(M)$. Specifically, the closest point projection $\kappa : M \to \mathrm{CC}(M)$ extends continuously to a map $\bar{\kappa} : \partial\overline{M} \to \partial\,\mathrm{CC}(M)$, and we have the following theorem of Canary-Bridgeman:

Proposition 2.5.1 (Canary-Bridgeman [15])**.** *For every $\epsilon > 0$ there exists $K > 0$ so that the following holds. Let M be a hyperbolic 3-manifold with finitely generated fundamental group, such that every component of $\partial\overline{M}$ has injectivity radius at least ϵ in the Poincaré metric. Then the closest point projection $\kappa : \partial\overline{M} \to \partial\,\mathrm{CC}(M)$ is K-lipschitz, where $\partial\overline{M}$ has the Poincaré metric and $\partial\,\mathrm{CC}(M)$ is considered with the path metric induced by its inclusion into M.*

The conformal boundary plays an important role in the deformation theory of hyperbolic 3-manifolds; in particular, a convex-cocompact hyperbolic 3-manifold is determined up to isometry by its topology and conformal boundary. A more precise statement of this can be derived as follows. Let M be the interior of a compact hyperbolizable 3-manifold $\overline{M}$ in which each boundary component has negative Euler characteristic. Define $\mathcal{CH}(M)$ to be the set of all convex-cocompact hyperbolic metrics on M, where two metrics are identified if they differ by an isometry isotopic to the identity map. It follows from Thurston's hyperbolization theorem [59] that $\mathcal{CH}(M)$ is nonempty, and it inherits a natural complex structure through its relation to the representation variety $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2\,\mathbb{C})$ (see [43, Section 4.3]). Then we have:

Theorem 2.5.2 (Ahlfors-Bers Parameterization, see [43])**.** *The map $\mathcal{CH}(M) \to \mathcal{T}(\partial\overline{M})$, induced from the map taking a convex-cocompact uniformization of M to its conformal boundary, is a biholomorphic equivalence. Therefore, $\mathcal{CH}(M)$ is a complex manifold of dimension*

$$-\frac{3}{2}\chi(\partial\overline{M}) = -3\chi(M).$$

The space $\mathcal{CH}(M)$ has a convenient naturality with respect to certain coverings. Specifically, if M' is a cover of M with finitely generated fundamental group then tameness and Canary's covering theorem imply that convex-cocompact metrics on M lift to convex-cocompact metrics on M'. In fact,

Lemma 2.5.3. *If M' is a 3-manifold with finitely generated fundamental group and $\tau : M' \to M$ is a covering map, there is a holomorphic map*

$$\tau^* : \mathcal{CH}(M) \to \mathcal{CH}(M')$$

induced by the map taking a hyperbolic structure on M to its pullback under τ.

The point of the holomorphy is that τ^* is related to the holomorphic map

$$\mathrm{Hom}(\pi_1(M'), \mathrm{PSL}_2\,\mathbb{C}) \to \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2\,\mathbb{C})$$

defined by precomposition with $\tau_* : \pi_1(M') \to \pi_1(M)$.

2.6 Geometric convergence

Recall that a sequence (G_i) of closed subgroups of $\mathrm{PSL}_2\,\mathbb{C}$ converges *geometrically* to a subgroup G if it does in the Chabauty topology. More concretely, (G_i) converges geometrically to G if G is the subgroup of $\mathrm{PSL}_2\,\mathbb{C}$ consisting precisely of those elements $g \in \mathrm{PSL}_2\,\mathbb{C}$ such that there are $g_i \in G_i$ with $g_i \to g$ in $\mathrm{PSL}_2\,\mathbb{C}$. In other words, G is the accumulation set of the groups G_i.

Most of our arguments are based on an interpretation of geometric convergence in terms of the quotient manifolds $\mathbb{H}^3/G_i$.

Definition. A sequence (M_i, ω_i) of pointed hyperbolic 3-manifolds converges *geometrically* to a pointed manifold $(M_\infty, \omega_\infty)$ if for every $K \subset M_\infty$ compact with $\omega_\infty \in K$ there is a sequence $\phi_i : K \to M_i$ of smooth maps with $\phi_i(\omega_\infty) = \omega_i$ converging in the C^k-topology to an isometric embedding for all $k \in \mathbb{N}$. We will refer to the maps ϕ_i as the *almost isometric embeddings provided by geometric convergence.*

Remark. Note that although the phrase 'converging in the C^k-topology to an isometric embedding' is suggestive and pleasing to the ear, it has no meaning. One way to formalize this would be to say that for each point $x \in K$ there are isometric embeddings

$$\beta_i : B(\phi_i(x), \epsilon) \to \mathbb{H}^3$$

from ϵ-balls around $\phi_i(x) \in M_i$ so that $\beta_i \circ \phi_i$ converges to an isometric embedding of some neighborhood of $x \in M_\infty$ into $\mathbb{H}^3$.

Recall that by our convention above, a pointed hyperbolic manifold is a manifold together with a base frame and that choosing a base frame $\omega_{\mathbb{H}^3}$ of hyperbolic space we obtain a bijection between the sets of discrete torsion free subgroups of $\mathrm{PSL}_2\,\mathbb{C}$ and of pointed hyperbolic 3-manifolds. Under this identification, the notions of geometric convergence of groups and manifolds are equivalent (see for instance [9, 37]):

Proposition 2.6.1. *Let $G_1, G_2, \ldots, G_\infty$ be discrete and torsion-free subgroups of $\mathrm{PSL}_2\,\mathbb{C}$ and consider for all $i = 1, \ldots, \infty$ the pointed hyperbolic 3-manifold (M_i, ω_i) where $M_i = \mathbb{H}^3/G_i$ and ω_i is the projection of the frame $\omega_{\mathbb{H}^3}$ of $\mathbb{H}^3$. The groups G_i converge geometrically to G_∞ if and only if the pointed manifolds (M_i, ω_i) converge geometrically to $(M_\infty, \omega_\infty)$.*

2.7 Algebraic convergence

Let Γ be a finitely generated group. Recall that a sequence (ρ_i) of representations $\rho_i : \Gamma \to \mathrm{PSL}_2\,\mathbb{C}$ converges *algebraically* to a representation ρ if for every $\gamma \in \Gamma$ we have $\rho_i(\gamma) \to \rho(\gamma)$ in $\mathrm{PSL}_2\,\mathbb{C}$. Jorgensen proved in [36] that if each image $\rho_i(\Gamma)$ is discrete and non-elementary then the same is true of $\rho(\Gamma)$. His argument also shows that torsion cannot suddenly appear in the limit, so we have the following theorem:

Proposition 2.7.1. *Let Γ be a finitely generated group. Then the subset $\mathcal{D}(\Gamma) \subset \mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ consisting of representations with discrete, torsion free and non-elementary image is closed with respect to the algebraic topology.*

When a sequence (ρ_i) converges both algebraically to a representation ρ and geometrically to some group G, it is easy to see that $\rho(\Gamma) \subset G$. In other words, the manifold $\mathbb{H}^3/\rho(\Gamma)$ covers the manifold $\mathbb{H}^3/G$. In particular, given a compact subset $C \subset \mathbb{H}^3/\rho(\Gamma)$ we can project it down to $\mathbb{H}^3/G$ and then map the image to M_i under the almost isometric embeddings given by geometric convergence. This produces maps $C \to M_i = \mathbb{H}^3/\rho_i(\Gamma)$ which look more and more like the restriction of a covering to C.

More generally, assume that H is a finitely generated subgroup of G containing $\rho(\Gamma)$. By the tameness theorem, the manifold $\mathbb{H}^3/H$ contains a standard compact core C_H. Composing the restriction to C_H of the covering $\mathbb{H}^3/H \to \mathbb{H}^3/G$ with the almost isometric embeddings given by geometric convergence, we obtain for sufficiently large i maps $C_H \to M_i$ similar to those described above. Using the induced homomorphisms $H \to \pi_1(M_i, \omega_i)$ one can then construct a sequence of representations $\sigma_i : H \to \mathrm{PSL}_2\,\mathbb{C}$ converging algebraically to the inclusion of $H \hookrightarrow G \hookrightarrow \mathrm{PSL}_2\,\mathbb{C}$. The assumption that $\rho(\Gamma) \subset H$ implies then that $\sigma_i(H) = \rho_i(\Gamma)$ for all i. In particular we deduce (compare with [45, Lemma 4.4]):

Proposition 2.7.2. *Let Γ be a finitely generated group, (ρ_i) a sequence in $\mathcal{D}(\Gamma)$ converging algebraically to a representation ρ_∞ and geometrically to a group $G \subset \mathrm{PSL}_2\,\mathbb{C}$. If $H \subset G$ is a finitely generated subgroup of G containing $\rho(\Gamma)$ then* *there is a sequence of representations* $\sigma_i : H \to \mathrm{PSL}_2\,\mathbb{C}$ *converging to the inclusion* $H \hookrightarrow \mathrm{PSL}_2\,\mathbb{C}$ *with* $\sigma_i(H) = \rho_i(\Gamma)$ *for all sufficiently large* i*. In particular, the groups* $\sigma_i(H)$ *converge geometrically to* G*.*

2.8 Roots

Recall that the elements in $\mathrm{PSL}_2\,\mathbb{C}$ are either hyperbolic, parabolic or elliptic depending on their dynamical behaviour. Every hyperbolic element $\gamma \in \mathrm{PSL}_2\,\mathbb{C}$ stabilizes a geodesic A in $\mathbb{H}^3$, and if $\alpha \in \mathrm{PSL}_2\,\mathbb{C}$ is a k-th root of γ, i.e. $\gamma = \alpha^k$, then $\alpha A = A$. It follows easily that the set of k-th roots of γ is finite. A similar argument applies in the parabolic and elliptic case, so we obtain the following well-known, and in this paper surprisingly important, fact:

Lemma 2.8.1. *For every* $k \in \mathbb{Z}$ *and nontrivial element* $\gamma \in \mathrm{PSL}_2\,\mathbb{C}$*, the set* $\{\alpha \in \mathrm{PSL}_2\,\mathbb{C} \mid \alpha^k = \gamma\}$ *is finite.*

Lemma 2.8.1 has the following immediate consequence:

Corollary 2.8.2. *Let* $\Gamma \subset \Gamma'$ *be groups and assume that* Γ' *contains a subgroup* H *such that (1)* Γ *and* H *generate* Γ' *and (2)* $\Gamma \cap H$ *has finite index in* H*. Then for every faithful representation* $\rho : \Gamma \to \mathrm{PSL}_2\,\mathbb{C}$*, the set* $\{\rho' : \Gamma' \to \mathrm{PSL}_2\,\mathbb{C} \mid \rho'|_\Gamma = \rho|_\Gamma\}$ *is finite.*

Also, Theorem 1.4.2 from the introduction follows almost directly from Lemma 2.8.1.

Theorem 1.4.2 (Anderson)**.** Assume that Γ is a finitely generated group and (ρ_i) is a sequence of faithful representations in $\mathcal{D}(\Gamma)$ that converges algebraically to a representation ρ and geometrically to a group G. Then maximal cyclic subgroups of $\rho(\Gamma)$ are maximally cyclic in G. In particular, if the image $\rho(\Gamma)$ of the algebraic limit has finite index in the geometric limit G, then $\rho(\Gamma) = G$.

Proof. If some maximal cyclic subgroup of $\rho(\Gamma)$ is not maximally cyclic in G, then there is some $g \in G \setminus \rho(\Gamma)$ that powers into it. So, $g^k = \rho(\eta)$ for some $\eta \in \Gamma$ and $k \geq 2$. Since $g \in G$ there are $\gamma_i \in \Gamma$ with $\lim_{i\to\infty} \rho_i(\gamma_i) = g$. Taking powers we have then

$$\lim_{i\to\infty} \rho_i(\gamma_i^k) = g^k = \rho(\eta) = \lim_{i\to\infty} \rho_i(\eta).$$

It follows from the Margulis lemma that $\rho_i(\gamma_i^k) = \rho_i(\eta)$ for sufficiently large i. Since the representations ρ_i are faithful, this implies that $\gamma_i^k = \eta$ for large i. But the group Γ embeds in $\mathrm{PSL}_2\,\mathbb{C}$, by say ρ_7, so Lemma 2.8.1 shows that each of its elements has only finitely many k^{th}-roots. This implies that up to passing to a subsequence we may assume that $\gamma_i = \gamma_j$ for all pairs (i, j), and hence g belongs to the algebraic limit. This is a contradiction, so the claim follows. □

We included the proof of Theorem 1.4.2 here because its failure for non-faithful sequences is the core of the examples presented in Section 6.1.

2.9 Maximal Cusps

Finally, we describe a class of hyperbolic manifolds that we will use as building blocks in constructing our examples in Section 6.1. A good reference for this section is [7].

Assume that M is a compact, orientable, irreducible and atoroidal 3-manifold with interior N and no torus boundary components. If N has a geometrically finite hyperbolic metric, then there is a collection C of disjoint simple closed curves in ∂M such that

$$\overline{N} \cong M \setminus \cup_{\gamma\in C}\gamma,$$

by a homeomorphism whose restriction to N is isotopic to the identity map. The curves in C are then determined up to isotopy, and correspond to the rank one cusps in N. We will say that the collection C has been *pinched.* Each curve in C is homotopically nontrivial in M and no two curves are freely homotopic in M. It follows from Thurston's Hyperbolization Theorem [37, 52] that any collection of curves on ∂M satisfying these two topological properties can be obtained as above from a hyperbolic structure on N; it is therefore said that such a collection is *pinchable.*

One says that a component $S \subset \partial M$ is *maximally cusped* if C contains a pants decomposition for S. In this case, any component of $\partial\,\mathrm{CC}(N)$ that faces S is a totally geodesic hyperbolic thrice-punctured sphere. If we have two hyperbolic 3-manifolds with maximally cusped ends, we can topologically glue their convex cores together along these thrice-punctured spheres. Moreover, every homeomorphism between hyperbolic thrice punctured spheres can be isotoped to an isometry, so by altering the identifications we can ensure that our gluing produces a hyperbolic 3-manifold. Here is a precise description of the result of this process.

Lemma 2.9.1 (Gluing Maximal Cusps)**.** *Let M_1, M_2 be compact, orientable 3-manifolds with interiors N_i that possess geometrically finite hyperbolic metrics. Assume that S_i are maximally cusped components of ∂M_i with pinched pants decompositions $P_i \subset S_i$. Then if $h : S_1 \to S_2$ is a homeomorphism with $h(P_1) = P_2$, there is a hyperbolic 3-manifold N with the following properties:*

- *$N \cong (M_1 \setminus P_1) \sqcup_h (M_2 \setminus P_2)$, so N has a rank 2 cusp corresponding to each element of P_1 (or P_2)*
- *N is the union of two disjoint subspaces isometric to $N_1 \setminus E_1$ and $N_2 \setminus E_2$, where E_i are the components of $N_i \setminus \mathrm{CC}(N_i)$ adjacent to S_i.*

CHAPTER 3
CARRIER GRAPHS

We develop here the technical tool mentioned in the introduction that allows for a geometric understanding of rank for fundamental groups of hyperbolic manifolds. In the following, assume M is a closed hyperbolic 3-manifold.

Definition 3.0.1. A *carrier graph* for M is a graph X and a map $f : X \to M$ which induces a surjection on fundamental groups.

Standing Assumption: In this paper we are interested in generating sets of minimal size, which correspond to carrier graphs with $\text{rank}(\pi_1(X)) = \text{rank}(\pi_1(M))$. From now on all carrier graphs will be assumed to have this property.

If a carrier graph $f : X \to M$ is rectifiable, we can pull back path lengths in M to obtain an pseudo-metric on X. Collapsing to a point each zero-length segment in X yields a new carrier graph with an actual metric; from now on we will assume all carrier graphs are similarly endowed. Define the *length* of a carrier graph to be the sum of the lengths of its edges, and a *minimal length carrier graph* to be a carrier graph which has smallest length (over all carrier graphs of minimal rank). An argument using Arzela-Ascoli's Theorem, [62], shows that minimal length carrier graphs exist in any closed hyperbolic 3-manifold.

The following Proposition, due to White, shows that minimal length carrier graphs are geometrically well behaved.

Proposition 3.0.2 (White, [62])**.** *Assume $f : X \to M$ is a minimal length carrier graph in a closed hyperbolic 3-manifold M. Then X is trivalent with $2(\text{rank}(\pi_1(M)) - 1)$ vertices and $3(\text{rank}(\pi_1(M)) - 1)$ edges, each edge in X maps to a geodesic segment in M, the angle between any two adjacent edges is $\frac{2\pi}{3}$, and the image of any simple closed path in X is an essential loop in M.*

White used this result to prove that a minimal length carrier graph must have a cycle with length bounded by some function of $\mathrm{rank}(\pi_1 M)$.

It is not too hard to see that any minimal length carrier graph must have at least one edge of universally bounded length. For since $\pi_1(M)$ is not free [33, Theorem 7.1], there is some cycle in the carrier graph that is null-homotopic in M. This must lift to a closed path in $\mathbb{H}^3$; in fact, the minimal length hypothesis implies that it must be a piecewise geodesic loop in $\mathbb{H}^3$ in which the corners have angle $\frac{2\pi}{3}$, [62]. Any such loop must have some short edge, though, for otherwise a standard argument in CAT(-1)-geometry shows that it tracks a geodesic, contradicting the fact that it is closed.

3.1 Chains of bounded relative length

We will devote the remainder of this chapter to strengthening White's guarantee of a short cycle to a geometric decomposition of minimal length carrier graphs into nested subgraphs, each of which is 'short' relative to the previous one. Proposition 1.2.1 from the introduction will be a consequence of this. The ideas for what follows were originally sketched in [56]; the purpose of this section is mainly to fill in some missing details.

Assume that $M = \mathbb{H}^3/\Gamma$ is a closed hyperbolic 3-manifold and $f : X \to M$ is a minimal length carrier graph. Choose an edge $e \subset X$ and a subgraph $Y \subset X$. Our first goal will be to provide a useful definition of the length of e relative to the subgraph Y. This should vanish when $e \subset Y$ and should agree with the hyperbolic length of $f(e)$ when neither of the vertices of e lies inside Y. If X is embedded as a subset of M with f the inclusion map, then relative length is similar to the length e has outside of the hyperbolic convex hulls of the components of Y that e touches, but we need to do our measurements in the universal cover and throw out sections of e that lie inside some of the thin parts of M.

To clarify this, fix a universal covering $\pi_X : \tilde{X} \to X$ and a lift $\tilde{f} : \tilde{X} \to \mathbb{H}^3$ of f. Assume that a vertex v of e lies in a connected component $Z_v \subset Y$ and choose lifts

$\hat{e}, \hat{Z}_v \subset \tilde{X}$ of e and Z that touch above v. Let $\Gamma_{\tilde{f}(\hat{Z}_v)}$ be the subgroup of Γ that leaves $\tilde{f}(\hat{Z}_v)$ invariant.

Definition 3.1.1 (Thick Convex Hulls)**.** The *thick convex hull of* $\tilde{f}(\hat{Z}_v)$*, written* $\mathrm{TCH}(\tilde{f}(\hat{Z}_v))$, is the radius-1 neighborhood of the smallest convex set K containing $\tilde{f}(\hat{Z}_v)$ such that for every $\gamma \in \Gamma_{\tilde{f}(\hat{Z}_v)}$ and $x \in \mathbb{H}^3 \setminus K$, we have $\mathrm{d}(\gamma(x), x) \geq 1$.

Definition 3.1.2 (Edge Length Relative to a Subgraph)**.** Define the *length of* e *relative to* Y*,* denoted $\mathrm{length}_Y(e)$, to be the length of the part of $\tilde{f}(\hat{e})$ that lies outside of $\mathrm{TCH}(\tilde{f}(\hat{Z}_v))$ for each vertex v of e contained in Y.

It is easy to see that the relative length of e is well-defined, independent of the lifts chosen above. The definition is a bit less complicated if we assume that X is embedded as a subset of M. For then we can lift e directly to $\mathbb{H}^3$ along with any connected components of Y that e touches, and then measure the length of e's lift outside of the thick convex hulls of the lifted subgraphs. In the proofs below, we will assume X to be embedded in order to remove a level of notational hinderance. The arguments will be exactly the same in the general case.

Although an edge can have very long absolute length while having short length relative to a subgraph Y, we can bound this difference if we have some control over the geometry of the covers of M corresponding to the fundamental groups of the components of Y.

Lemma 3.1.1. *Assume that* M *is a closed hyperbolic* 3*-manifold,* $f : X \to M$ *is a minimal length carrier graph,* Y *is a subgraph of* X *and* e *is an edge of* $X \setminus Y$*. Then* $\mathrm{length}(e)$ *is bounded above by a constant depending only on* $\mathrm{length}_Y(e)$*,* $\mathrm{length}(Y)$*,* $\mathrm{inj}(M)$*,* $\mathrm{rank}(\pi_1(M))$ *and the diameters of the convex cores of the covers of* M *corresponding to the components of* Y *that* e *touches.*

Proof. As mentioned above, we forget about f and assume that X is embedded as a subset of M. Suppose that e shares a vertex with a connected component $Z \subset Y$, and let $\tilde{e}, \tilde{Z} \subset \mathbb{H}^3$ be lifts that touch above that vertex. Since X is minimal length, $\tilde{e} \cap \mathrm{TCH}(\tilde{Z})$ must minimize the distance from $\tilde{e} \cap \partial\,\mathrm{TCH}(\tilde{Z})$ to $\tilde{Z}$. For otherwise, one could replace it by a minimizing segment; extending equivariantly gives a new carrier graph homotopic to X and of smaller length, violating the minimality assumption. Thus a bound on the Hausdorff distance between $\tilde{Z}$ and $\mathrm{TCH}(\tilde{Z})$ limits the length that $\tilde{e}$ can have inside of $\mathrm{TCH}(\tilde{Z})$, and we will show that this is bounded in terms of the quantities mentioned in the statement of the Lemma.

We first claim that the hyperbolic distance from $\tilde{Z}$ to $\mathrm{CH}(\Lambda(\Gamma_{\tilde{Z}}))$ is bounded above by a constant depending only on $\mathrm{inj}(M)$ and $\mathrm{rank}(\pi_1(M))$. Choose an infinite piecewise geodesic path $\gamma \subset \tilde{Z}$ that projects to a simple closed curve in Y and let $g \in \Gamma_{\tilde{Z}}$ be the corresponding deck transformation. Taking a maximal sequence of consecutive edges of γ that project to distinct edges in M yields a subpath γ' whose g-translates cover γ. Note that the orthogonal projection of γ' to $\mathrm{axis}(g)$ has length equal to the translation distance of g, which is at least $\mathrm{inj}(M)$. By Lemma 3.0.2, X has $3(\mathrm{rank}(\pi_1(M)) - 1)$ edges; the number of edges in γ' can certainly be no greater than this. Thus there is an edge of γ whose orthogonal projection to $\mathrm{axis}(g)$ has length at least $\frac{\mathrm{inj}(M)}{3(\mathrm{rank}(\pi_1(M))-1)}$. It follows from elementary hyperbolic geometry that there is a point on this edge whose distance from $\mathrm{axis}(g)$ is bounded above by a constant depending on that length. For instance, if one draws two lines l_1, l_2 orthogonal to $\mathrm{axis}(g)$ that are $\frac{\mathrm{inj}(M)}{3(\mathrm{rank}(\pi_1(M))-1)}$ apart, then the distance from the edge to $\mathrm{axis}(g)$ is at most the distance to $\mathrm{axis}(g)$ from either of the two geodesic lines that share one endpoint on $\mathbb{S}^1_\infty$ with l_1, the other with l_2, and lie on one side of $\mathrm{axis}(g)$. This proves the claim.

Now $\tilde{Z}$ and $\mathrm{CH}(\Lambda(\Gamma_{\tilde{Z}}))$ are both invariant under the action of $\Gamma_{\tilde{Z}}$ with quotients of bounded diameter, so our limit on the hyperbolic distance between them translates into a bound on their Hausdorff distance. But if $\tilde{Z}$ is Hausdorff-close to a convex set then it must also be Hausdorff-close to its convex hull, $\mathrm{CH}(\tilde{Z})$. Since the Hausdorff distance from $\mathrm{CH}(\tilde{Z})$ to $\mathrm{TCH}(\tilde{Z})$ is controlled by $\mathrm{inj}(M)$, we have a bound on the Hausdorff distance between $\tilde{Z}$ and $\mathrm{TCH}(\tilde{Z})$. □

For a subgraph $Z \subset X$, we define the *length of Z relative to Y* to be

$$\text{length}_Y(Z) = \sum_{\text{edges } e \subset Z} \text{length}_Y(e).$$

We are now ready to state the promised decomposition of minimal length carrier graphs.

Proposition 3.1.2 (Chains of Bounded Length)**.** *There is a universal constant L with the property that if M is a closed hyperbolic 3-manifold and $f : X \to M$ is a minimal length carrier graph then we have a sequence of (possibly disconnected) subgraphs*

$$\emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X$$

such that $\text{length}_{Y_i}(Y_{i+1}) < L$ *for all i.*

Proof. It is a standard fact in hyperbolic geometry that there exist a universal constant $C > 0$ with the following property:

(1) any path in $\mathbb{H}^3$ made of geodesic segments of length at least C connected with angles at least $\frac{\pi}{3}$ is a quasi-geodesic.

There is also a constant $D > C$ such that

(2) if $N \subset \mathbb{H}^3$ contains the axis of a hyperbolic isometry γ and $\mathrm{d}(x, \gamma(x)) \geq 1$ for all $x \in \mathbb{H}^3 \setminus N$, then $\mathrm{d}(x, \gamma(x)) \geq C$ for all $x \in \mathbb{H}^3 \setminus \mathcal{N}_D(N)$,

(3) any geodesic ray emanating from a convex subset $K \subset \mathbb{H}^3$ that leaves $\mathcal{N}_D(K)$ meets $\partial\mathcal{N}_D(K)$ in an angle of at least $\frac{\pi}{3}$,

and finally a constant $B > 0$ for which

(4) any geodesic exiting the radius-1 neighborhood of a convex subset $K \subset \mathbb{H}^3$ will exit $\mathcal{N}_D(K)$ after an additional length less than B.

We will show that if Y is any subgraph of X then there is an edge in $X \setminus Y$ of length at most $L = C + 2B$ relative to Y; applying this iteratively will give the chain of subgraphs in the statement of the Proposition.

So, suppose that Y is a subgraph of X. Observe that since the fundamental group of a closed hyperbolic manifold cannot be free, there is an essential closed loop $\gamma \subset X$ that is nullhomotopic in M. Furthermore, since $\pi_1(M)$ does not split as a free product, [33, Theorem 7.1], we can pick γ so that it has no subpath contained entirely in Y that is also a closed loop nullhomotopic in M. Lifting γ to $\mathbb{H}^3$ then gives a closed loop $\tilde{\gamma} \subset \mathbb{H}^3$ such that each time $\tilde{\gamma}$ touches a component of $\pi_M^{-1}(Y)$ it enters and leaves that component using different edges of $\pi_M^{-1}(X \setminus Y)$.

The first crucial observation is that one of the edges of γ must have length less that L. For otherwise, $\tilde{\gamma}$ is a closed path in $\mathbb{H}^3$ made up of geodesic segments of length at least L connected at $\frac{\pi}{3}$-angles, which is impossible by definition of L. If this short edge lies outside Y, then we are done. However, it very well might not, so in the remainder of the proof we will develop a version of this argument that runs relative to Y. This will produce a short edge outside Y, but we will be forced to measure its length relatively rather than absolutely.

So, consider a maximal segment of $\tilde{\gamma}$ that is contained in a component $\tilde{Z}$ of $\pi_M^{-1}(Y)$ and let e and f be the edges that $\tilde{\gamma}$ traverses before and after the segment in $\tilde{Z}$. If e or f has length less than L relative to Y, then we are done. Otherwise, the two edges have a length of at least L left after exiting $\mathrm{TCH}(\tilde{Z})$, so by (4) both of these edges must exit $\mathcal{N}_D(\mathrm{TCH}(\tilde{Z}))$; let e_0 and f_0 be the points where they meet $\partial\mathcal{N}_D(\mathrm{TCH}(\tilde{Z}))$. Assume for the moment that the distance between e_0 and f_0 is less than C. Then by (2), e and f project to different edges in X. Substituting $\pi_M(e \cap \mathcal{N}_D(\mathrm{TCH}(\tilde{Z}))) \subset X$ with the projection of the geodesic between e_0 and f_0 therefore yields a new carrier graph for M, and since the new edge has length less than C while the old has length at least D our new carrier graph has shorter length than X. This contradicts the minimality of X, so $\mathrm{d}(e_0, f_0) \geq C$.

We can now create a new closed path in $\mathbb{H}^3$ from $\tilde{\gamma}$ as follows: each time $\tilde{\gamma}$ traverses a component $\tilde{Z}$ of $\pi_M^{-1}(Y)$, replace the part of $\tilde{\gamma}$ that lies inside $\mathcal{N}_D(\mathrm{TCH}(\tilde{Z}))$

by the geodesic with the same endpoints. Then the new path is composed of geodesic segments of length at least C, and by (3), the segments intersect with angles at least $\frac{\pi}{3}$. Therefore it is a quasi-geodesic. Since it is also closed, this is impossible. $\square$

The following version of the above result, stated in the introduction, will be useful in proving Theorem 1.1.1.. It comes from combining the above statement with Lemma 3.1.1.

Proposition 1.2.1. Let M be a closed hyperbolic 3-manifold with $f : X \to M$ a minimal length carrier graph. Then we have a sequence of (possibly disconnected) subgraphs

$$\emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X$$

such that the length of any edge in $Y_{i+1} \setminus Y_i$ is bounded above by some constant depending only on $\text{inj}(M)$, $\text{rank}(\pi_1(M))$, $\text{length}(Y_i)$ and the diameters of the convex cores of the covers of M corresponding to $f_*(\pi_1(Y_i^j))$, where $Y_i^1, \ldots, Y_i^n$ are the connected components of Y_i.

CHAPTER 4
RANK AND FIBERED HYPERBOLIC 3-MANIFOLDS

This chapter concerns the relationship between the geometry of a hyperbolic 3-manifold fibering over the circle and the rank of its fundamental group. The main result here is the following, which we recall from the introduction.

Theorem 1.1.1. Given $\epsilon > 0$ and a closed orientable surface Σ_g, there is some $D > 0$ with the following property. Let M be an ϵ-thick hyperbolic 3-manifold fibering over S^1 with fiber Σ_g. If the diameter of M is at least D, then $\mathrm{rank}(\pi_1(M)) = 2g+1$.

This can also be phrased as a finiteness statement concerning the number of ϵ-thick manifolds that fiber with a given genus but have unexpected rank. The equivalence with the previous statement comes from the fact that there are only finitely many hyperbolic 3-manifolds with diameter less than any given constant [63].

Theorem. *Given $\epsilon, g > 0$ there are at most finitely many ϵ-thick hyperbolic 3-manifolds M fibering over $\mathbb{S}^1$ with fiber Σ_g for which rank$(\pi_1(M)) \neq 2g+1$.*

To contextualize the statements above, let us review some facts about the geometry of closed hyperbolic 3-manifolds M fibering over the circle with fiber Σ_g. It follows from work of Thurston, Canary [25] and Freedman-Hass-Scott [32] that passing through every point in M there is an embedded surface which has diameter bounded above by some constant depending only on $\mathrm{inj}(M)$ and g, and which is homotopic to the Σ_g-fibers. Therefore, in some sense the width of M in the fiber direction is bounded in terms of $g, \mathrm{inj}(M)$. If g and a lower bound for $\mathrm{inj}(M)$ are fixed, the diameter of M is then coarsely equivalent to the length of M in the 'circle direction'. Here is a precise definition of this length; the coarse equivalence with diameter is the content of Proposition 4.2.1.

Definition 4.0.3. The *waist length* of M, denoted waist(M), is the smallest length of a loop in M that projects nontrivially to $\pi_1(\mathbb{S}^1)$.

The discussion above implies that Theorem 1.1.1 can be rephrased with a condition on waist length instead of diameter. Specifically, its conclusion holds if M has large waist length relative to its width in the fiber direction, so in effect we are ensuring that rank$(\pi_1 M)$ is what you would expect from the fibration whenever M looks geometrically like a thin necklace.

Recall that the *Heegaard genus* of a closed 3-manifold M is the smallest $g = g(M)$ such that M can be obtained by gluing two genus g handlebodies along their boundaries. In [8], Bachman and Schleimer show that the Heegaard genus of a closed 3-manifold M fibering over the circle with fiber Σ_g is $2g + 1$, as long as the monodromy map of M has translation distance at least $2g+1$ in the curve complex of Σ_g. If M is hyperbolic, this translation distance is coarsely equivalent to M's waist length relative to its ϵ-thin parts. This is identical to the waist length defined above, except that when measuring the length of a loop one disregards the parts of the loop that lie inside of the ϵ-thin part of M. It is likely that the conclusion of Theorem 1.1.1 is true whenever this relative waist length is large, but it is not yet clear to us how to prove this.

Finally, let us perform a stylized version of the forthcoming proof of Theorem 1.1.1. Assume that M fibers over the circle and has large waist length relative to its injectivity radius and the genus of the fibering. Then as mentioned before M looks like a thin necklace. In this case, using Proposition 1.2.1 one can show that every minimal length carrier graph in M has one large edge that navigates the waist of M and a subgraph consisting of short edges that fills out the fundamental group of the fiber. Since $\pi_1(\Sigma_g)$ cannot be generated with less than $2g$ elements, the subgraph consisting of short edges must have rank at least that. Therefore the fundamental group of any minimal length carrier graph has rank at least $2g + 1$, implying that rank$(\pi_1(M)) = 2g + 1$.

The actual proof of Theorem 1.1.1 differs slightly from that sketched above in that we bypass a full characterization of minimal length carrier graphs inside of M

in favor of an argument more tailored to the statement of the Theorem.

The remainder of this chapter is broken into two sections. The first provides uniform bounds for the convex core diameters of certain covers of doubly degenerate hyperbolic 3-manifolds homeomorphic to $\Sigma_g \times \mathbb{R}$. Combined with Proposition 1.2.1, this will facilitate a relatively quick proof of Theorem 1.1.1, which is the subject of the second and last section.

4.1 Short Graphs in Doubly Degenerate $\Sigma_g \times \mathbb{R}$

Assume that M is a hyperbolic 3-manifold without cusps that is homeomorphic to $\Sigma_g \times \mathbb{R}$. Using Waldhausen's Cobordism Theorem [61] and Ahlfors' Finiteness Theorem [43], it is not hard to see that there is an explicit homeomorphism $M \cong \Sigma_g \times \mathbb{R}$ such that $\mathrm{CC}(M)$ sits inside M as either

- $\Sigma_g \times [0, 1]$, in which case M is convex cocompact
- $\Sigma_g \times [0, \infty)$, in which case M is called singly degenerate
- $\Sigma_g \times \mathbb{R}$, and then M is called doubly degenerate.

We mentioned in the introduction that Theorem 1.1.1 is an extension of an earlier theorem of Souto [57]. A key ingredient in Souto's proof was the following observation, which is a consequence of the Covering Theorem of Canary and Thurston [25]. It was originally proven by Scott-Swarup [55] in the case that M is the cyclic cover of a hyperbolic 3-manifold fibering over the circle.

Lemma 4.1.1 (Souto [57]). *Let M be a doubly degenerate hyperbolic 3-manifold homeomorphic to $\Sigma_g \times \mathbb{R}$ and let $\Gamma \subset \pi_1(M)$ be a proper subgroup of rank at most $2g$. Then Γ is free, infinite index and convex-cocompact.*

To prove Theorem 1.1.1, we need an improved version of Lemma 4.1.1 that gives a diameter bound for the convex core of $\mathbb{H}^3/\Gamma$ in terms of $\mathrm{inj}(M)$ and the length of a set of loops in M generating Γ. Our proof will be a compactness argument: we define a topology on the set of wedges of k bounded length loops in ϵ-thick doubly degenerate hyperbolic 3-manifolds homeomorphic to $\Sigma_g \times \mathbb{R}$, show that the resulting space is compact and then use continuity to show that there is an upper bound for the corresponding convex core diameters.

Definition 4.1.1. Define $\mathcal{G} = \mathcal{G}(\epsilon, L, k)$ to be the space of pairs (M, f), where

1. M is a doubly degenerate ϵ-thick hyperbolic 3-manifold homeomorphic to $\Sigma_g \times \mathbb{R}$
2. $f : \wedge_k \mathbb{S}^1 \to M$ is an L-lipschitz map from the wedge of k circles, endowed with some fixed metric.

We say that $(M_i, f_i) \to (M_\infty, f_\infty)$ if

1. $(M_i, \star_i)$ converges strongly to $(M_\infty, \star_\infty)$, and $\star_i$ is the wedge point of $f_i(\wedge_k \mathbb{S}^1)$
2. there is a sequence ϕ_i of almost isometric maps coming from the geometric convergence in (1) such that $\phi_i \circ f_i : \wedge_k \mathbb{S}^1 \to M$ converges pointwise to $f_\infty : \wedge_k \mathbb{S}^1 \to M_\infty$.

Here, (1) means that there are faithful representations $\rho_i : \pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbb{C})$ converging strongly to some ρ_∞ such that the quotient manifolds $\mathbb{H}^3/\rho_i(\pi_1(\Sigma_g))$ and $\mathbb{H}^3/\rho_\infty(\pi_1(\Sigma_g))$ are isometric to $(M_i, \star_i)$ and $(M_\infty, \star_\infty)$ as based hyperbolic manifolds, where the projections of some fixed point in $\mathbb{H}^3$ are taken as basepoints.

Proposition 4.1.2. *$\mathcal{G}$ is compact.*

Proof. Let (M_i, f_i) be a sequence in $\mathcal{G}$ and assume that $\star_i \in M_i$ is the wedge point of $f_i(\wedge_k \mathbb{S}^1)$. For each i, Canary's Filling Theorem [25] gives a simplicial hyperbolic surface in M_i with image passing through $\star_i$. Using the short markings of these surfaces provided by Corollary 2.2.2 we can construct representations $\rho_i : \pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbb{C})$ with $\mathbb{H}^3/\rho_i(\Sigma_g) \cong M_i$ so that a fixed base point $\star \in \mathbb{H}^3$ projects to each $\star_i$ and up to passing to a subsequence, ρ_i converges algebraically to some $\rho_\infty : \pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbb{C})$. Since our lower bound on injectivity radius persists through algebraic limits, $\rho_\infty(\pi_1(\Sigma_g))$ contains no parabolics. Work of Thurston and Bonahon then implies that $\rho_i \to \rho_\infty$ strongly. Specifically, one must trace through Thurston's proof of ([58], 9.2) with the hindsight provided by Bonahon's Tameness Theorem [13]. A statement of the resulting theorem is given by Canary in ([25], 9.1) as a prelude to a series of more general convergence theorems.

Set $M_\infty = \mathbb{H}^3/\rho_\infty(\Sigma_g)$ and let $\star_\infty \in M_\infty$ be the projection of $\star$. Then $(M_i, \star_i)$ converges geometrically to $(M_\infty, \star_\infty)$. The fundamental group of M_∞ is isomorphic to $\pi_1(\Sigma_g)$, so Bonahon's Tameness Theorem [13] implies that $M_\infty \cong \Sigma_g \times \mathbb{R}$. Moreover, it follows from strong convergence and [50, Thm 1.1] that the convex cores $\mathrm{CC}(M_i)$ converge geometrically to $\mathrm{CC}(M_\infty)$, thus M_∞ is doubly degenerate. We can construct a map $f_\infty : \wedge_k \mathbb{S}^1 \to M_\infty$ by applying Arzela-Ascoli's Theorem to the sequence of maps $\phi_i \circ f_i : \wedge_k \mathbb{S}^1 \to M_\infty$, where ϕ_i is a sequence of almost isometric maps coming from geometric convergence. Clearly (M_i, f_i) converges to (M_∞, f_∞) in $\mathcal{G}$. □

Corollary 4.1.3. *Let M be a doubly degenerate ϵ-thick hyperbolic 3-manifold homeomorphic to $\Sigma_g \times \mathbb{R}$ and let $p \in M$ be a basepoint. Assume that $\Gamma \subset \pi_1(M, p)$ is a proper subgroup that can be generated by $2g$ loops based at p of length less than L. Then Γ is convex cocompact and the diameter of the convex core of $\mathbb{H}^3/\Gamma$ is bounded above by some constant depending only on L, ϵ and g.*

Proof. Observe that Γ determines an element $(M, f) \in \mathcal{G} = \mathcal{G}(\epsilon, L, 2g)$, with the extra property that f is not π_1-surjective. Our goal then is to show that if $(M, f) \in \mathcal{G}$ and f is not π_1-surjective, then the diameter of the convex core of the cover $M_{\pi_1(f)}$ of M corresponding to the π_1 image of f is bounded above. If not, there is a sequence (M_i, f_i) of such pairs where these diameters grow without bound. By compactness, we may assume that $(M_i, f_i) \to (M_\infty, f_\infty)$ in $\mathcal{G}$. In fact, f_∞ cannot be π_1-surjective. To see this, note that by strong convergence there are a compact core $K \subset M_\infty$ and almost isometric embeddings $\phi_i : K \to M_i$ that are π_1-surjective for large i. Therefore, if f_∞ is π_1-surjective, then so is $\phi_i \circ f_\infty$, which is homotopic for large i to f_i. As f_i is never π_1-surjective, it follows that neither is f_∞. Lemma 4.1.1 then shows that $(M_\infty)_{\pi_1(f_\infty)}$ is convex-cocompact.

The manifolds $(M_i)_{\pi_1(f_i)}$ converge algebraically to $(M_\infty)_{\pi_1(f_\infty)}$, and since the limit is convex-cocompact, it follows from [43, Prop 7.39] that the convergence is strong and the convex cores of the manifolds in the sequence converge geometrically to that of the limit. Consequently, the diameters of these cores must also converge, and therefore must be bounded. This is a contradiction. $\square$

4.2 Proof of Theorem 1.1.1

Fix $\epsilon, g > 0$ and assume that M is an ϵ-thick hyperbolic 3-manifold fibering over the circle with fiber Σ_g. The goal of this section is to prove that there are only finitely many such M for which $\text{rank}(\pi_1(M)) \neq 2g+1$. We begin, however, with a quick computation concerning M's girth. Recall that $\text{waist}(M)$ is the smallest length of a loop in M that projects nontrivially to $\pi_1\mathbb{S}^1$.

Proposition 4.2.1 (Fibered 3-Manifolds Have High BMI)**.** *Let M be an ϵ-thick hyperbolic 3-manifold fibering over the circle with fiber Σ_g. Then*

$$2\,\text{diam}(M) - \frac{16}{\epsilon}(2g-2) \leq \text{waist}(M) \leq 2\,\text{diam}(M).$$

Proof. Assume that γ is a loop realizing the waist length of M. Canary's Filling Theorem [25] implies that every point in the cyclic cover of M corresponding to the fundamental group of the fiber lies in the image of a simplicial hyperbolic surfaces for which the inclusion map is a homotopy equivalence. Projecting down, this provides an exhaustion of M by simplicial hyperbolic surfaces in the homotopy class of the fiber. By homological considerations, any such surface must intersect γ. The Bounded Diameter Lemma (see Section 2) then implies that $\text{diam}(M) \leq \frac{1}{2}\,\text{waist}(M) + \frac{8}{\epsilon}(2-2g)$. This establishes the first inequality.

For the second, recall that the fundamental group of M is generated by the set of all loops in M of length less than $2\,\text{diam}(M)$. Any generating set for $\pi_1(M)$ must contain a loop that projects nontrivially to $\pi_1(\mathbb{S}^1)$, so the waist length of M is at most twice its diameter. $\square$

We are now ready to prove the main result of this chapter.

Theorem 1.1. Given $\epsilon > 0$ and a closed orientable surface Σ_g, there is some $D > 0$ with the following property. Let M be an ϵ-thick hyperbolic 3-manifold fibering over S^1 with fiber Σ_g. If the diameter of M is at least D, then $\mathrm{rank}(\pi_1(M)) = 2g + 1$.

Proof. Assume that M is an ϵ-thick hyperbolic 3-manifold fibering over the circle with fiber Σ_g and $\mathrm{rank}(\pi_1(M)) \leq 2g$. We will show that the waist length of M is bounded by some constant depending only on ϵ and g.

Let $f : X \to M$ be a minimal length carrier graph. By Proposition 1.2.1, there is a constant L and a chain of (possibly disconnected) subgraphs

$$\emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X$$

with $\mathrm{length}(Y_{i+1})$ bounded above by some constant depending only on ϵ, g and $\mathrm{length}(Y_i)$, and the diameters of the convex cores of the covers of M corresponding to the fundamental groups of the connected components of Y_i.

Assume for the moment that no connected component of Y_i runs all the way around M's waist, so that each lifts homeomorphically to the cyclic cover $M_{\pi_1(\Sigma_g)}$ of M. Since $\mathrm{rank}(\pi_1(X)) \leq 2g$, the components of Y_i have even smaller rank and thus cannot generate the fundamental group of $M_{\pi_1(\Sigma_g)}$. Therefore Corollary 4.1.3 applies to bound the diameters of the associated convex cores in terms of $\mathrm{length}(Y_i)$, ϵ and g. It follows that $\mathrm{length}(Y_{i+1})$ is also bounded above by $\mathrm{length}(Y_i)$, ϵ and g.

Applying this argument iteratively, we obtain a length bound for the first subgraph Y_i that contains a loop that projects nontrivially to $\pi_1(\mathbb{S}^1)$. The length bound depends on ϵ, g and the index of the subgraph, but since there are at most $3(\mathrm{rank}(\pi_1(M))-1)$ edges in X the number of subgraphs in our chain is also limited. Therefore we have that the waist length of M is bounded by a function of ϵ and g. $\square$

Under slight modifications, the proof of Theorem 1.1.1 shows that for mapping tori with large waist length there is only one Nielsen equivalence class of minimal size generating sets for $\pi_1(M)$. The interested reader may compare our proof with [56] for more details.

The arguments above also apply to hyperbolic 3-manifolds formed by gluing along their boundaries two twisted interval bundles over a non-orientable surface. One usually says that such a manifold M fibers over the orbifold $\mathbb{S}^1/(z \mapsto -z)$. The two embedded copies of the non-orientable surface are called the *singular fibers,* and the rest of M is foliated by *regular fibers,* which are orientable and doubly cover the singular fibers. We refer the reader to [33] for more information on the topology of such manifolds.

If $S_1, S_2 \subset M$ are the singular fibers and $p \in S_1$ is a basepoint, then $\pi_1(M, p)$ is generated by $\pi_1(S_1, p)$ and any loop freely homotopic into S_2, but not into S_1. Using this, one can check that if the regular fibers have genus g, then $\text{rank}(\pi_1(M)) \leq g+2$. We then have the following analogue of Theorem 1.1.1:

Theorem 4.2.2. *Given $\epsilon, g > 0$, there are at most finitely many hyperbolic* 3-*manifolds M fibering over* $\mathbb{S}^1/(z \mapsto -z)$ *with regular fiber* Σ_g *for which* $\text{rank}(\pi_1(M)) \neq g+2$.

The proof is nearly the same. One takes a minimal length carrier graph and shows that if the distance between the two singular fibers is large enough then one of the subgraphs given by Proposition 1.2.1 fills out the fundamental group of one of the singular fibers of M. This forces the fundamental group of the subgraph to have rank at least $g+1$, implying that the carrier graph has rank at least $g+2$. The finiteness statement follows because an upper bound for the distance between the two singular fibers provides a upper bound on the diameter, at worst by imitating the argument in Proposition 4.2.1.

CHAPTER 5

DEGENERATE ENDS, THE LAPLACIAN AND ARITHMETIC 3-MANIFOLDS

The focus of this chapter is the following proposition and its applications to the first eigenvalue of the Laplacian and arithmetic hyperbolic 3-manifolds.

Proposition 1.2.2. Assume that (M_i) is a sequence of pairwise distinct hyperbolic 3-manifolds with $\text{inj}(M_i) \geq \epsilon$ and $\text{rank}(\pi_1 M_i) \leq k$. Then there are base points $x_i \in M_i$ such that, up to passing to a subsequence, the pointed manifolds (M_i, x_i) converge in the pointed Gromov-Hausdorff topology to a pointed manifold (M_∞, x_∞) which has a degenerate end.

Recall that an end $\mathcal{E}$ of a non-compact hyperbolic 3-manifold M without cusps is *degenerate* if it has a neighborhood $\mathcal{U}$ homeomorphic to $\Sigma \times [0, \infty)$, where Σ is a closed surface, and there is a sequence of embedded surfaces S_i exiting the end $\mathcal{E}$, with bounded area and homotopic to $\Sigma \times \{0\}$ within $\mathcal{U}$.

The proof of Proposition 1.2.2 exploits the machinery of carrier graphs developed in Chapter 3. In fact, if we choose minimal length carrier graphs $X_i \to M_i$, then an easy consequence of Proposition 1.2.1 is that for each i there is a subgraph of X_i of universally bounded length whose image in $\pi_1(M_i)$ is non-abelian. The base points referenced in Proposition 1.2.2 can simply be chosen to lie on these subgraphs.

Our work here is organized as follows. The first section is devoted to a proof of Proposition 1.2.2. In the second, we introduce the first eigenvalue λ_1 of the Laplacian and discuss its relationship with injectivity radius and the rank of the fundamental group. Finally, we discuss some applications to arithmetic hyperbolic 3-manifolds.

5.1 Limits with degenerate ends

In this section we prove Proposition 1.2.2. The key step in the proof is the following application of Proposition 1.2.1.

Lemma 5.1.1. *Assume that* (M_i) *is a sequence of pairwise distinct hyperbolic* 3-*manifolds with* $\mathrm{inj}(M_i) \geq \epsilon$ *and* $\mathrm{rank}(\pi_1(M_i)) \leq k$. *Then there are a constant* L *and a sequence* (Y_i) *of metric graphs with 1-Lipschitz maps* $(f_i : Y_i \to M_i)$ *such that*

1. $\mathrm{rank}(\pi_1(Y_i)) \leq k$,
2. $\mathrm{length}(Y_i) \leq L$, *and*
3. $\lim_{i\to\infty} \mathrm{diam}(CC(\mathbb{H}^3/(f_i)_*(\pi_1(Y_i)))) = \infty$.

Here $\mathrm{CC}(\mathbb{H}^3/(f_i)_*(\pi_1(Y_i)))$ *is the convex core of the cover of* M_i *corresponding to the image of the homomorphism* $(f_i)_* : \pi_1(Y_i) \to \pi_1(M_i)$.

Proof of Lemma 5.1.1. To begin with, fix ϵ, k and a sequence of hyperbolic 3-manifolds (M_i) as in the statement. For each i fix a minimal length carrier graph

$$f_i : X_i \to M_i$$

with $\mathrm{rank}(\pi_1(X_i)) = \mathrm{rank}(\pi_1(M_i)) = k$.

Assume for the moment that the sequence $(\mathrm{length}(X_i))$ is bounded from above by some positive number L. In other words, the graphs X_i themselves satisfy (1) and (2). On the other hand, we have by definition that $(f_i)_*(\pi_1(X_i)) = \pi_1(M_i)$ and hence

$$\mathrm{CC}(\mathbb{H}^3/(f_i)_*(\pi_1(X_i))) = M_i$$

Since the sequence (M_i) consists of pairwise distinct manifolds with $\mathrm{inj}(M_i) \geq \epsilon$ we obtain, for example, from Wang's finiteness theorem that $\mathrm{diam}(M_i) \to \infty$. This means that the carrier graphs $f_i : X_i \to M_i$ themselves satisfy also (3). This concludes the proof if the sequence $(\mathrm{length}(X_i))$ is bounded.

We treat now the general case. In the light of the above, we may assume without loss of generality that $\mathrm{length}(X_i) \to \infty$. Consider for each i the chain

$$\emptyset = Y_0^i \subset Y_1^i \subset \cdots \subset Y_{n_i}^i = X_i \tag{5.1.1}$$

provided by Proposition 1.2.1. Since $\mathrm{length}(Y_0^i) = 0$, $\mathrm{length}(X_i) \to \infty$ and the length n_i of each chain is bounded independently of i, we can choose a sequence (m_i) with

(a) $0 \le m_i \le n_i - 1$,

(b) $\limsup_{i\to\infty} \mathrm{length}(Y_{m_i}^i) < \infty$, and

(c) $\lim_{i\to\infty} \mathrm{length}(Y_{m_i+1}^i) = \infty$.

Observe that by condition (b), any of the connected components $Z_1^i, \ldots, Z_{r_i}^i$ of $Y_{m_i}^i$ satisfies (1) and (2) for any $L < \infty$ with

$$\limsup_{i\to\infty} \mathrm{length}(Y_{m_i}^i) < L$$

By Proposition 1.2.1, $\mathrm{length}(Y_{m_i+1}^i)$ is bounded in terms of k, L and

$$\max_{j=1,\ldots,r_i} \{\mathrm{diam}(CC(\mathbb{H}^3/(f_i)_*(\pi_1(Z_j^i))))\}$$

Since $\mathrm{length}(Y_{m_i+1}^i)$ tends to ∞ by condition (c), we obtain that there is a sequence of component of $Y_{m_i}^i$, say Z_1^i, with

$$\lim_i \mathrm{diam}(CC(\mathbb{H}^3/(f_i)_*(\pi_1(Z_1^i)))) = \infty$$

In other words, the sequence of maps $f_i|_{Z_1^i} : Z_1^i \to M_i$ satisfies (3). This concludes the proof of Lemma 5.1.1 □

We are now ready to prove Proposition 1.2.2. Before starting, we recall the statement once more.

Proposition 1.2.2. Assume that (M_i) is a sequence of pairwise distinct hyperbolic 3-manifolds with $\mathrm{inj}(M_i) \geq \epsilon$ and $\mathrm{rank}(\pi_1(M_i)) \leq k$. Then there are base points $x_i \in M_i$ such that, up to passing to a subsequence, the pointed manifolds (M_i, x_i) converge in the pointed Gromov-Hausdorff topology to a pointed manifold (M_∞, x_∞) which has a degenerate end.

Proof. For each i, let $f_i : Y_i \to M_i$ be a sequence of graphs as provided by Lemma 5.1.1. We choose base points $y_i \in Y_i$ and set $x_i = f_i(y_i) \in M_i$; we also choose for all i an orthonormal frame of the tangent space $T_{x_i}M_i$ and, abusing notation, refer to it by x_i as well.

Choosing a base frame $x_{\mathbb{H}^3}$ of hyperbolic space we obtain for each i a unique discrete torsion-free subgroup $\Gamma_i \subset \mathrm{Isom}_+(\mathbb{H}^3)$ such that the hyperbolic manifolds M_i and $\mathbb{H}^3/\Gamma_i$ are isometric by an isometry mapping the base frame x_i to the projection of the base frame $x_{\mathbb{H}^3}$. From now on we identify $M_i = \mathbb{H}^3/\Gamma_i$.

The assumption that $M_i = \mathbb{H}^3/\Gamma_i$ has at least injectivity radius ϵ implies that the sequence of groups Γ_i contains a subsequence, say the whole sequence, which converges in the Chabauty topology to a discrete and torsion free group Γ_∞. It is well-known that this is equivalent to the convergence in the pointed Gromov-Hausdorff topology of the pointed manifolds (M_i, x_i) to the manifold $M_\infty = \mathbb{H}^3/\Gamma$ (see [9]).

The assumption that the manifolds M_i are pairwise distinct implies that M_∞ is not compact. In particular, in order to show that it has a degenerate end, it suffices by Canary's extension of Thurston's covering theorem [25] to find a manifold $\tilde{M}_\infty$ which has a degenerate end and covers M_∞. This is our goal.

Passing to a subsequence, we may assume that the group $\pi_1(Y_i, y_i)$ is isomorphic to the free group $\mathbb{F}_m$ of rank $m \leq k$ for each i. Moreover, since the subgraphs graphs Y_i have length bounded by some universal constant L, we may choose the identification $\mathbb{F}_m \simeq \pi_1(Y_i, y_i)$ in such a way that each element of the standard basis is represented by a loop of at most length $2L$. The composition of this identification, the homomorphism $(f_i)_* : \pi_1(Y_i, y_i) \to \pi_1(M_i, x_i)$ and the identification

$\pi_1(M_i, x_i) \simeq \Gamma_i$ yields a representation

$$\rho_i : \mathbb{F}_m \to \mathrm{Isom}_+(\mathbb{H}^3)$$

in such a way that if $e_j \in \mathbb{F}_m$ is an element of the standard basis then for all i we have

$$d_{\mathbb{H}^3}((\rho_i(e_j))(x_{\mathbb{H}^3}), x_{\mathbb{H}^3}) \leq 2L$$

This implies that, up to passing to a further subsequence, the sequence of representations (ρ_i) converges to a representation ρ_∞ of $\mathbb{F}_m$ into $\mathrm{Isom}_+(\mathbb{H}^3)$, meaning that for each $\gamma \in \mathbb{F}_m$ we have $\lim_i \rho_i(\gamma) = \rho_\infty(\gamma)$. The image of ρ_∞ is a subgroup of Γ_∞; in particular, it is discrete and the manifold $\tilde{M}_\infty = \mathbb{H}^3/\rho_\infty(\mathbb{F}_m)$ covers M_∞. Observe that $\mathrm{inj}(\tilde{M}_\infty) \geq \epsilon$ and hence $\rho_\infty(\mathbb{F}^m)$ does not contain parabolic elements.

We claim that $\tilde{M}_\infty$ has a degenerate end. Otherwise, its convex-core $\mathrm{CC}(\tilde{M}_\infty)$ is compact by the work of Agol [2], Calegari-Gabai [23] and Canary [24]. Marden's stability theorem [42] implies then that there are bi-Lipschitz maps (defined for large enough i)

$$\tilde{\phi}_i : \mathbb{H}^3/\rho_\infty(\mathbb{F}_m) \to \mathbb{H}^3/\rho_i(\mathbb{F}_m)$$

whose bi-Lipschitz constants tends to 1. This implies that

$$\lim_{i\to\infty} \mathrm{diam}(CC(\mathbb{H}^3/\rho_i(\mathbb{F}_m))) = \mathrm{diam}(CC(\mathbb{H}^3/\rho_\infty(\mathbb{F}_m))) < \infty$$

contradicting that by Lemma 5.1.1 we have

$$\lim_{i\to\infty} \mathrm{diam}(CC(\mathbb{H}^3/\rho_i(\mathbb{F}_m))) = \lim_{i\to\infty} \mathrm{diam}(CC(\mathbb{H}^3/(f_i)_*(\pi_1(Y_i)))) = \infty$$

We have proved that $\tilde{M}_\infty$ has at least a degenerate end $\tilde{\mathcal{E}}$. As remarked above, Thurston's covering theorem [25] implies that M_∞ has a degenerate end as well. We have proved Proposition 1.2.2. $\square$

5.2 An Application to Eigenvalues of the Laplacian

Let M be a closed hyperbolic 3-manifold. Given a smooth function f and a smooth vector field X on M let ∇f and $\operatorname{div}(X)$ be their gradient and divergence respectively. The *Laplacian* Δf of a function $f \in C^\infty(\mathbb{H}^3)$ is then defined as

$$\Delta f = -\operatorname{div} \nabla f$$

The Laplacian extends to a self-adjoint linear operator with discrete spectrum on the Sobolev space $H_{1,2}(M)$. By the spectral theorem, there is a Hilbert basis of $H_{1,2}(M)$ consisting of eigenfunctions of Δ. Let

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \dots$$

be the eigenvalues of Δ in increasing order. In [27], Cheeger introduced the so-called Cheeger constant

$$h(M) = \inf_{U \subset M} \frac{\operatorname{vol}(\partial U)}{\min\{\operatorname{vol}(U), \operatorname{vol}(M \setminus U)\}}$$

where the infimum is taken over smooth 3-dimensional submanifolds with boundary inside M. Cheeger showed that $h(M)$ can be used to bound $\lambda_1(M)$ from below; later, Buser [22] showed that $\lambda_1(M)$ can be estimated from above using the Cheeger constant and lower-bounds on the Ricci-curvature. For hyperbolic 3-manifolds, their formulas combine as follows:

$$\frac{1}{4} h(M)^2 \leq \lambda_1(M) \leq 4h(M)^2 + 10h(M) \tag{5.2.1}$$

The next lemma follows directly from the definition.

Lemma 5.2.1. *Assume that a hyperbolic 3-manifold M without cusps has a degenerate end. For every η positive we can find two disjoint compact 3-dimensional submanifolds $U_1, U_2 \subset M$ with $\frac{\operatorname{vol}(\partial U_j)}{\operatorname{vol}(U_j)} < \eta$ for $j = 1, 2$.*

We are now ready to prove the following theorem, stated in the introduction.

Theorem 1.3.1. For every $\epsilon, \delta, k > 0$, there are only finitely many isometry classes of hyperbolic 3-manifolds M with $\mathrm{inj}(M) \geq \delta$, first eigenvalue of the Laplacian $\lambda_1(M) \geq \delta$ and $\mathrm{rank}(\pi_1(M)) \leq k$.

Proof. Seeking a contradiction, assume that for some ϵ, δ and k positive, there is a sequence (M_i) of pairwise distinct hyperbolic 3-manifolds with $\mathrm{inj}(M_i) \geq \epsilon$, $\lambda_1(M_i) \geq \delta$ and $\mathrm{rank}(\pi_1(M_i)) \leq k$. Passing to a subsequence and choosing base points, assume that (M_i, x_i) converges to a manifold (M_∞, x_∞) as indicated in Proposition 1.2.2.

Since the manifold M_∞ has a degenerate end we have by Lemma 5.2.1 that for each $\eta > 0$, there are two disjoint compact submanifolds $U_1, U_2 \subset M$ with $\frac{\mathrm{vol}(\partial U_j)}{\mathrm{vol}(U_j)} < \eta$. Choose R with $U_1, U_2 \subset B_R(M_\infty, x_\infty)$ and let ψ_i be a sequence of bi-Lipschitz embeddings with bi-Lipschitz constant closer and closer to 1. We have then

$$\lim_{i\to\infty} \frac{\mathrm{vol}(\partial(\psi_i(U_j)))}{\mathrm{vol}(\psi_i(U_j))} = \frac{\mathrm{vol}(\partial U_j)}{\mathrm{vol}(U_j)} < \eta$$

for $j = 1, 2$. Taking into account that $\psi_i(U_1) \cap \psi_i(U_2) = \emptyset$, we deduce that

$$\limsup_{i\to\infty} h(M_i) \leq \eta$$

Buser's inequality (5.2.1) implies that

$$\limsup_{i\to\infty} \lambda_1(M_i) \leq 4\eta^2 + 10\eta$$

Since $\eta > 0$ was arbitrary we obtain that $\lim_i \lambda_1(M_i) = 0$, contradicting the assumption that $\lambda_1(M_i) > \delta$ for all i. This concludes the proof of Theorem 1.3.1. □

5.3 Rank and Arithmetic 3-manifolds

In this final section we sketch the proofs of some additional results which can be obtained if one considers only arithmetic hyperbolic 3-manifolds. See [41] for definitions.

Assume that M_i is a sequence of pairwise distinct, closed arithmetic hyperbolic 3-manifolds with $\text{inj}(M_i)$ and $\text{rank}(\pi_1(M_i)) \leq k$. A deep result of Vigneras, combined with a lemma of Long-Reid (see Agol [1]), asserts that each of the arithmetic manifolds M_i covers some hyperbolic orbifold $\mathcal{O}_i$ with $\lambda_1(\mathcal{O}_i) \geq \frac{3}{4}$. By Proposition 1.2.2 we can find base points $x_i \in M_i$ such that, up to passing to a subsequence, the manifolds (M_i, x_i) converge in the pointed Gromov-Hausdorff topology to a hyperbolic manifold M_∞ which has a degenerate end. Denote by $\hat{x}_i$ the projection of the base point x_i under the covering

$$\tau_i : M_i \to \mathcal{O}_i$$

provided by Vigneras' theorem. Passing again to a subsequence we may assume that the orbifolds $(\mathcal{O}_i, \hat{x}_i)$ and the coverings τ_i converge in the Gromov-Hausdorff topology to an orbifold $\mathcal{O}_\infty$ and a covering

$$\tau_\infty : M_\infty \to \mathcal{O}_\infty$$

respectively. By Canary's extension of Thurston's covering theorem [26], we deduce that either $\mathcal{O}_\infty$ has a degenerate end or is compact. The former case is ruled out as in the proof of Theorem 1.3.1 using that $\lambda_1(\mathcal{O}_i) \geq \frac{3}{4}$ for all i. In particular, $\mathcal{O}_\infty$ is compact and hence there is i_0 with $\mathcal{O}_i = \mathcal{O}_\infty$ for all $i \geq i_0$. We have proved that, up to passing to a subsequence, all the manifolds M_i cover some fixed orbifold and in particular, they are commensurable. Hence:

Corollary 1.3.2. For all ϵ and k positive, there are only finitely many commensurability classes of closed arithmetic hyperbolic 3-manifolds M with $\text{inj}(M) \geq \epsilon$ and $\text{rank}(\pi_1(M)) \leq k$.

Using the information given by Thurston's covering theorem more carefully, we obtain also that all but finitely many arithmetic hyperbolic 3-manifolds M with $\text{inj}(M) \geq \epsilon$ and $\text{rank}(\pi_1(M)) = k$ fiber over one of the two 1-dimensional orbifolds

$\mathbb{S}^1$ and $\mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z})$ with fiber of bounded genus. Combining this observation with the main results of [11] it is not difficult to deduce the following corollary:

Corollary 5.3.1. *Given $\epsilon, k > 0$, there are only finitely many arithmetic closed hyperbolic* 3*-manifolds with* $\mathrm{rank}(\pi_1 M) = k$ *and* $\mathrm{inj}(M) \geq \epsilon$ *that do not fiber over the circle with fiber $\Sigma_{\frac{k-1}{2}}$ (k odd), or fiber over the orbifold $\mathbb{S}^1/(z \mapsto -z)$ with non-singular fiber Σ_{k-2} (k even).*

We also obtain the two following consequences, the second of which is due to Agol in the case that $\mathrm{rank}(\pi_1(M)) = 2$.

Corollary 5.3.2. *For every ϵ and k there are only finitely many closed arithmetic hyperbolic 3-manifolds M with* $\mathrm{inj}(M) \geq \epsilon$ *and* $\mathrm{rank}(\pi_1(M)) \leq k$ *which have the same $\mathbb{Z}/2\mathbb{Z}$-homology as $\mathbb{S}^3$.*

Corollary 5.3.3. *For every ϵ and k, there are only finitely many closed arithmetic hyperbolic 3-manifolds M with* $\mathrm{inj}(M) \geq \epsilon$ *and* $\mathrm{rank}(\pi_1(M)) \leq 3$.

CHAPTER 6
ALGEBRAIC AND GEOMETRIC LIMITS

In this chapter, we study the relationship between algebraic and geometric limits of sequences of representations into $\mathrm{PSL}_2\,\mathbb{C}$. As mentioned in the introduction, our primary goal is to investigate the extent to which results concerning faithful sequences of representations fail in the unfaithful setting.

The layout of the chapter is as follows. In the first section, we construct Examples 1.4.1, 1.4.2 and 1.4.3, which exhibit ways in which the algebraic and geometric limits of sequences of unfaithful representations can differ. The next section provides a proof of Theorem 1.4.3, our generalization of Anderson-Canary's theorem concerning limits of sequences of faithful representations. After that, we prove a technical result, Proposition 6.3.1, which supplies conditions under which a covering $M \to N$ of hyperbolic 3-manifolds factors as a tower $M \to M' \to N$ with $\chi(M') > \chi(M)$. Proposition 6.3.1 and a slightly stronger version of Theorem 1.4.3 are the key ingredients in the proof of Theorem 1.4.4 in section 6.4. In section 6.5 we discuss extensions of Theorem 1.4.4 and Theorem 1.4.3 to the case that the algebraic limit has cusps; for instance, this permits us to recover Evans' general version of Anderson-Canary's Theorem 1.4.1. In the last section, we briefly describe to which extent other well-known theorems about faithful representations remain true if the condition of faithfulness is dropped.

6.1 Examples

We construct here the three examples of sequences with different algebraic and geometric limits that were promised in the introduction. All our constructions follow the same general strategy: we construct an appropriate sequence of representations

$\sigma_i : \hat{\Gamma} \to \mathrm{PSL}_2\,\mathbb{C}$ where the algebraic and geometric limits match and then restrict them to a subgroup $\Gamma < \hat{\Gamma}$ so that this is no longer the case. More specifically, assume that (σ_i) converges algebraically to a faithful representation σ_∞ and geometrically to $\sigma_\infty(\hat{\Gamma})$. Assume that $\Gamma < \hat{\Gamma}$ is a proper subgroup with $\sigma_i(\Gamma) = \sigma_i(\hat{\Gamma})$ for all $i \in \mathbb{N}$ and let $\rho_i = \sigma_i|_\Gamma$. By construction, ρ_i converges algebraically to $\rho_\infty = \sigma_\infty|_\Gamma$ and geometrically to $\sigma_\infty(\Gamma)$. Since Γ is a proper subgroup of $\hat{\Gamma}$ and σ_∞ is faithful, we obtain that the algebraic limit group $\rho_\infty(\Gamma) = \sigma_\infty(\Gamma)$ is a proper subgroup of the geometric limit $\sigma_\infty(\hat{\Gamma})$.

Here is the first example mentioned in the introduction.

Example 1.4.1. Let Γ be the fundamental group of a closed orientable surface of genus 3. There is a sequence of representations (ρ_i) in $\mathcal{D}(\Gamma)$ that converges algebraically to a faithful representation ρ and geometrically to a group G such that

- G has no parabolics
- $\rho(\Gamma)$ has index 2 in G.

It will be apparent from the proof that there is nothing very special about genus 3. In fact, the argument will work whenever Γ is the fundamental group of a closed orientable surface that nontrivially covers another such surface.

A Sequence of Convex-Cocompact Handlebodies: Let (H, S) be a genus g handlebody with boundary surface S. Recall that the Masur domain $O_H \subset \mathcal{PML}(S)$ is the set of all projective measured laminations on S that intersect positively with every element of $\mathcal{PML}(S)$ that is a limit of meridians. Since it is open and the attracting laminations of pseudo-anosov elements of $\mathrm{Mod}(S)$ are dense in $\mathcal{PML}(S)$, we may choose a pseudo-anosov diffeomorphism $f : S \to S$ with attracting lamination $\lambda \in O_H$; denote by $\bar{\lambda}$ the repelling lamination of f. Theorem 2.5.2 implies that the deformation space of convex-cocompact hyperbolic metrics on a hyperbolizable 3-manifold is parameterized by the Teichmuller space of its boundary. So, we can produce a sequence of convex-cocompact metrics on H

corresponding to an orbit of f on $\mathcal{T}(S)$. Concretely, after fixing some base conformal structure X on S, we have convex-cocompact hyperbolic manifolds $N_i, i = 1, 2, \ldots$ and homeomorphisms $h_i : (H, S) \to (\overline{N_i}, \partial\overline{N_i})$ such that $h_i \circ f^i : S \to \partial\overline{N_i}$ is conformal with respect to X.

Good Markings: By Proposition 2.5.1, the nearest point projection $\eta_i : \partial\overline{N_i} \to \partial\,\mathrm{CC}(N_i)$ is K-lipschitz for some constant K independent of i. Considering S with its Poincaré metric, the map

$$\sigma_i := \eta_i \circ h_i \circ f^{-i} : S \to N_i$$

is K-lipschitz as well. Since it is π_1-surjective, we may use σ_i to mark the fundamental group of N_i with $\pi_1(S)$. Specifically, after taking a base point $p \in S$, the surjections $(\sigma_i)_* : \pi_1(S, p) \to \pi_1(N_i, \sigma_i(p))$ determine up to conjugacy a sequence of representations

$$\rho_i : \pi_1(S, p) \to \mathrm{PSL}(2, \mathbb{C}),\ \mathbb{H}^3/\rho_i(\pi_1(S, p)) = N_i.$$

Since all σ_i are uniformly lipschitz, we there is some subsequence (ρ_{i_j}) that converges algebraically to a representation ρ_∞

$$\rho_{i_j} \ \to\ \rho_\infty : \pi_1(S, p) \to \mathrm{PSL}(2, \mathbb{C})$$

Set $N_\infty = \mathbb{H}^3/\rho_\infty(\pi_1(S, p))$.

Characterization of the Limit: The goal here is to verify the following description of the algebraic limit ρ_∞.

> The representation $\rho_\infty : \pi_1(S, p) \to \mathrm{PSL}(2, \mathbb{C})$ is faithful and purely loxodromic, so $N_\infty \cong S \times \mathbb{R}$ and has no cusps. One of the ends of N_∞ is convex-cocompact with associated conformal structure X and the other is degenerate with ending lamination $\bar{\lambda}$; here X is the base conformal structure on S fixes above and $\bar{\lambda}$ is the repelling lamination of f. In

particular, the limit ρ_∞ does not depend on the chosen subsequence (ρ_{i_j}) and the convergence $\rho_i \to \rho_\infty$ is strong.

We first need to show that the marking of $\pi_1(N_\infty)$ given by ρ is induced by an embedding $\sigma_\infty : S \to N_\infty$. Assume by passing to a subsequence that $\rho_{i_j}(\pi_1(S,p)) \to G \subset \mathrm{PSL}(2,\mathbb{C})$ geometrically, and let $N_G = \mathbb{H}^3/G$. If base points are chosen for N_{i_j} on $\partial\,\mathrm{CC}(N_{i_j})$, the resulting based manifolds converge in the Gromov Hausdorff topology to N_G. Our marking maps σ_{i_j} converge to a map $\sigma_G : S \to N_G$ (the existence of the limit can be shown by applying Arzela-Ascoli's Theorem to the composition of σ_{i_j} with a sequence of almost isometric maps from N_{i_j} to N_G coming from geometric convergence). There is a natural covering map $\pi : N_\infty \to N_G$ corresponding to the inclusion $\rho_\infty(\pi_1(S,p)) \subset G$, and σ_G lifts to a map $\sigma_\rho : S \to N_\infty$ inducing the marking given by ρ_∞.

Claim. The image of σ_∞ bounds a neighborhood E' of a convex-cocompact end of N_∞. Furthermore, $E' \cong \Sigma_g \times \mathbb{R}$ and $\pi|_{E'} : E' \to N_G$ is an embedding.

Proof. We will show that the image of σ_G bounds a convex-cocompact end in N_G homeomorphic to $\Sigma_g \times \mathbb{R}$. It will follow immediately that this end lifts to an end E' as desired.

Let $K \subset N_G$ be some Gromov-Hausdorff limit of the sequence of convex cores $\mathrm{CC}(N_{i_j})$, passing to another subsequence as necessary. Note that K is convex and contains $\mathrm{CC}(N_G)$. Its boundary ∂K is the limit of the boundaries $\partial\,\mathrm{CC}(N_{i_j})$, and is therefore the image of σ_G. Using convexity, it is then not hard to see that the image of σ_G is homeomorphic to Σ_g. Setting $E = N_G \setminus K$, we obtain a convex-cocompact end of N_G. As $\partial E \cong \Sigma_g$, the nearest point retraction gives a homeomorphism $N_G \setminus K \cong \Sigma_g \times (0,\infty)$. □

Recall that the sequence (N_{i_j}) is marked by precomposing a homeomorphism $\overline{h_{i_j}} : S \to \partial\overline{N_{i_j}}$ that extends to the handlebodies with a pseudo-anosov map $f : S \to S$ that has attracting lamination $\lambda \in O_H$. The following claim is the reason we chose λ in the Masur domain O_H.

Claim. ρ_∞ is faithful.

Proof. An equivalent statement of the claim is that the map $\sigma_\infty : S \to N_\infty$ is π_1-injective. So by the Loop Theorem, it suffices to check that ρ_∞ is injective on elements of $\pi_1(S,p)$ representable by simple loops on S.

Let $\gamma \subset S$ be a simple closed curve. Then $f^{i_j}(\gamma) \to \lambda$ in $\mathcal{PML}$; since $\lambda \in O_H$, which is an open subset of $\mathcal{PML}$, for large i_j we have $f^{i_j}(\gamma) \in O_H$ as well. In particular, $f^{i_j}(\gamma)$ is not compressible in H. It follows that

$$\sigma_{i_j}(\gamma) = \eta_{i_j} \circ h_{i_j} \circ f^{i_j}(\gamma)$$

is incompressible in N_{i_j}. Therefore $\rho_{i_j}(\gamma) \neq \mathrm{Id}$ for sufficiently large i_j. A standard application of the Margulis Lemma shows that $\rho_\infty(\gamma) \neq \mathrm{Id}$ as well (see, e.g.[43, Theorem 7.1]). □

Geometrically, the reason that ρ_∞ is faithful is that for large i all compressible curves on $\partial\,\mathrm{CC}(N_i)$ are very long. Since a fixed generating set for $\pi_1(S,p)$ maps under our markings to a set of loops on $\partial\,\mathrm{CC}(N_i)$ with uniformly bounded length, this implies that elements of $\pi_1(S,p)$ representing compressible curves in N_i must for large i be expressible only with very large words in the generators. So in the limit, there are no compressible curves.

As $\rho_\infty : \pi_1(S,p) \to \mathrm{PSL}(2,\mathbb{C})$ is faithful, the Tameness Theorem implies that $N_\infty \cong S \times \mathbb{R}$. From above, we know that one of the two topological ends of N_∞ is convex-cocompact with associated conformal structure X. To analyze the geometry of the other end, we must first prove the following:

Claim. The lamination $\bar{\lambda}$ is not realized in N_∞. In particular, N_∞ has a degenerate end with ending lamination λ.

Proof. Fix a meridian m. By definition, the curve $f^{-i_j}(m)$ is in the kernel of the representation ρ_{i_j} for all i_j. On the other hand, the sequence $(f^{-i_j}(m))$ converges to $\bar{\lambda}$ in $\mathcal{PML}(S)$. This implies that $\bar{\lambda}$ is not realized; see for instance [49, Section 4]. □

Since N_∞ is homeomorphic to $S \times \mathbb{R}$ and has a convex-cocompact end and a degenerate end, we deduce that ρ_∞ is purely loxodromic. Moreover, the resolution of the ending lamination conjecture by Minsky [46] and Brock-Canary-Minsky [18] implies that the limit ρ_∞ does not depend on the chosen subsequence. The next observation concludes our analysis.

Claim. ρ_i converges to ρ_∞ strongly.

Proof. We need to show that the covering $\pi : N_\infty \to N_G$ is trivial. There is a neighborhood of the degenerate end of N_∞ that is completely contained in the convex core $\mathrm{CC}(N_\infty)$. So, $N_\infty \setminus E'$ is the union of $\mathrm{CC}(N_\infty)$ and some compact set. A result of Thurston [39] and Bonahon [13] implies that the injectivity radius of N_∞ is bounded above inside of its convex core, so by extension there is an upper bound K for the injectivity radius of N_∞ outside of E'. Pick a point $x \in N_G$ deep enough inside its convex-cocompact end so that $\mathrm{inj}(x, N_G) > K$. Then no element of $N_\infty \setminus E'$ can project to x. Since π is injective on E', $|\pi^{-1}(x)| = 1$. Therefore π is trivial. □

Restricting the Markings: Finally, we show that we can restrict our representations (ρ_i) to a subgroup of $\pi_1(S, p)$ to create a sequence with the properties desired for our example.

Claim. After passing to a subsequence, there exists an index 2 subgroup $\Gamma \subset \pi_1(S, p)$ such that for each $i > 0$,

$$(\sigma_i)_*|_\Gamma : \Gamma \to \pi_1(N_i(\sigma_i(p)))$$

is a surjection.

Proof. The proof consists of two simple observations. First, if M is a handlebody then there is an index 2 subgroup $\Gamma \subset \pi_1(\partial M)$ that surjects onto $\pi_1(M)$. One can take, for example, the kernel of the map taking an element of $\pi_1(\partial M)$ to its mod-2 intersection number with any longitude in a standard meridian-longitude basis for

$H_1(\partial M)$. Therefore, we can construct for each i and index 2 subgroup Γ_i with $(\sigma_i)_* : \Gamma_i \to \pi_1(N_i, \sigma_i(p))$ a surjection. Since there are only finitely many index 2 subgroups of $\pi_1(S,p)$, we can pass to a subsequence so that a single $\Gamma \subset \pi_1(S,p)$ works for all i. □

Consider now the sequence of representations $\rho_i|_\Gamma : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$. The claim implies that $\rho_i(\Gamma) = \rho_i(\pi_1(S,p))$, so

$$\rho_i(\Gamma) \to \rho_\infty(\pi_1(S,p))$$

geometrically. Now $\rho_i \to \rho_\infty$ algebraically, so $\rho_i|_\Gamma \to \rho_\infty|_\Gamma$. Therefore, since ρ_∞ is faithful, $\rho_\infty(\Gamma)$ has index 2 in $\rho_\infty(\pi_1(S,p))$. We have therefore provided a sequence of representations converging algebraically to a faithful representation whose image is an index 2 subgroup of the geometric limit, and so that the geometric limit has no parabolics. This concludes our example.

An Alternate Method: The difficult part of the example above is constructing a sequence of hyperbolic 3-manifolds (N_i), each homeomorphic to the interior of a genus g handlebody, that converges geometrically to a hyperbolic 3-manifold N_∞ homeomorphic to $\Sigma_g \times \mathbb{R}$. Such a sequence can be assembled differently by working backwards from the limit.

Fix a Bers slice $X \subset AH(\pi_1(\Sigma_g))$. There is a marked hyperbolic 3-manifold $N_\infty \in \partial X$ with no cusps, one degenerate end and one convex-cocompact end. One can construct N_∞, for instance, as the algebraic limit of a sequence obtained by iterating a pseudo-anosov mapping class on X, [20]. McMullen has shown, [44], that any point on the boundary of a Bers slice can be approximated by one-sided maximal cusps. This means that there is a sequence of marked hyperbolic 3-manifolds $M_i \in \partial X$ converging algebraically to N_∞, where each M_i has a maximally cusped end. As N_∞ has no cusps, Theorem 1.4.1 implies that there are base points $p_i \in M_i$ so that $(M_i, p_i) \to (N_\infty, p_\infty)$ geometrically. Furthermore, p_i can be chosen to lie on the component of $\partial\,\mathrm{CC}(M_i)$ facing the convex-cocompact end of M_i. Note that since the thrice-punctured sphere components of $\partial\,\mathrm{CC}(M_i)$ disappear in the geometric limit, their distances to p_i grow without bound.

Passing to a subsequence, we may assume that the pinched pants decompositions on the maximally cusped ends of M_i all have the same topological type. In other words, we may choose a pants decomposition $P \subset \Sigma_g$ and homeomorphisms

$$\overline{M}_i \cong (\Sigma_g \times [0,1]) \setminus (P \times \{0\})$$

for all i. Pick an identification of Σ_g with the boundary of a genus g handlebody H so that P is a pinchable collection of curves on ∂H; the latter condition can be ensured, for instance, by composing any fixed identification with a high power of a pseudo-anosov homeomorphism of ∂H whose attracting lamination lies in the Masur domain. This allows us to endow H with a geometrically finite hyperbolic metric in which P has been pinched. Then both $\partial\,\mathrm{CC}(H)$ and the bottom boundary components of $\mathrm{CC}(M_i)$ are identified with $\Sigma_g \backslash P$, so we may use Lemma 2.9.1 to glue them together. This produces a sequence of hyperbolic 3-manifolds N_i' equipped with isometric embeddings of $\mathrm{CC}(H)$ and the subset $K_i \subset M_i$ that is the union of the convex core of M_i and its convex-cocompact end; we will denote the inclusion of the latter by $\iota_i : K_i \to N_i'$. Since the frontier of K_i in M_i consists of the thrice-punctured sphere components of $\partial\,\mathrm{CC}(M_i)$, its distance to the base point $p_i \in M_i$ tends to infinity with i. The same holds for the distances from $\iota_i(p_i)$ to the frontier of $\iota_i(K_i) \subset N_i'$. Therefore, the sequence of based manifolds $(N_i', \iota_i(p_i))$ converges geometrically to the same limit, (N_∞, p_∞), as our original sequence.

Observe that N_i' is homeomorphic to the manifold obtained from H by pushing $P \subset \partial H$ into the interior of H and then drilling it out. The curves in P correspond to rank 2 cusps of N_i', so we may choose (x, y)-coordinates for the Dehn filling space of each cusp so that $(1, 0)$ corresponds to filling a curve that is contractible in H and $(0, 1)$ represents filling a curve homotopic into P. Then the manifold $N_{i,n}'$ obtained from $(1, n)$-Dehn filling on each cusp of N_i' is homeomorphic to H (compare with [39, Section 3]). If n is large, an extension of Thurston's Dehn filling theorem due to to Bonahon-Otal [14] and Comar [28] implies that $N_{i,n}'$ is hyperbolic. Furthermore, there are base points $p_{i,n} \in N'_{i,n}$ such that $(N'_{i,n}, p_{i,n}) \to (N'_i, \iota_i(p_i))$ geometrically. An appropriate sequence (n_i) can then be chosen so that (N'_{i,n_i}, p_{i,n_i}) converges geometrically to (N_∞, p_∞). Setting $N_i = N'_{i,n_i}$ finishes our work.

Example 1.4.2. Let Γ be the fundamental group of a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3. There is a sequence (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to a representation ρ and geometrically to a group G such that:

- G does not contain any parabolic elements.
- $\rho(\Gamma)$ has infinite index in G.

The representations here are constructed from those in the previous example by using Klein-Maskit combination:

The Combination Theorem (see [43]). Let G_1 and G_2 be two discrete and torsion free subgroups of $\mathrm{PSL}_2\,\mathbb{C}$. Suppose that there exist fundamental domains $D_i \subset \Omega(G_i)$ for G_i, each containing the exterior of the other. Then $G = \langle G_1, G_2 \rangle$ is discrete, torsion free and is isomorphic to $G_1 \star G_2$. Moreover, if the groups G_i do not contain parabolics then the same is true for G.

Recall our notation from the previous example: (ρ_i) is a sequence of representations in $\mathcal{D}(\pi_1(\Sigma_2))$ converging strongly to a faithful representation ρ_∞ without parabolics, and $\Gamma \subset \pi_1(\Sigma_2)$ is an index 2 subgroup with $\rho_i(\Gamma) = \rho_i(\pi_1(\Sigma_2)$ for all i. For convenience, set $G_i = \rho_i(\pi_1(\Sigma_2))$ and $G_\infty = \rho_\infty(\pi_1(\Sigma_2))$.

Our previous analysis implies that the convex cores of $\mathbb{H}^3/G_i$ converge to the convex core of $\mathbb{H}^3/G_\infty$ in the Gromov-Hausdorff topology, so the limit sets $\Lambda(G_i)$ converge to $\Lambda(G_\infty)$ in the Hausdorff topology. Since $\Omega(G_\infty) \neq \emptyset$, it follows that there exist fundamental domains $D_i \subset \Omega(G_i)$ for the action of G_i converging to a fundamental domain D_∞ for the action of G_∞ on $\Omega(G_\infty)$. Pick some loxodromic element $\alpha \in \mathrm{PSL}(2, \mathbb{C})$ with fixed points contained in the interior of D_∞. Moreover, assume that its translation distance is large enough so that there is a fundamental set for $\Omega(\langle \alpha \rangle)$ whose complement is entirely contained within D_∞. After discarding a finite number of terms, its complement will also be contained in D_i for all i. We now construct new representations

$$\rho_i' : \pi_1(\Sigma_2) \star \mathbb{Z} \to \mathrm{PSL}(2, \mathbb{C})$$

from ρ_i by sending $1 \in \mathbb{Z}$ to α. If $G_i' \subset \mathrm{PSL}(2, \mathbb{C})$ is the image of ρ_i', the Klein-Maskit combination theorem implies that $G_i' = G_i \star \langle \alpha \rangle$ and is discrete, torsion free and has no parabolics. The same statements apply when $i = \infty$, showing in particular that ρ_i' is faithful.

It follows, for instance, from the argument in [5, Proposition 10.2] that $\rho_i' \to \rho_\infty'$ strongly. Consider the subgroup $\Gamma' = \Gamma \star \mathbb{Z} \subset \pi_1(\Sigma_2) \star \mathbb{Z}$. Then $\rho_i'|_{\Gamma'}$ converges algebraically to $\rho_\infty'|_{\Gamma'}$. However, since $\rho_i(\Gamma) = \rho_i(\pi_1(\Sigma_2))$ for all i, $\rho_i'(\Gamma') = \rho_i'(\pi_1(\Sigma_2) \star \mathbb{Z})$ as well. So, $\rho_i'(\Gamma')$ converges geometrically to G_∞'. Since ρ_∞' is faithful, $\rho_\infty'(\Gamma')$ has infinite index in G_∞'. Therefore $\rho_i' : \Gamma' \to \mathrm{PSL}(2, \mathbb{C})$ is a sequence of representations for which the algebraic limit has infinite index in the geometric limit. As mentioned above, the geometric limit Γ' has no parabolics. Since Γ' is isomorphic to the fundamental group of a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3, we have provided the desired example.

The geometry of Example 1.4.2: The manifolds $N_i' = \mathbb{H}^3/G_i'$ are all convex-cocompact hyperbolic 3-manifolds homeomorphic to the interior of a genus 3 handlebody. The strong limit $N_\infty' = \mathbb{H}^3/G_\infty'$ is then a compression body with genus 3 exterior boundary and connected, genus 2 interior boundary. Its exterior end is convex-cocompact and its interior end is degenerate. After restricting our representations to Γ', the manifolds in our sequence and their geometric limit remain unchanged, but the algebraic limit is now a cover of N_∞' homeomorphic to a compression body with genus 4 exterior boundary and connected, genus 3 interior boundary.

Example 1.4.3. Let Γ be the fundamental group of a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3. There is a sequence (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to a representation ρ and geometrically to a group G such that:

- $\rho(\Gamma)$ has does not contain any parabolic elements.
- G is infinitely generated.

In [60], Thurston exhibited a sequence of representations of a closed surface group into $\mathrm{PSL}(2, \mathbb{C})$ converging geometrically to a group that is not finitely generated. However, the algebraic limit of these representations contains parabolic elements. The idea here is to attach pieces of the handlebodies in Example 1.4.2 to the manifolds in his sequence so that the parabolics are hidden from the algebraic limit.

To facilitate such a combination, we must build a variant of Example 1.4.2 in which each of the handlebodies in the sequence is maximally cusped. Let M be a compression body with genus 3 exterior boundary S_E and connected, genus 2 interior boundary S_I. Assume that $P_E \subset S_E$ is a pants decomposition consisting of curves in the Masur domain of M.

Claim. There is a sequence of maximally cusped pointed hyperbolic 3-manifolds (N_i), each homeomorphic to the interior of a genus 3 handlebody, that converges geometrically to a hyperbolic 3-manifold N_∞ homeomorphic to the interior of M in which P_E has been pinched. Moreover, the subsets $\mathrm{CC}(N_i) \subset N_i$ converge to $\mathrm{CC}(N_\infty)$ geometrically.

Proof. The sequence (N_i) will be constructed in two steps. First, we will produce a sequence of hyperbolic 3-manifolds M_i homeomorphic to the interior of M in which both ends are maximally cusped. The sequence will converge geometrically to the manifold N_∞ referenced in the statement of the claim. We will then use the same gluing trick exploited in Example 1.4.1 to cap off the interior ends of each M_i without changing the sequence's geometric limit, thus producing the desired sequence of handlebodies (N_i).

Fix pants decompositions P_E and P_I for the boundary components of M, and assume that every curve in P_E lies in the Masur domain of M. If we choose a pseudo-anosov diffeomorphism $f : S_I \to S_I$, then for each i we have a new pants decompositions $f^i(P_I)$ for S_I. It is not hard to check that for each i, $P_E \cup f^i(P_I)$ is a pinchable collection of curves on ∂M (see Section 2.9). So, there is a sequence of marked hyperbolic 3-manifolds $M_i \in AH(M)$ in which $P_E \cup f^i(P_I)$ have been pinched. Note that in fact M_i lies in the deformation space $AH(M, P_E)$ of hyperbolic structures on the interior of M in which the curves of P_E represent parabolics. Since P_E consists of curves lying in the Masur domain, the pared manifold (M, P_E) has incompressible and acylindrical boundary. It follows from a theorem of Thurston, [59, Theorem 7.1], that $AH(M, P_E)$ is compact. So after passing to a subsequence, we may assume that (M_i) converges algebraically to some $N_\infty \in AH(M, P_E)$.

We claim that the only parabolic loops in N_∞ are those that are freely homotopic into P_E. The end of N_∞ facing S_E is maximally cusped by P_E, so there is no room for additional cusps there. It therefore suffices to show that the end facing S_I has no cusps. Work of Thurston, Bonahon and Brock, implies that there is a continuous map

$$\text{length} : AH(M) \times \mathcal{ML}(S_I) \to \mathbb{R}$$

that extends the function that assigns to an element $N \in AH(M)$ and a simple closed curve $\gamma \in S_I$ the shortest length of a curve in N homotopic to γ (see [17]). Since $\text{length}_{M_i}(f^i(P_I)) = 0$ for all i, in the limit we have $\text{length}_{M_\infty}(\lambda) = 0$, where λ is the attracting lamination of f. This implies that λ cannot be geodesically realized by a pleated surface in N_∞ homotopic to S_I. Let $\hat{M}_\infty$ be the cover of N_∞ corresponding to $\pi_1(S_I)$. The end of N_∞ facing S_I lifts homeomorphically to an end $\mathcal{E}$ of $\hat{N}_\infty$. The other end of $\hat{N}_\infty$ has no cusps and is therefore convex-cocompact by the Tameness Theorem and Thurston's covering theorem. Since λ is filling and unrealizable in $\hat{N}_\infty$, the argument in [43, Theorem 6.34] then shows that $\mathcal{E}$ is degenerate with ending lamination λ. In particular, $\mathcal{E}$ has no cusps. Projecting down, the same is true for the end of N_∞ facing S_I.

This shows that the convergence $M_i \to N_\infty$ is type preserving, so Theorem 1.4.1 implies that the convergence is strong. So, base frames for M_i can be chosen so that the sequence converges geometrically to N_∞. The rest of the argument follows that given at the end of Example 1.4.1. Fix a genus 2 handlebody H and choose a hyperbolic metric on its interior in which a pants decomposition $P \subset \partial H$ with the same topological type as $P_I \subset S_I$ has been pinched. We can then create for each i a hyperbolic 3-manifold N_i' by removing from M_i the component of $M_i \setminus \mathrm{CC}(M_i)$ facing S_I and gluing $\mathrm{CC}(H)$ in its place. As in Example 1.4.1, the sequence (N_i') converges geometrically to N_∞. Performing an appropriate Dehn filling on each N_i' yields a sequence of hyperbolic 3-manifolds (N_i), each homeomorphic to the interior of a genus 3 handlebody, that also converges geometrically to N_∞.

We must show that $\mathrm{CC}(N_i) \to \mathrm{CC}(N_\infty)$ geometrically. First, every component of $\partial\,\mathrm{CC}(N_\infty)$ is contained in a geometric limit of some sequence of components of $\partial\,\mathrm{CC}(N_i)$. These are all thrice punctured spheres, however, so in fact we have that for large i there are components of $\partial\,\mathrm{CC}(N_i)$ that closely approximate each component of $\partial\,\mathrm{CC}(N_\infty)$. However, $\partial\,\mathrm{CC}(N_i)$ and $\partial\,\mathrm{CC}(N_\infty)$ both have 6 components, so for large i they must almost coincide. From this, it is easy to check that $\mathrm{CC}(N_i) \to \mathrm{CC}(N_\infty)$ geometrically. $\square$

Fix a geometrically finite hyperbolic structure on $\Sigma_3 \times \mathbb{R}$ in which both ends are maximally cusped, and let C be its convex core. Then there are pants decompositions $P_{+,-} \subset \Sigma_3$ so that

$$C \cong (\Sigma_3 \times [-1,+1]) \setminus (P_- \times \{-1\} \cup P_+ \times \{+1\}),$$

and we label the components of ∂C as positive and negative accordingly, so that $\partial C = \partial_+ C \cup \partial_- C$. Let E be the component of the complement of C that faces its positive boundary components. Assume that the pairs $(\Sigma_3, P_{+,-})$ and (S_E, P_E) have the same topological type, and that every curve in P_+ intersects some curve in P_-.

We now glue these pieces to the manifolds N_i and N_∞ from the previous Lemma. To begin with, let

$$N_\infty' = \mathrm{CC}(N_\infty) \sqcup_h C \sqcup_g C \sqcup_g \cdots .$$

Here, the gluing maps $h : \partial\,\mathrm{CC}(N_\infty) \to \partial_- C$ and $g : \partial_- C \to \partial_+ C$ can be any isometries that extend to maps $(S_E, P_E) \to (\Sigma_3, P_-)$ and $(\Sigma_3, P_+) \to (\Sigma_3, P_-)$. This ensures that N'_∞ is constructed as in Lemma 2.9.1. Note that the inclusion map $\mathrm{CC}(N_\infty) \to N'_\infty$ is π_1-injective.

Next, for large i geometric convergence and the gluing map h determine an identification $h_i : \partial\,\mathrm{CC}(N_i) \to \partial_- C$. Define

$$N'_i = \mathrm{CC}(N_i) \sqcup_{h_i} \underbrace{C \sqcup_g \cdots \sqcup_g C}_{\text{i times}} \sqcup_{id} E.$$

Recall that $\mathrm{CC}(N_i) \to \mathrm{CC}(N_\infty)$ geometrically. From this it follows that if N'_i is given the base frame of N_i produced in the previous Lemma, then (N'_i) converges geometrically N'_∞.

As in Example 1.4.1, performing $(1, n)$-Dehn filling on each of the cusps in N'_i produces, for large n, a new hyperbolic manifold $N'_{i,n}$ homeomorphic to the interior of a genus 3 handlebody. Also, an appropriate diagonal sequence $N^*_i = N'_{i,n_i}$ can be chosen to converge geometrically to N'_∞. In summary, we have proven the following claim.

Claim. There is a sequence of convex-cocompact pointed hyperbolic 3-manifolds N^*_i, each homeomorphic to the interior of a genus 3 handlebody, that converges geometrically to a hyperbolic 3-manifold N'_∞ with infinitely generated fundamental group.

We now obtain the sequence of representations advertised in the statement of this example by marking the manifolds N^*_i appropriately. Recall that the fundamental group of the compression body M splits as a free product

$$\pi_1(M) = \pi_1(S_I) \star \langle \alpha \rangle \cong \pi_1(\Sigma_2) \star \mathbb{Z},$$

for some element $\alpha \in \pi_1(M)$. The inclusion map $M \cong \mathrm{CC}(N_\infty) \to N'_\infty$ is π_1-

injective, so it determines an embedding

$$\rho_\infty : \pi_1(\Sigma_2) \star \mathbb{Z} \to \pi_1(N'_\infty).$$

Then ρ_∞ identifies a finite generating set for $\pi_1(\Sigma_2) \star \mathbb{Z}$ with a finite set of loops in N'_∞. For large i, geometric convergence provides an almost isometric embedding of these loops into N_i^*; therefore, there are induced homomorphisms

$$\rho_i : \pi_1(\Sigma_2) \star \mathbb{Z} \to \pi_1(N_i^*).$$

In fact, ρ_i is surjective, and therefore is a marking of $\pi_1(N_i^*)$.

The sequence of marked hyperbolic manifolds (N_i^*, ρ_i) converges algebraically to the cover of N'_∞ corresponding to the image of ρ_∞, and converges geometrically to N'_∞ (as noted above). Note that $\pi_1(N'_\infty)$ is not finitely generated. We are not quite done, however, because the algebraic limit here has cusps. To hide the cusps, we will use the finite index trick exploited in Example 1.4.1, but for this to work the pants decomposition P_E used above must be chosen more carefully.

Claim. There is a pants decomposition $P_E \subset S_E$ consisting of curves in the Masur domain, none of which are conjugate into a subgroup of $\pi_1(M)$ of the form $\Gamma \star \langle\alpha\rangle$, where $\Gamma < \pi_1(S_I, p)$ has index 2.

Deferring the proof for a moment, pick as in Examples 1.4.1 and 1.4.2 an index 2 subgroup $\Gamma \subset \pi_1(\Sigma_2)$ so that $\rho_i|_{\Gamma\star\mathbb{Z}}$ surjects onto $\pi_1(N_i^*)$. If in constructing (N_i^*), the pants decomposition P_E is chosen as indicated in the above claim, there will be no parabolics in the algebraic limit of $(\rho_i|_{\Gamma\star\mathbb{Z}})$. However, since this is still a sequence of markings for N_i^*, the geometric limit will be N'_∞, which has infinitely generated fundamental group. This finishes the example.

Proof of Claim. Although the claim is purely topological, the proof we give uses 3-dimensional hyperbolic geometry. It would be nice to give a more straightforward proof; also, it is possible that the the second part of the conclusion is satisfied by any pants decomposition of curves in the Masur domain.

To begin, construct by some means a hyperbolic manifold N homeomorphic to the interior of M that has no cusps and in which both ends are degenerate. One way to produce N is as follows. First, find a hyperbolic manifold homeomorphic to $\Sigma_3 \times \mathbb{R}$ that has one degenerate end and one maximally cusped end. This can be done using an argument similar to the construction of N_∞ above. We can then glue its convex core to the convex core of N_∞ as in Lemma 2.9.1 and appropriately fill the resulting cusps to create a totally degenerate hyperbolic manifold N homeomorphic to the interior of M.

Fix an index 2 subgroup $\Gamma \subset \pi_1(S_I, p)$, and let N_Γ be the cover of N corresponding to $\Gamma \star \langle\alpha\rangle$. Then N_Γ has a degenerate end homeomorphic to $\Sigma_3 \times [0, \infty)$ that double covers the genus 2 end of N. Adjoining a loop in N_Γ representing α to a level surfaces of this end and thickening produces a compact core for N_Γ homeomorphic to the interior of a compression body with genus 4 exterior boundary and connected, genus 3 interior boundary. The tameness theorem of Agol [2] and Calegari-Gabai [23] implies that N_Γ is itself homeomorphic to the interior of such a compression body, and a theorem of Canary [24] implies that its genus 4 end, $\hat{\mathcal{E}}$, is either degenerate or convex-cocompact. It cannot be that both ends of N_Γ are degenerate, for then Canary's covering theorem [25] would imply that $\Gamma \star \langle\alpha\rangle$ is finite index in $\pi_1(N)$. Therefore, $\hat{\mathcal{E}}$ is convex-cocompact.

Since the genus 3 end, $\mathcal{E}$, of N is degenerate, it has an ending lamination $\lambda \subset S_E$. Canary has shown that λ lies in the Masur domain of M (see [24] for a proof of this and the uniqueness of λ). Choose a pants decomposition $P_E \subset S_E$ consisting of curves that lie close to λ in $\mathcal{PML}(S_E)$. The Masur domain of M is an open subset of $\mathcal{PML}(S_E)$, so we may assume that each curve in P_E lies inside of it. Furthermore, their geodesic representatives in N lie very deep inside of $\mathcal{E}$. As $\hat{\mathcal{E}}$ is convex-cocompact, the convex core of N_Γ covers a subset of N that has bounded intersection with $\mathcal{E}$. So, we may assume that the geodesic representatives of curves in P_E do not intersect its image. If some curve in P_E were conjugate into $\Gamma \star \langle\alpha\rangle$, then its geodesic representative in N would lift to a closed geodesic in N_Γ. Every closed geodesic in N_Γ is contained in its convex core, so this is impossible. $\square$

6.2 Proof of Theorem 1.4.3

Before beginning the bulk of the proof, we will present a technical lemma whose proof requires a bit of differential geometry. Afterwards, Theorem 1.4.3 will follow from purely synthetic arguments.

Recall from Proposition 2.4.2 that a hyperbolic 3-manifold M with finitely generated fundamental group and no cusps contains a compact core $C \subset M$ for which each component of ∂C facing a convex-cocompact end of M is smooth and strictly convex. For convenience, let S_{cc} be the union of those components of ∂C facing convex cocompact ends and E_{cc} the union of the adjacent components of $M \setminus C$. Then E_{cc} is homeomorphic to $S_{cc} \times (0, \infty)$ via 'radial coordinates':

$$R : S_{cc} \times (0, \infty) \to E_{cc}, \;\; R(x, t) = \exp_x(t\nu(x)),$$

where ν is the outer unit-normal vectorfield along S_{cc}.

If $f : C \to N$ is a smooth immersion into some complete hyperbolic 3-manifold N, it has a natural *radial extension* $\bar{f} : C \cup E_{cc} \to N$. Namely, there is a radial coordinate map along the image of S_{cc}:

$$R_f : S_{cc} \times (0, \infty) \to N, \;\; R_f(x, t) = \exp_x(t\nu_f(x)),$$

where $\nu_f(x)$ is the unit vector in $TN_{f(x)}$ orthogonal to $df_x(TS_x)$ that points away from $f(C)$, and one can then define

$$\bar{f}(p) = \begin{cases} R_f \circ R^{-1}(p) & p \in E_{cc} \\ f(p) & p \in C. \end{cases}$$

Observe that $\bar{f}$ is continuous, and differentiable everywhere but on S_{cc}.

In the situations where we will find radial extensions useful, the map f will be very close in the C^2 topology to a *Riemannian* immersion. In particular, the (strict) convexity of the surface S_{cc} will persist in the image. This implies a convenient regularity in the radial extension:

Lemma 6.2.1. *If $f : C \to N$ is a smooth immersion with $f(S_{cc})$ convex, there exists some $L > 0$ so that for all $p \in E_{cc}$ and $v \in TM_p$,*

$$\frac{1}{L} \leq \frac{||d\bar{f}_p(v)||}{||v||} \leq L.$$

The following global statement comes from applying Lemma 6.2.1 and a compactness argument on C.

Corollary 6.2.2 (Radial Extensions are Locally Bilipschitz)**.** *If $f : C \to N$ is a smooth immersion with $f(S_{cc})$ convex, there exists some $L > 0$ so that every $p \in C \cup E_{cc}$ has a neighborhood on which f is L-bilipschitz.*

Proof of Lemma 6.2.1. Consider a component $E \subset E_{cc}$, and let $S \subset S_{cc}$ be the adjacent component of ∂C. Here, $\bar{f}$ is the composition $R_f \circ (R)^{-1}$ of radial coordinate maps, so to prove the Lemma it suffices to find a constant L so that for all $(x,t) \in S \times (0,\infty)$ and $v \in TS_x \times \mathbb{R}$,

$$\frac{1}{L} \leq \frac{||(dR_f)_{(x,t)}(v)||}{||dR_{(x,t)}(v)||} \leq L. \tag{6.2.1}$$

Since the ratio is one when v is contained in the $\mathbb{R}$ factor, from now on we will assume $v \in TS_x$.

We first estimate $\|dR_{(x,t)}(v)\|$. Given $x \in S$ and $v \in TS_x$, let $g(s)$ be a curve in S with $g(0) = x$ and $g'(0) = v$, and consider the geodesic variation

$$\gamma_s(t) := R(g(s),t) = \exp_{g(s)}(t\nu(g(s)))$$

The corresponding Jacobi field $J_{x,v}(t) = \frac{d}{ds}R(g(s),t)|_{s=0}$ along the geodesic $\gamma_0(t)$ then satisfies $J_{x,v}(t) = dR_{(x,t)}(v)$.

The Jacobi-field $J_{x,v}(t)$ is determined by its initial conditions

$$\begin{aligned} J_{x,v}(0) &= dR_{(x,0)}(v), \text{ and} \\ \frac{\nabla}{dt} J_{x,v}(t)|_{t=0} &= \frac{\nabla}{dt}\frac{d}{ds} \exp_{g(s)} \big(t\nu(g(s))\big)|_{t=0} \\ &= \frac{\nabla}{ds}\frac{d}{dt} \exp_{g(s)} \big(t\nu(g(s))\big)|_{t=0} \\ &= \frac{\nabla}{ds}\nu(g(s)) = \nabla_{dR_{(x,0)}(v)}\nu \end{aligned}$$

Therefore, we have

$$J_{x,v}(t) = \cosh(t)E_1(t) + \sinh(t)E_2(t),$$

where $E_1(t)$ and $E_2(t)$ are the parallel vector fields along γ_0 with $E_1(0) = dR_{(x,0)}(v)$ and $E_2(0) = \nabla_{dR_{(x,0)}(v)}\nu$. That the right-hand side satisfies the Jacobi equation follows quickly from the fact that $E_1(t)$ and $E_2(t)$ are both orthogonal to $\gamma_0'(t)$.

The triangle inequality, together with the fact that the vector fields E_1 and E_2 have constant length, shows that

$$\|J_{x,v}(t)\| \leq \cosh(t)(\|dR_{(x,0)}(v)\| + \|\nabla_{dR_{(x,0)}(v)}\nu\|).$$

On the other hand we have

$$\begin{aligned} \|J_{x,v}(t)\|^2 = &\cosh(t)^2\|dR_{(x,0)}(v)\|^2 + \sinh(t)^2\|\nabla_{dR_{(x,0)}(v)}\nu\|^2 \\ &+ 2\cosh(t)\sinh(t)\langle dR_{(x,0)}v, \nabla_{dR_{(x,0)}(v)}\nu\rangle \end{aligned}$$

By convexity of S_{cc}, the last term in the sum is positive. A little bit of algebra and the fact that $J_{x,v}(t) = dR_{(x,t)}(v)$ yields

$$\frac{\sinh(t)}{2} \leq \frac{\|dR_{(x,t)}(v)\|}{\|dR_{(x,0)}(v)\| + \|\nabla_{dR_{(x,0)}(v)}\nu\|} \leq \cosh(t) \tag{6.2.2}$$

Since $f(S_{cc})$ is convex, a similar computation shows

$$\frac{\sinh(t)}{2} \leq \frac{\|d(R_f)_{(x,t)}(v)\|}{\|d(R_f)_{(x,t)}(v)\| + \|\nabla_{d(R_f)_{(x,t)}v}\nu_f\|} \leq \cosh(t) \tag{6.2.3}$$

By compactness the ratio between the denominators in (6.2.2) and (6.2.3) is uniformly bounded from above and below. In other words, there is some positive constance c with

$$\frac{\sinh(t)}{2c\cosh(t)} \leq \frac{\|d(R_f)_{(x,t)}(v)\|}{\|dR_{(x,t)}(v)\|} \leq \frac{2c\cosh(t)}{\sinh(t)} \tag{6.2.4}$$

for all $(x,t) \in S \times (0,\infty)$ and $v \in T_xS$. If t is constrained away from zero then the lower and upper bounds in (6.2.4) are bounded by positive numbers from below and above respectively. When $t = 0$ both $dR_{(x,t)}$ and $d(R_f)_{(x,t)}$ have maximal rank, so constant positive bounds arise from a compactness argument. This yields (6.2.1) and concludes the proof of the Lemma. □

We are now ready to prove the main result of this section.

Theorem 1.4.3. Let Γ be a finitely generated group and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$. Assume that (ρ_i) converges algebraically to a representation $\rho \in \mathcal{D}(\Gamma)$ and geometrically to a subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$. If

- $\rho(\Gamma)$ does not contain parabolic elements,
- maximal cyclic subgroups of $\rho(\Gamma)$ are maximal cyclic in G,

then $G = \rho(\Gamma)$.

Before going further, set $M_A = \mathbb{H}^3/\rho(\Gamma)$, $M_G = \mathbb{H}^3/G$ and $M_i = \mathbb{H}^3/\rho_i(\Gamma)$ for $i = 1, 2, \ldots$, choose a base frame $\omega_{\mathbb{H}^3}$ for hyperbolic space $\mathbb{H}^3$ and let ω_i, ω_A and ω_G be the corresponding base frames of M_i, M_A and M_G respectively. By Proposition 2.6.1, the pointed manifolds (M_i, ω_i) converge geometrically to (M_G, ω_G). We may assume without loss of generality that M_G has infinite volume. Otherwise, it is either compact, in which case the sequence (M_i) is eventually stable, or it is noncompact and (M_i) is obtained by performing Dehn filling on M_G with larger and larger coefficients [9, Theorem E.2.4]. In the latter case, it is not hard to see that there must be parabolics in the algebraic limit $\rho(\Gamma)$.

There is a covering map $\pi : M_A \to M_G$ induced by the inclusion $\rho(\Gamma) \subset G$, and our goal is to show that this is a homeomorphism. Recall that Proposition 2.4.2 provides an exhaustion of M_A by compact cores $C \subset M_A$ such that

(1) if a component S of ∂C faces a convex cocompact end of M_A then S is smooth and strictly convex,

(2) if a component S of ∂C faces a degenerate end of M_A then the restriction $\pi|_S : S \to \pi(S)$ is a finite covering onto an embedded surface in M_G.

Fixing a compact core $C_0 \subset M_A$, we may also assume that all C are large enough to contain C_0 and satisfy the following property:

(3) $\pi(C_0) \cap \pi(S) = \emptyset$ for any component $S \subset \partial C$ facing a degenerate end of M_A.

Then to prove that π is a homeomorphism, it clearly suffices to show that $\pi|_C$ is injective for all such compact cores $C \subset M_A$.

Fix a compact core $C \subset M_A$ as described above. The geometric convergence $(M_i, \omega_i) \to (M_G, \omega_G)$ supplies for sufficiently large i an almost isometric embedding $\phi_i : \pi(C) \hookrightarrow M_i$, so for large i, we have a map

$$f_i : C \to M_i,\ \ f_i = \phi_i \circ \pi$$

that behaves much like the restriction of a nearly Riemannian covering map. In fact, f_i is π_1-surjective. For if $S \subset \Gamma$ is a finite generating set then C contains loops based at ω_A representing the elements of $\rho(S)$; f_i then maps these loops to loops in M_i representing $\rho_i(S)$, which generate $\pi_1(M_i) \cong \rho_i(\Gamma)$. We aim to show that f_i is actually an embedding, as the same will then be true for $\pi|_C$.

We first consider the case where M_A is convex-cocompact, as the proof is particularly simple. In this case, every component of ∂C is strictly convex and faces a convex-cocompact end of M_A, so f_i radially extends (as in the beginning of the section) to a globally defined map

$$\bar{f}_i : M_A \to M_i.$$

We claim that this is a covering map for $i >> 0$. To see this, note that when i is large f_i is C^2-close to a local isometry, so the strict convexity of ∂C persists after applying f_i. Therefore Corollary 6.2.2 applies to show that $\bar{f}_i$ is (uniformly) locally bilipschitz. It is well-known that any locally isometric map between complete Riemannian manifolds is a covering map, and in fact the same proof applies to locally bilipschitz maps. So, $\bar{f}_i : M_A \to M_i$ is a covering map. However, f_i is π_1-surjective, so its extension $\bar{f}_i$ is a π_1-surjective covering map, and therefore a homeomorphism. This shows that f_i is injective, and in particular $\pi|_C$ is as well.

In the general case, the argument needs modification because one cannot radially extend f_i into the degenerate ends of M_A. To deal with this, we will alter the problematic parts of M_A so that an extension of f_i is obvious.

Claim. If $S \subset \partial C$ faces a degenerate end of M_A, then the restriction of f_i to S is an embedding with image a separating surface in M_i.

Proof. Our choice of C ensures that $\pi|_{S_d}$ is a finite covering onto its image. The assumption that maximally cyclic subgroups of $\rho(\Gamma)$ are maximally cyclic in G then implies that $\pi|_{S_d}$ is an embedding, for otherwise there would be a loop in $\pi(S_d)$ that does not lift to M_A, but that has a power which does.

Next, property (3) above implies that every component $S \subset f_i(S_d)$ is disjoint from $f_i(C_0)$. However, the argument given to show that $f_i|C$ is π_1-surjective also applies to $f_i|C_0$, so every loop in M_i is homotopic into $f_i(C_0)$ and therefore has trivial algebraic intersection with S. Therefore S is separating. □

For each such S, let P_i^S be the closure of the component of $M_i \setminus f_i(S)$ containing $f_i(C_0)$. Then if E_{cc} is the union of the components of $M_A \setminus C$ that are neighborhoods of convex cocompact ends, one can construct a new 3-manifold

$$M'_A = (C \cup E_{cc}) \sqcup_{f_i} \big(\bigcup_S P_i^S\big)$$

by gluing each P_i^S to $C \cup E_{cc}$ along S. The map f_i extends naturally to a continuous map $\bar{f}_i : M'_A \to M_i$; the extension into E_{cc} is radial and on P_i^S we use the natural inclusion into M_i. It is easy to see that $\bar{f}_i$ is a π_1-surjective covering map, so the proof ends the same way it did in the previous case.

We would like to observe that we did not really use that every maximal cyclic group in $\rho(\Gamma)$ is maximal cyclic in G. We namely proved the following less aesthetically pleasant but more general theorem:

Theorem 6.2.3. *Let Γ be a finitely generated group and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$. Assume that (ρ_i) converges algebraically to a representation ρ and geometrically to a subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$. If*

- *$\rho(\Gamma)$ does not contain parabolic elements, and*
- *every degenerate end of $\mathbb{H}^3/\rho(\Gamma)$ has a neighborhood which embeds under the covering $\mathbb{H}^3/\rho(\Gamma) \to \mathbb{H}^3/G$,*

then $G = \rho(\Gamma)$.

Before concluding this section, observe that Theorem 1.4.3 together with Theorem 1.4.2 imply the Anderson-Canary Theorem 1.4.1 mentioned in the introduction.

6.3 Attaching roots

In this section we prove:

Proposition 6.3.1. *Let M and N be hyperbolic 3-manifolds with infinite volume and let $\tau : M \to N$ be a covering. Assume that N has no cusps, that $\pi_1(M)$ is finitely generated and that M has a degenerate end which does not embed under the covering τ. Then there is a hyperbolic 3-manifold M' with finitely generated fundamental group, with $\chi(M') > \chi(M)$ and coverings $\tau' : M \to M'$ and $\tau'' : M' \to N$ with $\tau = \tau'' \circ \tau'$.*

Continuing with the notation above, by Proposition 2.4.2 the manifold M has a standard compact core C with the property that if a component S of ∂C faces a degenerate end of M then the restriction of τ to S is a covering onto an embedded surface in N. The assumption that M has a degenerate end which does not embed under the covering τ implies that there is actually a component S_0 of ∂C such that

$$\tau|_{S_0} : S_0 \to \tau(S_0)$$

is a non-trivial covering. Observe that by the covering theorem the embedded surface $\tau(S_0) \subset N$ faces a degenerate end of N.

Choosing a base point $* \in S_0$, we set

$$\Gamma = \pi_1(M, *) \quad \text{and} \quad H = \pi_1(\tau(S_0), \tau(*)).$$

The desired manifold M' will be the cover of N corresponding to the subgroup

$$\Gamma' = \langle \tau_*(\Gamma), H \rangle \subset \pi_1(N, \tau(*)).$$

By construction, $\pi_1(M') \cong \Gamma'$ is finitely generated and there are covering maps $\tau' : M \to M'$ and $\tau'' : M' \to N$ with $\tau = \tau'' \circ \tau'$, so it remains only to prove that $\chi(M') > \chi(M)$.

For our purposes, the most useful way to interpret the Euler characteristic will be through its relation to the dimension of the deformation spaces $\mathcal{CH}(M)$ and $\mathcal{CH}(M')$ of convex-cocompact hyperbolic structures on M and M' (see Section 2.5). Observe that since M and M' have finitely generated fundamental group and no cusps, they are homeomorphic by the tameness theorem to the interiors of compact hyperbolizable 3-manifolds $\bar{M}$ and $\bar{M}'$ whose boundary components have negative Euler characteristic. It follows from Section 2.5 that $\mathcal{CH}(M)$ and $\mathcal{CH}(M')$ are complex manifolds of $\mathbb{C}$-dimensions $-3\chi(M)$ and $-3\chi(M)$, respectively, and that there is a holomorphic map

$$(\tau')^* : \mathcal{CH}(M') \to \mathcal{CH}(M)$$

defined by lifting hyperbolic structures using $\tau' : M \to M'$. We will prove:

Claim. $(\tau')^*$ has discrete fibers and is not open.

Since any holomorphic map with discrete fibers is open unless the dimension of the domain is smaller than the dimension of the image, we deduce from the claim that

$$-3\chi(M') = \dim_{\mathbb{C}} \mathcal{CH}(M') < \dim_{\mathbb{C}} \mathcal{CH}(M) = -3\chi(M).$$

Therefore, $\chi(M') > \chi(M)$ and hence Proposition 6.3.1 will follow once we prove the claim.

The first part of the claim is almost immediate. If $\tau'_* : \Gamma \to \Gamma'$ is the inclusion induced by the covering $\tau' : M \to M'$ and $H < \Gamma'$ is as above, then by construction

- Γ' is generated by $\tau'_*(\Gamma)$ and H
- $\tau'_*(\Gamma) \cap H$ has finite index in H.

It follows from Corollary 2.8.2 that a representation $\Gamma \to \mathrm{PSL}_2\,\mathbb{C}$ has only finitely many extensions to Γ'. Therefore, hyperbolic structures on M' that map under $(\tau')^*$ to the same element of $\mathcal{CH}(M)$ have only finitely many options for holonomy representations, up to conjugacy. However, the elements of $\mathcal{CH}(M')$ with holonomy in any fixed conjugacy class form a discrete subset of $\mathcal{CH}(M')$ (see page 154, [43]), so $(\tau')^*$ must have discrete fibers.

To show that it is not open, we use the Ahlfors-Bers parameterization to produce from $(\tau')^*$ a holomorphic map

$$\beta : \mathcal{T}(\partial \bar{M}') \cong \mathcal{CH}(M') \xrightarrow{(\tau')^*} \mathcal{CH}(M) \cong \mathcal{T}(\partial \bar{M}).$$

The Teichmuller spaces of $\partial \bar{M}'$ and $\partial \bar{M}$ split as products of the Teichmuller spaces of their connected components; let $S \subset \partial \bar{M}$ be the component adjacent to the degenerate end that the surface S_0 faces. The covering theorem implies that τ'

extends to a nontrivial cover $\bar{\tau} : S \to S'$ onto some connected component $S' \subset \partial \bar{M}'$. With respect to the decompositions

$$\mathcal{T}(\partial \bar{M}) = \mathcal{T}(S) \times \mathcal{T}(\partial \bar{M} \setminus S), \quad \mathcal{T}(\partial \bar{M}') = \mathcal{T}(S') \times \mathcal{T}(\partial \bar{M}' \setminus S')$$

the map β can be written as

$$\beta(\sigma_1, \sigma_2) = (\bar{\tau}^* \sigma_1, \hat{\beta}(\sigma_1, \sigma_2))$$

where $\bar{\tau}^* : \mathcal{T}(S') \to \mathcal{T}(S)$ is the map induced by the covering $\bar{\tau} : S \to S'$. This covering is non-trivial, so the image of $\bar{\tau}^*$ has positive codimension and hence the same holds for the image of β. Therefore β is not open, implying the same for $(\tau')^*$.

Remark. A purely homological computation yields that under the same assumptions as in Proposition 6.3.1 we have $b_1(M') < b_1(M)$ where $b_1(\cdot)$ is the first Betti number with $\mathbb{R}$-coefficients. However, this homological argument does not seem to work in the relative case that we will discuss in section 6.5. This is why we choose to work with Euler characteristics and deformation spaces instead.

6.4 Proof of Theorem 1.4.4

Recall the statement of Theorem 1.4.4:

Theorem 1.4.4. Let Γ be a finitely generated group and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$. Assume that (ρ_i) is algebraically convergent and converges geometrically to a subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$. If G does not contain parabolic elements, then G is finitely generated.

If the hyperbolic 3-manifold $\mathbb{H}^3/G$ is compact then G is obviously finitely generated. Assume from now on that this is not the case; since G does not contain parabolic elements, this assumption impies that $\mathbb{H}^3/G$ has infinite volume.

Among all finitely generated subgroups of G which contain $\rho(\Gamma)$, choose H such that the associated hyperbolic 3-manifold $\mathbb{H}^3/H$ has maximal Euler characteristic $\chi(\mathbb{H}^3/H)$. Since $H \subset G$ we have a covering $\mathbb{H}^3/H \to \mathbb{H}^3/G$. Using the maximality assumption, we obtain from Proposition 6.3.1 that every degenerate end of $\mathbb{H}^3/H$ embeds under this cover.

On the other hand, the assumption that H contains $\rho(\Gamma)$ implies, by Proposition 2.7.2, that there is a sequence of representations $\sigma_i : H \to \mathrm{PSL}_2\,\mathbb{C}$ converging algebraically to the inclusion of H in G such that the groups $\sigma_i(H)$ converge geometrically to G. Theorem 6.2.3 implies now that $H = G$. In particular, G is finitely generated. This concludes the proof of Theorem 1.4.4.

6.5 Parabolics

It is a well established fact that most theorems in the deformation theory of Kleinian groups that hold in the absence of parabolics hold also, in some probably weaker form, in the presence of parabolics. It is also well known that proofs in the case with parabolics are much more cumbersome and technical but follow the same arguments as the proofs in the purely hyperbolic case. For the sake of clarity and transparency of exposition, we decided to prove Theorem 1.4.4 only in the absence of parabolics. We state now the general results and discuss what changes have to be made in the arguments given above.

Throughout this section we assume that the reader is familiar with basic geometric facts about hyperbolic manifolds with cusps, [43].

As mentioned in the introduction, Evans [30] obtained the following extension of Theorem 1.4.1.

Theorem 6.5.1 (Evans[30])**.** *Assume Γ is a finitely generated group and that (ρ_i) is a sequence of faithful representations in $\mathcal{D}(\Gamma)$ converging algebraically to some representation ρ. If the convergence $\rho_i \to \rho$ is weakly type preserving, then ρ_i converges geometrically to $\rho(\Gamma)$.*

Recall that an algebraically convergent sequence $\rho_i \to \rho$ is *weakly type preserving* if for every $\gamma \in \Gamma$ with $\rho(\gamma)$ parabolic, there is i_γ with $\rho_i(\gamma)$ parabolic for all $i \geq i_\gamma$.

Theorem 1.4.3 can be extended to the weakly type preserving setting as follows:

Theorem 6.5.2. *Let Γ be a finitely generated group, and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$ converging algebraically to a representation ρ and geometrically to a subgroup G of* $\mathrm{PSL}_2\,\mathbb{C}$. *If*

- *the convergence $\rho_i \to \rho$ is weakly type preserving, and*
- *maximal cyclic subgroups of $\rho(\Gamma)$ are maximally cyclic in G,*

then $G = \rho(\Gamma)$.

To begin with, set $M_i = \mathbb{H}^3/\rho_i(\Gamma)$, $M_A = \mathbb{H}^3/\rho(\Gamma)$, $M_G = \mathbb{H}^3/G$ and let $\pi : M_A \to M_G$ be the covering induced by the inclusion $\rho(\Gamma) \subset G$. We first recall the basic idea of the proof in the case without cusps. Since maximal cyclic subgroups of $\rho(\Gamma)$ are maximally cyclic in G, there are arbitrarily large standard compact cores C of M_A such that the components of ∂C facing convex cocompact ends of M_A are strictly convex and the components facing degenerate ends embed under the covering $\pi : M_A \to M_G$. Composing the restriction $\pi|_C$ with the almost isometric embeddings $\pi(C) \to M_i$ supplied by geometric convergence, we obtain maps $f_i : C \to M_i$ such that

1. if a component S of ∂C faces a degenerate end of M_A then $f_i|_S$ is an embedding for all large i,
2. if $S \subset \partial C$ faces a convex cocompact end then $f_i|_S$ is a convex immersion.

We then construct for large i a 3-manifold N_i containing C and a covering $\bar{f}_i : N_i \to M_i$ with $\bar{f}_i|_C = f_i$. This covering is π_1-surjective and hence a diffeomorphism, so in particular C embeds under the covering $\pi : M_A \to M_G$. Since C can be chosen to be arbitrarily large, this proves that the covering π is trivial and hence $G = \rho(\Gamma)$.

If there are parabolics in the algebraic limit, the ends of M_A are more complicated and refinements of the tools above are needed. The natural replacement of the compact core C is a submanifold $C \subset M_A$ with the following properties:

1. if a component S of ∂C faces a degenerate NP-end of M_A then S embeds under the covering $M_A \to M_G$,
2. if S faces a geometrically finite NP-end then S is strictly convex,
3. the complement in C of the μ-cuspidal part $M_A^{\text{cusp}<\mu}$ of M_A is a standard compact core, where μ is the Margulis constant.

The construction of such a submanifold is the same as that used to produce the compact cores above: one takes a large metric neighborhood of the convex core $\text{CC}(M_A)$ and deletes standard neighborhoods of the degenerate NP-ends of M_A. In this case, however, the resulting manifold C will contain parts of the cusps of M_A, and will therefore be noncompact. This is a problem, since $\pi(C) \subset M_G$ will also be noncompact and the almost isometric embeddings from M_G to M_i provided by geometric convergence are only defined on compact sets. However, we still can use the almost isometric embeddings to produce locally bilipschitz maps $f_i : C \setminus M_A^{\text{cusp}<\mu} \to M_i$, and we will show that if the convergence $\rho_i \to \rho$ is weakly type preserving then these can be extended to locally bilipschitz maps

$$\hat{f}_i : C \to M_i$$

converging uniformly on compact sets to the restriction $\pi|_C$. The proof of Theorem 6.5.2 is then word-by-word the same as the proof Theorem 1.4.3 with the maps $\hat{f}_i$ playing the role of f_i.

It remains to construct $\hat{f}_i$. Consider the maps

$$f_i : C \setminus M_A^{\text{cusp}<\mu} \to M_i$$

described above, and let ϵ be a small positive constant. If i is large then f_i is locally $(1+\epsilon)$-bilipschitz; combined with the fact that the convergence $\rho_i \to \rho$ is weakly type preserving, this implies that f_i sends loops homotopic into the cusps of M_A to parabolic loops in M_i with nearly the same length. It follows that $C \cap \partial M_A^{\text{cusp}<\mu}$ is mapped under f_i into a small neighborhood of $\partial M_i^{\text{cusp}<\mu}$. After a small perturbation, we can then arrange that

1. $f_i(C \cap \partial M_A^{\text{cusp}<\mu}) \subset \partial M_i^{\text{cusp}<\mu}$,
2. Df_i sends vectors orthogonal to $\partial M_A^{\text{cusp}<\mu}$ to vectors orthogonal to $\partial M_i^{\text{cusp}<\mu}$.

Brock and Bromberg [21, Lemma 6.16] show how to accomplish such a perturbation with a bit of finesse; in particular, their argument shows that we can still assume that f_i is locally bilipschitz.

Recall that C was constructed by removing standard neighborhoods of the degenerate NP-ends of M_A from its convex core. These neighborhoods can be chosen so that their intersections with $M_A^{\text{cusp}<\mu}$ are foliated by geodesic rays orthogonal to $\partial M_A^{\text{cusp}<\mu}$; the intersection $C \cap M_A^{\text{cusp}<\mu}$ then enjoys the same property. Combining this with (1) and (2) above allows us to extend f_i to a locally bilipschitz map $\hat{f}_i : C \to M_i$ as follows: define $\hat{f}_i$ to coincide with f_i on $C \setminus M_A^{\text{cusp}<\mu}$ and map geodesic rays in $C \cap M_A^{\text{cusp}<\mu}$ orthogonal to $\partial M_A^{\text{cusp}<\mu}$ isometrically to geodesic rays orthogonal to $\partial M_i^{\text{cusp}<\mu}$. A quick computation in the upper half space model for $\mathbb{H}^3$ verifies that the extension $\hat{f}_i$ is also locally $(1+\epsilon)$-bilipschitz. It follows that $\hat{f}_i \to \pi|_C$ uniformly on compact subsets.

Observe that as in the case without parabolics, we can replace the second hypothesis of Theorem 6.5.2 with the condition that every degenerate end of M_A embeds in M_G.

A version of Proposition 6.3.1 to the setting with cusps is also readily established:

Proposition 6.5.3. *Let M and N be hyperbolic 3-manifolds with infinite volume and let $\tau : M \to N$ be a covering. Assume that $\pi_1(M)$ is finitely generated and that M has a degenerate end which does not embed under the covering τ. Then there is a hyperbolic 3-manifold M' with finitely generated fundamental group, with*

$$3\chi(M') + \#\{\textit{cusps in } M'\} > 3\chi(M) + \#\{\textit{cusps in } M\}$$

and coverings $\tau' : M \to M'$ and $\tau'' : M' \to N$ with $\tau = \tau'' \circ \tau'$.

The proof is the same as that of Proposition 6.3.1, except that instead of considering the deformation space $\mathcal{CH}(M)$ of convex-cocompact hyperbolic structures on M one uses geometrically finite metrics whose parabolic loci coincide with that of the original hyperbolic structure on M. The space of such metrics, up to isometries isotopic to the identity map, is a complex manifold of $\mathbb{C}$-dimension

$$-3\chi(M) - \#\{\text{cusps in } M\} \geq 0.$$

After these remark the proof of Proposition 6.5.3 is the same as the proof of Proposition 6.3.1.

Having provided versions of Theorem 1.4.3 and Proposition 6.3.1 that apply to representations with parabolics, we are almost ready to discuss the general form of Theorem 1.4.4. Before doing so, we need a definition:

Definition. Assume that a sequence of subgroups (G_i) of $\mathrm{PSL}_2\,\mathbb{C}$ converges geometrically to a subgroup G. We say that the convergence $G_i \to G$ is *geometrically weakly type preserving* if for every $g \in G$ parabolic there is a sequence (g_i) with $g_i \in G_i$ converging to g and with g_i parabolic for all but finitely many i.

In the terminology of [16], a geometrically convergent sequence of subgroups converges in a geometrically weakly type preserving manner if and only if the associated sequence of pointed manifolds has *uniform length decay*.

The general version of Theorem 1.4.4 reads now:

Theorem 6.5.4. *Let Γ be a finitely generated group and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$ that converges algebraically to a representation ρ and geometrically to a subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$. If the convergence $\rho_i \to G$ is geometrically weakly type-preserving, then G is finitely generated.*

As in the proof of Theorem 1.4.4 we choose a finitely generated subgroup $H \subset G$ containing $\rho(\Gamma)$ and maximizing the quantity $3\chi(\mathbb{H}^3/H) + 2\#\{\text{cusps in } \mathbb{H}^3/H\}$.

We claim that $G = H$. As before, there is a sequence of representations $\sigma_i : H \to \mathrm{PSL}_2\,\mathbb{C}$ converging algebraically to the inclusion of H in G and with $\sigma_i(H) = \rho_i(\Gamma_i)$ for all i. In particular, G is the geometric limit of the groups $\sigma_i(H)$. The assumption that the convergence $\sigma_i(H) = \rho_i(\Gamma) \to G$ is geometrically weakly type-preserving implies that the algebraic convergence of σ_i to the inclusion of H into G is (algebraically) weakly type preserving. In particular we deduce from Theorem 6.5.2 that either $G = H$ or a degenerate end of $\mathbb{H}^3/H$ does not embed under the cover $\mathbb{H}^3/H \to \mathbb{H}^3/G$. The second possibility is ruled out by the choice of H and Proposition 6.5.3, so $G = H$ and therefore is finitely generated. This concludes the proof of Theorem 6.5.4.

6.6 Pre-compactness and eventually faithful sequences

As mentioned in the introduction, we discuss here to which extent some other results concerning faithful representations remain true if the condition of faithfulness is relaxed.

Definition. A sequence $(\rho_i)_i$ of representations is *eventually faithful* if for all $\gamma \in \Gamma$ there is some i_γ such that $\gamma \notin \mathrm{Ker}(\rho_i)$ for all $i \geq i_\gamma$.

In some sense, every algebraically convergent sequence is eventally faithful. The following Lemma formalizes this; its proof is a simple application of the Margulis Lemma.

Lemma 6.6.1. *Let Γ be a finitely generated group and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$ converging algebraically to a representation ρ. Consider the quotient group $\hat{\Gamma} = \Gamma/\mathrm{Ker}(\rho)$ and let $\pi : \Gamma \to \hat{\Gamma}$ be the associated projection. Then there is an eventually faithful sequence of representations $\sigma_i : \hat{\Gamma} \to \mathrm{PSL}_2\,\mathbb{C}$ converging algebraically to a representation σ such that $\rho_i = \sigma_i \circ \pi$ and $\rho = \sigma \circ \pi$.*

We assume from now on that (ρ_i) is an eventually faithful sequence of representations in $\mathcal{D}(\Gamma)$. If $S \subset \Gamma$ is a finite generating set, then each representation ρ_i

determines a convex function

$$d_{\rho_i} : \mathbb{H}^3 \to \mathbb{R},\ d_{\rho_i}(x) = \sum_{\gamma \in S} d_{\mathbb{H}^3}(x, \rho_i(\gamma)x).$$

Conjugating our representations by $\mathrm{PSL}_2\,\mathbb{C}$ if necessary, we may assume that some base point $0 \in \mathbb{H}^3$ is the unique minimum of each d_{ρ_i}. In this case we say that the sequence (ρ_i) consists of *normalized representations* and we set $d_{\rho_i} = d_{\rho_i}(0)$. It is well-known that the sequence (ρ_i) contains an algebraically convergent subsequence if

$$\liminf d_{\rho_i} < \infty.$$

Otherwise, the sequence of actions of Γ via ρ_i on the scaled hyperbolic spaces $\frac{1}{d_{\rho_i}}\mathbb{H}^3$ contains a subsequence which converges in the equivariant Gromov-Hausdorff topology to a non-trivial action $\Gamma \curvearrowright T$ on some real tree T. Recall that an action on a tree is called trivial if it has global fixed points. Morgan-Shalen [47], Paulin [53] and Bestvina [10] proved that if the representations ρ_i are faithful the action $\Gamma \curvearrowright T$ is *small*, meaning that the stabilizers of non-degenerate segments in T are virtually abelian. Their arguments still apply if the sequence is only eventually faithful, see [49]:

Theorem 6.6.2. *Every eventually faithful sequence of normalized representations in $\mathcal{D}(\Gamma)$ has a subsequence that either converges algebraically in $\mathcal{D}(\Gamma)$ or converges in the Gromov Hausdorff topology to a nontrivial small action $\Gamma \curvearrowright T$ on a $\mathbb{R}$-tree T.*

It is theorem of Morgan-Shalen [47] that if the fundamental group of a compact, irreducible and atoroidal 3-manifold M admits a nontrivial small action on a real tree, then either ∂M is compressible or there are essential properly embedded annuli in $(M, \partial M)$. Combining this fact with Theorem 6.6.2 we obtain the following result, essentially due to Thurston:

Corollary 6.6.3. *Assume that Γ is the fundamental group of a compact 3-manifold* *with incompressible and acylindrical boundary. Then every eventually faithful sequence of normalized representations in $\mathcal{D}(\Gamma)$ contains a convergent subsequence.*

Many other results, for instance Thurston's double limit theorem [51] or the results in [40], ensuring the existence of convergent subsequences of sequences of faithful representations can be reduced to the non-existence of certain actions of groups on trees; all these results still hold with obvious variations in the statements for eventually faithful sequences. It may be therefore surprising that some convergence results for sequences of faithful representations completely fail in our more general setting.

In [59], Thurston proved a generalization of Corollary 6.6.3 to the case that Γ is the fundamental group of a compact 3-manifold M with incompressible boundary. More precisely, the so-called *only-windows-break theorem* asserts that whenever (ρ_i) is a sequence of faithful representations of $\Gamma = \pi_1(M)$ and $N \subset M$ is a component of the complement of the characteristic manifold, then the sequence $(\rho_i|_{\pi_1(N)})$ has, up to conjugacy, a convergent subsequence. Leaving the interested reader to consult [34] for information about the characteristic manifold, we limit ourselves to the following concrete example.

Let H be a handlebody of genus 2 and $\gamma \subset \partial H$ a simple closed curve on the boundary of H such that $\partial H \setminus \gamma$ is incompressible and acylindrical; for instance, γ can be taken in the Masur domain of H [40]. We consider the manifold N obtained by doubling H along $\mathcal{N}(\gamma)$, where $\mathcal{N}(\gamma)$ is a regular neighborhood of γ in ∂H. The manifold N has incompressible boundary and there is, up to isotopy, a single properly embedded essential annulus $A \subset N$, the annulus along which we have glued. The annulus A cuts N open into two copies H_1 and H_2 of H. In this particular example, Thurston's only-windows-break theorem asserts:

Theorem 6.6.4 (Thurston)**.** *Let N be as above and (ρ_i) a sequence of discrete and faithful representations of $\pi_1(N)$ into $\mathrm{PSL}_2\,\mathbb{C}$. Then the sequence of restrictions*

$$\rho_i|_{\pi_1(H_1)} : \pi_1(H_1) \to \mathrm{PSL}_2\,\mathbb{C}$$

has a subsequence that converges up to conjugacy in $\mathrm{PSL}_2\,\mathbb{C}$.

We claim that there is an eventually faithful sequence of discrete representations of $\pi_1(N)$ for which the claim of Theorem 6.6.4 fails.

By construction, the manifold N is the double of H along $\mathcal{N}(\gamma)$. Let

$$\tau : N \to H$$

be that map given by "folding" along $\mathcal{N}(\gamma)$. Identifying H with H_1, one of the two pieces of N, we choose a base point $p \in H$ and hence we have a homomorphism

$$\tau_* : \pi_1(N,p) \to \pi_1(H,p)$$

We consider also the Dehn-twist

$$\delta : N \to N$$

along the annulus A. It is easy to see that the sequence

$$\tau_* \circ \delta_*^n : \pi_1(N,p) \to \pi_1(H,p)$$

is eventually faithful, meaning that every element is contained in the kernel of $\tau_* \circ \delta_*^n$ for finitely many n. By construction, $\tau_* \circ \delta_*^n$ is the identity on $\pi_1(H,p)$ for all n.

Remark. Essentially we are saying that $\pi_1(N)$ is fully residually free. This fact holds for every group obtained by doubling a free group along a cyclic subgroup.

Since H is a handlebody, $\pi_1(H,p)$ is a free group. Hence, we can choose some sequence of faithful representations $(\sigma_n) \in \mathcal{D}(\pi_1(H))$ which cannot be conjugated to obtain convergent subsequences. Setting $\rho_n = \sigma_n \circ \tau_* \circ \delta_*^n$ we obtain an eventually faithful representation which does not contain any subsequence whose restriction to $\pi_1(H)$ converges up to conjugacy.

Theorem 6.6.5. *The only-windows-break theorem fails for eventually faithful sequences of representations.*

An alternative statement of Theorem 6.6.5 could be that when one is not absolutely faithful more than the windows can get broken.

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