College Admission with Multidimensional Reserves: The Brazilian Affirmative Action Case*

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Abstract

In August 2012 the Brazilian federal government passed a law mandating the implementation of reserves in public federal universities for candidates from racial minorities, low income families and those coming from public high schools. In this problem, individuals may be part of one or more of those classifications, and it is possible for students not to reveal their type (being low income, minority or public high school graduate) if they want. This turns out to be a problem not studied in the literature. We first design a new choice function that is incentive compatible, fair and respecting reserves whenever it is possible and there is enough low-income minority student application. Next, we suggest a stable, strategy-proof and fair mechanism to implement this problem. Then, we show that the current choice functions being implemented are not incentive compatible, fair or respecting reserves, even when there is enough poor-minority student application. Also, any stable mechanism that uses current choice functions is neither strategy-proof nor fair.

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1 Introduction

Affirmative action policies in societies with heterogeneous populations are getting popular and are often considered necessary for equal opportunities for certain demographic groups. The United States and Brazil are examples of countries with heterogeneous populations in terms of wealth level and racial backgrounds. One way to solve the problem of inequality between individuals who belong to different racial or gender groups or come from families with different income levels is affirmative action. Affirmative action is a method of positive discrimination in favor of certain groups of people to close socioeconomic gaps that exist between different groups as a result of historic discrimination practices. This paper studies affirmative action in college admission in Brazil where the goal is to give underrepresented groups increased chances of attending better universities.

The Brazilian federal higher education system comprises of 59 universities and 38 institutes of education, science and technology, with an annual inflow of about one million students to its undergraduate programs. Following an increasing role for affirmative action for students of African descent and of low-income families in terms of access to public universities, the Brazilian congress passed in August 2012 a law establishing the implementation of a series of affirmative actions throughout said system.

The law established that 50% of the seats in each program offered in those institutions should be reserved for affirmative action. In order to take advantage of those reserves, a student must complete the three years of high-school in a public institution (being it local, state or federal). Of those seats, at least 50% should be reserved for low-income students (as defined in the law). Additionally, a number of seats in the same proportion of the aggregate number of blacks, browns and indians (here referred to as "minorities") in the state in which the institution is, should be reserved for those demographic groups, also in aggregation. For example, in a state in which minorities constitute 25% of the population, a program with

\[ \text{Reserved seats} = 0.5 \times \text{Aggregate minority population} \]

1For detailed information about history of affirmative action in Brazil, check Moehlecke (2003).
3In Brazil, like Turkish system studied in Baliski and Sonmez (1999), students apply directly to a specific program in the university, differently from other countries like the US where students simply apply to the university and once there chooses majors or programs to pursue.
capacity of 80 will have 40 seats reserved for public high school graduate students. At least 20 of those should be reserved for low-income students, and 10 of those for minorities.

In October of the same year, Brazil’s Ministry of Education published an ordinance\(^4\) specifying some details on the implementation of the reserves law as well as a suggested mechanism for choosing students while satisfying those reserves. Starting in the student selection processes of 2013, based on our observations, those recommendations were widely adopted as the new selection criteria.

The key distinctive issue presented by the reserves proposed in the law is the fact that they are multidimensional. That is, students may be part of one or more of the reserves specified. For instance, low-income white student from public high school qualifies for the low-income reserve but not for the minority reserve. Although the literature for affirmative action from a mechanism design perspective has seen many important contributions, as in Abdulkadiroglu and Sonmez (2003), Westkamp (2012), Kominers and Sonmez (2013) and Hafalir, Yenmez and Yildirim (2012), to the best of our knowledge none of them are able to respond to the challenge introduced by these kind of reserves.

Another issue that is outside the scope of those papers is identification problem. In Brazil, students are not obligated to declare their income level, the name of the high school they graduated from, or their racial status if they do not benefit from an affirmative action policy. Therefore, some students may have incentive to hide their type, depending on the mechanism used for assignment.

Starting in 2010, a new centralized system\(^5\) was put in place to match students to federal universities. Although the study of the characteristics of that system is outside of the scope of this paper, the problems identified here are still present in it, and moreover it shows that there is a tendency for centralization of that process. Methods that could improve upon the current system in a centralized way (as the one that we present in this paper,) may therefore have a direct application and impact.

Division of indivisible goods problem in the absence of money is studied in many papers

\(^4\)Normative Ordinance number 18, of October 11th 2012.

\(^5\)The Unified System of Selection, denoted SISU.
starting from the seminal paper by Gale and Shapley (1962). They study a college admission market where students have preferences over colleges and colleges have preferences over sets of students to be admitted. They define the stability notion in marriage market and college admission problems. The stability notion they defined, with small variations, is still in use and considered as one of the most important goals of mechanism designers for matching problems. They also introduce celebrated student-proposing deferred acceptance algorithm (DA) to find stable allocation. The DA mechanism is also utilized in many applied and theoretical papers in matching literature. The centralized algorithm we suggest in this paper, cumulative offer algorithm, is also a variation of DA algorithm.

The school choice with affirmative action problem consists of two parts. The first part is schools’ strategies of choosing students which we call choice function. Choice function provides a set of students for any possible set of students that apply for a given school. The second part is the algorithm that the central authority uses to allocate school seats to students with respect to schools’ choice functions.

The first approach on this problem from mechanism design perspective is the work of Abdulkadiroglu and Sonmez (2003). They study the school choice problem in Boston. The system in Boston, called Boston Mechanism, gives students higher priorities in schools stated in their neighborhoods or in schools in which students have a sibling attending. By giving these priorities, the Boston mechanism positively discriminates some students for certain schools. Abdulkadiroglu and Sonmez (2003) propose two celebrated algorithms, DA and top trading cycles (TTC) as alternatives of Boston school choice algorithm while keeping priorities of schools as given. They show that DA allocates school seats in a stable way and the outcome is efficient in students’ perspective. Also, DA is not manipulable, i.e. no student can manipulate the mechanism and gets a better school assignment. Subsequently, Abdulkadiroglu (2005) considers college admission problem with affirmative action policy and he shows that two assumptions on school preferences are sufficient to recover the properties of DA algorithm.

In a recent paper, Westkamp (2013) studies the German university admission system in which reserved seats are transferred on different subpopulations in case of lack of application. In this matching with complex constraints problem, Westkamp specifies a method for schools to
choose set of students in any given case and designs a mechanism that gives a stable allocation under this method. Also, in another recent paper, Kamada and Kojima (2011) study the Japanese Residency Matching Program, where there are quotas for regions in order to help rural regions attract more residents. In the mechanism they study, the government sets a target capacity for each hospital to implement these quotas. They show that using target capacities may result in inefficiencies such that violating these targets may improve the inefficiencies.

In 2012, Kojima (2012) showed that in affirmative action problems with two groups (majorities and minorities), using quotas for even one side may be inefficient and hurt the minority group which is the positive discrimination is addressed to. In a subsequent paper, Hafalir, Yenmez and Yildirim (2013) study the school choice problems with affirmative action for minorities. They show deficiencies of utilizing quotas for school choice problems with affirmative action and its remedies such as switching the system to DA with minority reserves instead of quotas for both group or majority quotas only. With minority reserves, schools give higher priority to minority students for the seats reserved for them. Therefore, reserved seats are kept available to any students and they show this mechanism with minority reserves is remedy for not only wasted seats but also welfare loss due to the fact that minorities have less available seats in schools under majority quota.

Our model is built upon matching with montracx model described by Hatfield and Milgrom (2005). Hatfield and Milgrom (2005) connect the matching problem of indivisible goods and labor market model. They show that the foundations of labor market model where workers can be hired by many alternative contracts by Kelso and Crawford (1982) are also achievable in matching markets. This paper is very important because it not only subsumes and unifies these two problems but also relates DA algorithm with fixed point techniques in lattice theory. In our problem, students do not have to declare their demographic status, i.e. a minority student can be admitted as a non-minority student. Hence, as in matching with contracts problem, students can be admitted in many different ways to schools.

In section 2, we present the mechanism suggested by the Ministry of Education and currently used by the universities surveyed. In section 3, we introduce the matching with contracts model that is applicable with the school choice problem with affirmative action. In section
4, we introduce Multidimensional Brazil Reserves Choice Function and we build upon the choice function defined to describe a mechanism, Student Optimal Stable Mechanism, that matches students with colleges in a centralized way, satisfies the stability, is strategy-proof and fair. In section 5, we show that even for a single college, Brazil Reserves Choice Function induces a game with multiple Nash Equilibria in which strategically sophisticated students may obtain advantage by misrepresenting their demographic type. We also show that the current mechanism is not fair and cannot guarantee satisfying reserves when it is possible. In section 6, we conclude. All the proofs are given in the Appendix section.

2 Brazilian Reserves Choice Function

For the most part, until 2010, college admissions in Brazil worked in a completely decentralized way. Students applied for a single program in each university they desire to (Ex: History or Biology). By using some combination of scores in a national exam and sometimes exams particular to those programs, the universities ranked them and accepted the top applicants to each program up to the programs’ capacities, putting the remaining ones in a waiting list.

Among those accepted, typically some would not enroll because they were also accepted by other universities and courses of their preference. The universities would then proceed to a second round, accepting students from the waitlist following their ranking. Depending on the university this might be followed by third and fourth rounds.

The introduction of the reserves law has not changed the decentralized nature of the system yet. But the centralized online system used for some universities gives a strong signal that officials in charge of college admissions in Brazil are open to utilize a centralized method, which is shown in many papers that improves efficiency and reduces wasted seats in colleges. On the other hand, the reserves law changed the choice rules of universities in each step in an attempt to satisfy the reserves. The rules used by the universities surveyed in this work are, essentially, strict implementations (or small variations) of the one suggested by Brazil’s Ministry of Education. This rule tells the set of students to be chosen from any set of applicants and will be denoted as the class of Brazil Reserves Choice Function (BRCF). It suggests that the
seats for each program should be split into five subsets. For any program with capacity \( Q \), five distinct subsets are:

- A set \( Q_{mi} \) with \( \lceil \frac{Q}{4} r^m \rceil \) seats reserved for low-income minorities from public high schools\(^6\)
- A set \( Q_{Mi} \) with \( \lceil \frac{Q}{4} (1 - r^m) \rceil \) seats reserved for low-income majorities from public high schools
- A set \( Q_{mi} \) with \( \lceil \frac{Q}{4} r^m \rceil \) seats reserved for non-low-income minorities from public high schools
- A set \( Q_{MI} \) with \( \lceil \frac{Q}{4} (1 - r^m) \rceil \) seats reserved for non-low-income majorities from public high schools
- A set \( Q_- \) with the remaining seats, open to any student

where \( r^m \) is the ratio of minorities in the state that program (college) belongs.

Given the students who apply for each of those, the ones better ranked on the entrance exam are accepted up to the capacity of the set. If there are enough applicants for each of those sets, the reserves as described by the law are satisfied. In case the number of students who apply for some of those sets is smaller than their capacity, those seats are filled following the following priority structure:

- If there are seats available in \( Q_{mi} \), those are made available:
  - to low-income majorities from public schools, then
  - to non-low-income minorities from public schools, then
  - to non-low-income majorities from public schools, then
  - to any student

- If there are seats available in \( Q_{MI} \), those are made available:

\(^6\)[\( \lceil \cdot \rceil \)] is ceiling function. \( \lfloor a \rfloor \) gives the closest integer greater than or equal to \( a \).
– to low-income minorities from public schools, then
– to non-low-income minorities from public schools, then
– to non-low-income majorities from public schools, then
– to any student

• If there are seats available in $Q_{MI}$, those are made available:

  – to non-low-income majorities from public schools, then
  – to low-income minorities from public schools, then
  – to low-income majorities from public schools, then
  – to any student

• If there are seats available in $Q_{MI}$, those are made available:

  – to non-low-income minorities from public schools, then
  – to low-income minorities from public schools, then
  – to low-income majorities from public schools, then
  – to any student

It is not specified, however, in which order those seats are filled following those priorities.

3 Model

We are dealing with a student-program matching problem where programs have complex reserve structures and students have more than one way to attend a program. Due to the characteristics of the problem we use components of matching with contracts model. There are finite sets $S = \{s_1, \ldots, s_n\}$ and $P = \{p_1, \ldots, p_m\}$ of students and programs. The set $S^p \subset S$ contains all students in $S$ from public schools, $S^m \subset S^p$ contains the racial minority students from public schools and $S^i \subset S^p$ contains the low-income students from public schools. Each program $p$
has its own capacity level $Q_p$ and minority reserve ratio $r_p^m$. Each student $s$ has a vector of exam scores $z(s) = (z_{p_1}(s), \ldots, z_{p_m}(s))$ such that $z_p(s)$ indicates score of student $s$ for program $p$. For any student couple $s$ and $s'$, $z_p(s)$ and $z_p(s')$ are assumed to be different. Formally, $\forall s, s' \in S$ and $p \in P, z_p(s) = z_p(s') \iff s = s'$. The set $T$ of demographic types is such that $T = \{mi, mI, Mi, MI, Pr\}$. Each student $s$ has a type $t_s \in T_s \subset T$ where $T_s$ is the set of demographic types available to student $s$. For this problem $mi$ refers to being a low-income minority student from public high school, $mI$ refers to being a non-low-income minority student from public high school, $Mi$ refers to being a low-income non-minority student from public high school, $MI$ refers to being a non-low-income non-minority student from public high school, $Pr$ refers to being a student from non-public high school. In Brazilian system, if a student applies as a non-low-income, non-minority or non-public high school graduate then she is not asked to prove any of these. On the other hand, if a student applies as a low income, minority or public high school graduate then she is required to prove her credentials. Therefore, some students may not reveal their true income, race and educational background by applying as non-low-income, non-minority or non-public high school graduates. Hence, $T_s$ may include more than one type. For example, if a student is a low-income non-minority student from public high school, then $t_s = Mi$ and $T_s = \{Mi, MI, Pr\}$.

Throughout this section we will make use of the matching with contracts notation. A contract $x \in X$, in this context, is a tuple $(s, p, t)$, where $s \in S$, $p \in P$ and $t \in T$. For a contract $x$, $x_s$, $x_p$ and $x_T$ represent student, program and demographic type in contract $x$ respectively. Let $X$ be the set of all contracts. For ease of notation, for a set of contracts $Y$, $Y_i$ is the subset of $Y$ that contains only the contracts that include $i \in S \cup P$. Let $s(Y)$, moreover, be the set of students with contracts in $Y$, that is, $s(Y) = \{s \in S : \exists(s, p, t) \in Y\}$. An allocation is a set of contracts $X' \subset X$, such that for every $s \in S$ and every $p \in P$, $|X'_s| \leq 1$ and $|X'_p| \leq Q_p$. Let $\chi$ be set of all possible allocations.

The null contract, meaning that the student has no contract, is denoted by $\emptyset$. Students have complete preferences, $\succeq$, over contracts and the null contract. These preferences are derived from students’ strict preferences, $\succ^*$, over programs and being unmatched, in addition to the
fact that they consider irrelevant how they are accepted to a program:

\[ \forall s \in S, \forall p, p' \in P \text{ and } t, t' \in T_s : (s, p, t) \succ_s (s, p', t') \iff p \succ^*_s p' \]

Also, let \( \succ^t \) be strict preferences of students over contracts with demographic type \( t \) and null contract, which is also derived from students strict preferences, \( \succ^* \), over colleges and being unmatched:

\[ \forall s \in S, \forall p, p' \in P \text{ and } t \in T_s : (s, p, t) \succ^t_s (s, p', t) \iff p \succ^*_s p' \]

Next, choice function of program \( p, C_p : 2^X \to 2^X \) is a function that chooses a subset for any given set of contracts and for any given set of contracts \( Y \subset X \), \( C_p(Y) \subset Y \) is a subset of contracts with cardinality at most \( Q_p \) and in which there is at most one contract of each student. The assumption about student preferences we mentioned above is one of the main differences of our paper with the current matching with contracts literature, since our model allows indifferences among contracts which is assumed to be strict so far. Due to indifferences of students between some contracts, we can not derive choice functions of students defined in many to one matching with contracts models. As a result, instead of choice functions of students, we are going to use student preferences over contracts which makes the model with choice functions a subcase of our model. Therefore, primitives of our model are student preferences over contracts and programs’ choice functions.

A mechanism is a strategy space \( \Delta_s \) for each student \( s \) along with an outcome function \( \varphi : (\Delta_{s_1}, \Delta_{s_2}, \ldots, \Delta_{s_m}) \to \chi \) that selects an allocation for each strategy vector \( (\delta_{s_1}, \delta_{s_2}, \ldots, \delta_{s_m}) \in (\Delta_{s_1}, \Delta_{s_2}, \ldots, \Delta_{s_m}) \). Given a student \( s \) and a strategy profile \( \delta_s \in \Delta_s \), let \( \delta_{-s} \) denote the strategy of all students except student \( s \). A direct mechanism is a mechanism where strategies are demographic types of students, \( t \), along with preferences over contracts with those demographic types, \( \succ^t \). Hence a direct mechanism is simply a function \( \psi : \prod_{s \in S} (T_s \times (\succ^t_{t \in T_s})) \to \chi \) that selects an allocation for each demographic type and preference profile.
4 Student Optimal Stable Mechanism

4.1 The Multidimensional Brazil Reserves Choice Function

One of our objectives is to find a choice function that satisfies reserves for each program, gives incentive to students for revealing their true demographic types and guarantees the existence of a stable allocation. We also aim to design a mechanism that carries out our choice function’s properties and finds a stable allocation.

We are proposing a choice function, Multidimensional Brazil Reserves Choice Function (or MCF), in order to allocate students to their reserves. Unlike BRCF, our choice function, which is denoted as \( C^{MCF} \) hereafter, keeps incentive compatibility by giving priority for a reserved seat to any student that can mimic relevant demographic type. Also, by doing this, choice function satisfies, another important criterion, namely fairness.

The centralized mechanism in Brazil provides some notion of consistency for contracts offered to programs. In the Brazil case, although students may not reveal their true demographic type, once they decide on one, they need to submit same demographic type at each application. That is why we assume a student cannot apply to a program more than once with different contracts. Hence, we design the choice function as if there is one possible type for each student.

Let \( q_p \) be the reserve for students in \( S^i \cap S^m \), for program \( p \). For any given set of contract \( Y \), algorithm of the choice function are the following:

**Phase 0**: Program \( p \) rejects each contract that does not include itself (\( x_p \neq p \implies x \notin C_p(X) \)).

**Phase 1**: Program \( p \) considers only contracts with \( x_T = m_i \). Program \( p \) accepts contracts including students with highest scores \( z_p \) one at a time. Program continues until either all contracts are considered or \( q_p \) contracts are chosen. In any case, program proceeds with Phase 2. Let \( \theta \) be \( q_p - |\{\text{contracts accepted in Phase1}\}| \).

Possible constraints to bind

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Phase 2: Program $p$ considers remaining contracts with $x_T \in \{mi, Mi, mI\}$. Program $p$ accepts contracts including students with highest scores $z_p$ one at a time. During the process, if constraint (1) or (2) binds program $p$ tentatively rejects all the remaining contracts with demographic type in the relevant set. Then, the program continues accepting contracts one by one in the order of student scores. Phase 2 ends if all contracts are considered or $r_p M + 2 q_p = q_p$ contracts are accepted. Then, the program proceeds with Phase 3.

Phase 3: In this phase, program considers all tentatively rejected contracts and all the remaining contracts with $x_T \in \{mi, Mi, mI, MI\}$. Program $p$ accepts contracts including students with highest scores $z_p$ one at a time. Program continues until either all contracts are considered or $Q_p$ students are chosen. In any case, program proceeds with Phase 4.

Phase 4: In this phase, program considers all the remaining contracts. Program $p$ accepts contracts including students with highest scores $z_p$ one at a time. Program continues until either all contracts are considered or $Q_p$ students are chosen. Then program terminates the procedure and rejects all the remaining contracts, if there is any.

4.2 Stability

As with the Gale and Shapley (1962) paper and most of the matching literature, we are interested in stable allocations. Intuitively, an allocation is stable if students or programs cannot improve upon the chosen allocation by either walking away from it or by bilaterally making arrangements outside of the allocation. One can see that if students have strict preferences over contracts then our stability definition and stable allocation definition in current literature would be equivalent.

**Definition 1** An allocation $X'$ is **stable** if

1. for all $s \in S$ and for all $p \in P$, $X'_s \succ_s \emptyset$ and $C_p(X') = X'_p$; and
ii. \( (p, s) \in P \times S \), and contract \( x \in X \setminus X' \), such that
\[
x \in C_p((X' \setminus X_s') \cup \{x\}) \text{ and } x \succ_s X_s'.
\]

In order to show the existence of stability, we utilize the substitutes and law of aggregate demand properties defined by Hatfield and Milgrom (2005) and irrelevance of rejected contracts defined by Aygün and Sönmez (2013).

### 4.3 Substitutes, IRC and Law of Aggregate Demand

In this section, we define the three properties which are sufficient for existence of stability in our college admission problem and show that \( C^{MCF}() \) has these properties.

**Definition 2** Elements of \( X \) are substitutes for program \( p \) if for all \( Y' \subset Y'' \subset X \) we have
\[
x \in Y' \setminus C_p(Y') \implies x \in Y'' \setminus C_p(Y'').
\]

The substitutes condition simply states that if a contract \( x \) is rejected, not chosen, in a set of contracts \( Y' \) then adding any other contract to \( Y' \) cannot make \( x \) desirable or \( x \) should remain rejected in bigger sets that contain \( Y' \).

**Lemma 1** Elements of \( X \) are substitutes for each program \( p \) under choice function \( C^{MCF}() \).

**Definition 3** Choice function \( C() \) satisfies Law of Aggregate Demand for program \( p \) if for all \( Y' \subset Y'' \subset X \) we have \( |C_p(Y')| \leq |C_p(Y'')| \).

Under the law of aggregate demand, when more contracts are added to a set of contracts, the size of the chosen set never shrinks. Since, in any phase, unfilled seats are transferred to the next phases, and any student is acceptable to programs, we can state the following lemma.

**Lemma 2** Choice function \( C^{MCF}() \) satisfies Law of Aggregate Demand for each program \( p \).

For many to one matching problems that use choice functions of programs as primitive, Aygün and Sönmez (2013) shows that substitutes condition is not sufficient to guarantee existence of stable allocations. Therefore, since our primitive of the model for programs is choice
functions rather than preferences, we use the Irrelevance of Rejected Contracts\(^7\) condition defined by Aygün and Sönmez (2013) along with substitutes condition.

**Definition 4** Given a set of contracts \( X \), a choice function \( C() \) satisfies the *Irrelevance of Rejected Contracts* (IRC) if

\[
\forall Y \subset X, \forall x \in X \setminus Y \quad x \notin C(Y \cup \{x\}) \implies C(Y) = C(Y \cup \{x\}).
\]

IRC condition simply states that an outcome of the choice function should not be affected by removing rejected contracts. With the help of this condition, Aygün and Sönmez (2013) show that we can guarantee the existence of stability without the need for strict preferences of programs over sets of contracts.

**Lemma 3** Choice function \( C^{MCF}(\cdot) \) satisfies Irrelevance of Rejected Contracts for each program \( p \).

Finally, with the help of those conditions above, we can state that it is guaranteed to find a stable allocation for our student-program matching problem.

**Proposition 1** If all programs use \( C^{MCF}(\cdot) \) set of stable allocations for student-program matching problem is not empty.

### 4.4 Incentive Compatibility, Fairness and Reserves

An ideal choice function should also satisfy incentive compatibility and fairness. Incentive compatibility suggests that no student should be worse off by revealing her true demographic type. With this property, we can state that for any school, students do not have to gather information and strategize their application processes. Hence, we can level the playing field for students.

\(^7\)Irrelevance of Rejected Contracts condition first defined as "Consistency" in Alkan and Gale (2001).
Definition 5 Given a set of contracts $X$, a choice function $C : 2^X \rightarrow 2^X$ is incentive compatible if for any given set of contracts $Y \subset X$, and any student $s$ with no contract in $Y$,

$$(s, p, t_s) \notin C_p(Y \cup \{(s, p, t_s)\}) \implies (s, p, t') \notin C_p(Y \cup \{(s, p, t')\}), \forall t' \in T_s.$$ 

Proposition 2 Choice function $C^{MCF}()$ is incentive compatible.

Unlike BRCF, the choice function we design gives students no incentive to hide their true types. This property will have an important role in strategic properties of the mechanism we suggest.

Definition 6 Given a set of contracts $X$, a choice function $C : 2^X \rightarrow 2^X$ is fair if for any given subset $Y \subset X$ and any program $p$,

$$x \notin C_p(Y) \implies \forall y \in C(Y), \text{ either } z_p(y_S) > z_p(x_S) \text{ or } y_T \notin T_{x_S}.$$ 

Fairness of the choice function as we use here indicates that, if a contract is not chosen this means chosen contracts either include students with higher test scores or they are chosen due to affirmative action policy.

Proposition 3 Choice function $C^{MCF}()$ is fair.

The new law issued in Brazil requires some reserve structure on the sets chosen by programs. This means that if there is a student of a certain type and a reserved seat for her type then she should be chosen in order to satisfy the reserve structure. So, one of the natural objectives we are going to be interested in satisfying is the reserve structure.
Definition 7 A choice function $C_p : 2^X \rightarrow 2^X$ satisfies reserves at program $p$ if $\forall Y \subseteq X$:

$$
|\{x \in Y : x_T \in \{mi, Mi, mI, MI\}\}| \geq \frac{Q_p}{2},
$$

$$
|\{x \in Y : x_T \in \{mi, Mi\}\}| \geq \frac{Q_p}{4},
$$

and $|\{x \in Y : x_T \in \{mi, mI\}\}| \geq \frac{r^m Q_p}{2}$ implies

$$
|\{x \in C_p(Y) : x_T \in \{mi, Mi, mI, MI\}\}| \geq \frac{Q_p}{2},
$$

$$
|\{x \in C_p(Y) : x_T \in \{mi, Mi\}\}| \geq \frac{Q_p}{4},
$$

and $|\{x \in C_p(Y) : x_T \in \{Mi, mi\}\}| \geq \frac{r^m Q_p}{2}$.

The definition above states that a choice function must choose a sufficient number of students from all demographic types that are subject to affirmative action, whenever it is possible. It is obvious that when $q_p = 0$ our choice function satisfies reserves. However, when $q_p = 0$ and $r^m = \frac{1}{2}$, all the seats reserved for public high school graduates will be reserved for only low-income and minority students from public high schools. In this case, non-low-income non-minority students from public high schools cannot have seats reserved for public high school graduates unless there are not enough applications from other types. Also, low-income minority students from public high schools may not enjoy this advantage unless they have high scores. Current guidelines set by the Brazilian government give priority to non-low-income non-minority and low-income minority students from public high schools for some seats. Due to this fact, one can argue that there is an implicit objective that programs should give priority to each demographic type for some seats. Since giving priority to each group may cause incentive compatibility problems, our choice function, as a second best, reserves seats to each demographic type along with all students who can mimic the type. For a given program $p$, let $q_p$ be number of reserved seats to low-income minority students from public high schools. Therefore, if a program receives at least $q$ contracts with type $mi$, program should accept at least $q$ contracts with type $mi$. Otherwise, program should accept all contracts with type $mi$.

Definition 8 A choice function $C_p : 2^X \rightarrow 2^X$ satisfies weak-reserves at program $p$ if
\[ \forall Y \subseteq X: \]

\[ |\{ x \in Y : x_T \in \{mi, Mi, mI, MI\} \}| \geq \frac{Q_p}{2}, \]

\[ |\{ x \in Y : x_T \in \{mi, Mi\} \}| \geq \frac{Q_p}{4}, \]

\[ |\{ x \in Y : x_T \in \{mi, mI\} \}| \geq \frac{r_p^m Q_p}{2} \]

and \[ |\{ x \in Y : x_T \in \{mi\} \}| \geq q_p \] implies

\[ |\{ x \in C_p(Y) : x_T \in \{mi, Mi, mI, MI\} \}| \geq \frac{Q_p}{2}, \]

\[ |\{ x \in C_p(Y) : x_T \in \{mi, Mi\} \}| \geq \frac{Q_p}{4} \]

and \[ |\{ x \in C_p(Y) : x_T \in \{mi, mI\} \}| \geq \frac{r_p^m Q_p}{2}. \]

The second version of satisfying reserves includes reserved seats for low-income minorities. Weak-reserves requires satisfying reserves only in situations where we have at least enough applications from low-income minority students from public high school to fill reserved seats for them and satisfying all reserves is possible.

**Proposition 4** Choice function \( C^{MCF}(\cdot) \) satisfies weak-reserves at any program \( p \).

Although \( C^{MCF} \) does not choose set of contracts that satisfy reserves for any set of contracts, \( q_p \) can be determined differently for different programs. While programs that set low \( q_p \) minimize the number of cases that fail to give enough seats to certain demographic types, programs that set \( q_p \) higher give more opportunity to non-low-income non-minority students from public high schools. One possible way for setting \( q_p \) is to construct an expected number of applications from low-income-minority students from public high schools based on past years’ applications.

### 4.5 Student Proposing Cumulative Offer Algorithm and Student Optimal Stable Mechanism

The choice function defined above only shows how a school should behave for a given situation. Now, with the help of choice function, we are ready to introduce Student Optimal Stable
Mechanism, $\psi^{SOSM}$. SOSM is a direct mechanism. Therefore, students submit demographic type $t$ and preferences $\succ^t$. Then, we use the student proposing cumulative offer algorithm with submitted demographic type $t$, preferences $\succ^t$ and $C^{MCF}()$ for each program. The cumulative offer algorithm description we use here, was previously introduced by Hatfield and Kojima (2010).

Step 1: One randomly selected student $s_1$ offers her first choice contract $x^1$, according to her preferences $\succ^t_{s_1}$. The program that receives the offer, $p_1 = x^1_p$, holds the contract. Let $A_{p_1}(1) = x^1$, and $A_p(1) = \emptyset$ for all $p \neq p_1$.

In general,

Step $k \geq 2$: One of the students for whom no contract is currently held by a program, say $s_k$, offers the most preferred contract, according to her preferences $\succ^t_{s_k}$, that has not been rejected in previous steps. Call the new offered contract, $x^k$. Let $p_k = x^k_p$, hold $C_{p_k}(A_{p_k}(k-1) \cup \{x^k\})$ and reject all other contracts. Let $A_{p_k}(k) = A_{p_k}(k-1) \cup \{x^k\}$, and $A_p(k) = A_p(k-1)$ for all $p \neq p_k$.

The algorithm terminates when either every student is matched to a program or every unmatched student has no contract left to offer. The algorithm terminates in some finite number $K$ of steps due to a finite number of contracts. At that point, the algorithm produces $X' = \bigcup_{p \in P} C'_p(A_p(K))$, i.e., the set of contracts that are held by some program at the terminal step $K$.

Now, we explain the properties of the mechanism we just introduced. We have already shown that set of stable allocations is not empty in case of choice functions with substitutes condition. Our first result shows that the student optimal stable mechanism gives us a stable allocation which is one of the main desired properties of a mechanism in matching literature.

**Proposition 5** The Student Optimal Stable Mechanism, $\psi^{SOSM}$, produces a stable allocation for any given problem.

The next property we introduce is fairness. We already defined a fairness notion for choice functions. Mechanisms are fair if they consider solely student scores or affirmative action criteria defined by the Brazilian constitution when they choose an allocation.
Definition 9 An allocation $X'$ is fair if for any given pair of contracts $x, y \in X'$

$$y_P \succ^*_x x_P \implies \text{either } z_{y_P}(y_S) > z_{y_P}(x_S) \text{ or } y_T \notin T_{x_S}.$$ 

A mechanism is fair if for any given problem it chooses a fair allocation.

In previous school choice and student placement literature, like Balinski and Sonmez (1999), it is shown many times that stability is sufficient for fairness of allocation. This idea comes from the fairness of responsive preferences of schools. As opposed to the previous school choice and student placement literature, programs in our model do not have responsive preferences. The non existence of responsive preferences may result in allocations that are not fair. Therefore, in our problem, the stability of the mechanism is not sufficient for fairness. That is the reason why fairness in our mechanism comes from the fairness of the choice function.

Proposition 6 The Student Optimal Stable Mechanism, $\psi^{SOSM}$, is fair.

The next property we discuss here is the strategy-proofness of the mechanism, which is a desired element for mechanism design problems. Strategy-proofness can be described as the non-manipulability of a mechanism via preferences or type. In our problem, students' strategy spaces do not consist only of preferences over schools but also include demographic types. Therefore, although it is tempting to conclude that the strategy proofness of the SOSM immediately follows as a corollary, due to the wider strategy space for students it is not correct for our setup.

Definition 10 A mechanism is strategy-proof if

$$\forall s \in S, \delta_{-s} \in \prod_{j \in S \setminus \{s\}} \Delta_j, (t_s, \succ_s^{(s)}), \delta' \in \Delta_s, \text{ such that } \psi((t_s, \succ_s^{(s)}), \delta_{-s}).$$

Explicitly, for any student we consider, no matter what her true preferences are or what her demographic type is it will be her best interest to reveal her true demographic credentials and preferences. Any manipulation can not make her better off. This is valid for any allocation problem and any strategies other students report.
Proposition 7  The Student Optimal Stable Mechanism, $\psi^{SOSM}$, is strategy-proof.

5  Current Mechanism Revisited

So far, we introduced some desired properties that a choice function and a mechanism should satisfy. In this section, first, we are going to formally describe two of the choice functions which are implementation of guidelines and currently used by two of the biggest federal universities in Brazil. Next, we are going to show deficiencies of those choice functions and any stable mechanism that uses these choice functions.

5.1  Two Examples of BRCF

Since the specification given by law allows for different choice procedures, different universities use different procedures. We will describe two instances: the choice function used by the Federal University of Minas Gerais (UFMG) and by the Federal University of Rio Grande do Sul (UFRGS).

For both choice functions, the set of seats is divided into five groups, $Q_{mi}$, $Q_{Mi}$, $Q_{mI}$, $Q_{MI}$ and $Q_-$, with capacities $\frac{Q}{4}r^m$, $\frac{Q}{4}(1 - r^m)$, $\frac{Q}{4}r^m$, $\frac{Q}{4}(1 - r^m)$ and $\frac{Q}{2}$ respectively. Among these seats $Q_{mi}$ is the seats reserved for low-income minority students from public high schools, $Q_{mI}$ for non-low-income minority students from public high schools, $Q_{Mi}$ for low-income non-minority students from public high schools, $Q_{MI}$ for non-low-income non-minority students from public high schools and $Q_-$ for universal access seats. Each group of seats has a certain priority ordering shared among them, and the function fills those seats sequentially, following a pre-defined ordering. Priority orderings and sequence of seats are given in the next subsection.

5.1.1  Matching Problem with Slot Specific Priorities

In this part, we analyze the properties of implementations of guidelines by UFMG and UFRGS. These implementations by UFMG and UFRGS are in the class of choice functions described in Westcamp (2012) and Kominers and Sonmez (2012). This relationship is helpful to analyze
these properties. Therefore, first, we introduce some notations about the matching problem
with slot specific priorities as described in Kominers and Sonmez (2013) and examine if these
choice functions guarantee stable allocations. Next, we look for properties such as fairness,
incentive compatibility and satisfying reserves.

Each program $p$ has set of slots $I_p$ such that each slot can be assigned to at most one student.
Set of slots can be partitioned into five sets, $Q^p_{mi}$, $Q^p_{Mi}$, $Q^p_{mI}$, $Q^p_{MI}$ and $Q^p$.
Slots $i \in I_p$ have (linear) priority orders $\Pi^i$ over $X_p$. Next, the slots in $I_p$ are ordered according to an order of
precedence $\triangleright^p$ such that if $i \triangleright^p i'$ then slot $i$ is, whenever possible, filled before $i'$. For a program
$p$, we can denote seats as $I_p = \{i^1_p, i^2_p, \ldots, i^Q_p\}$ in a way that $i^l_p \triangleright^p i^{l+1}_p$. The class of choice
functions of programs for matching problem with slot specific priorities are as following:

- For any given set of contracts, $Y$, reject all contracts but contracts in $Y_p$

- Assign the best contract in $Y_p$ according to $\Pi^{i^1}_p$, to $i^1_p$

- Among the remaining contracts assign the best contract according to $\Pi^{i^2}_p$, to $i^2_p$

\vdots

- Among the remaining contracts assign the best contract according to $\Pi^{i^Q}_p$, to $i^Q_p$

- Reject all the remaining contracts.

Let $x^\alpha$ be a generic contract with type $\alpha \in (mi, Mi, mI, MI, Pr)$, i.e. $x^\alpha_T = \alpha$. According
to this notation, UFMG and UFRGS use following priorities and order of precedence:

\[
\begin{align*}
\text{UFMG slot Priorities} & \\
i_p \in Q^p_{mi} & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & \text{if } x_T = x'_T \\
    x^m i \Pi x_{mi} i^p x_{mi} x_{mi} x_M i^p x_{mi} x_{mi} x_P^r & \text{otherwise}
\end{cases} \\
i_p \in Q^p_{Mi} & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & \text{if } x_T = x'_T \\
    x^M i \Pi x_{mi} i^p x_{mi} x_{mi} x_M i^p x_{mi} x_{mi} x_P^r & \text{otherwise}
\end{cases} \\
i_p \in Q^p_{mi} & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & \text{if } x_T = x'_T \\
    x^M i \Pi x_{mi} i^p x_{mi} x_{mi} x_M i^p x_{mi} x_{mi} x_P^r & \text{otherwise}
\end{cases} \\
i_p \in Q^p_{MI} & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & \text{if } x_T = x'_T \\
    x^M i \Pi x_{mi} i^p x_{mi} x_{mi} x_M i^p x_{mi} x_{mi} x_P^r & \text{otherwise}
\end{cases} \\
i_p \in Q^p_0 & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & \text{if } x_T = x'_T \\
    x^p i^p x_{mi} i^p x_{mi} x_{mi} x_M i^p x_{mi} x_{mi} x_P^r & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{UFMG order of precedence} & \\
\text{First } |Q^p_{mi}| \text{ seats are in } Q^p_{mi} & \\
\text{Next } |Q^p_{Mi}| \text{ seats are in } Q^p_{mi} & \\
\text{Next } |Q^p_{mi}| \text{ seats are in } Q^p_{mir} & \\
\text{Last } |Q^p_0| \text{ seats are in } Q^p_0 & \\
\end{align*}
\]

\[
\begin{align*}
\text{UFRGS slot Priorities} & \\
i_p \in Q^p_{mi} & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & \text{if } x_T = x'_T \\
    x^m i \Pi x_{mi} i^p x_{mi} x_{mi} x_M i^p x_{mi} x_{mi} x_P^r & \text{otherwise}
\end{cases} \\
i_p \in Q^p_{Mi} & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & \text{if } x_T = x'_T \\
    x^m i \Pi x_{mi} i^p x_{mi} x_{mi} x_M i^p x_{mi} x_{mi} x_P^r & \text{otherwise}
\end{cases} \\
i_p \in Q^p_{mi} & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & \text{if } x_T = x'_T \\
    x^m i \Pi x_{mi} i^p x_{mi} x_{mi} x_M i^p x_{mi} x_{mi} x_P^r & \text{otherwise}
\end{cases} \\
i_p \in Q^p_{MI} & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & \text{if } x_T = x'_T \\
    x^m i \Pi x_{mi} i^p x_{mi} x_{mi} x_M i^p x_{mi} x_{mi} x_P^r & \text{otherwise}
\end{cases} \\
i_p \in Q^p_0 & \quad \begin{cases} 
    x_i^p x' \iff z_p(x_S) > z_p(x'_S) & 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{UFRGS order of precedence} & \\
\text{First } |Q^p_{mi}| \text{ seats are in } Q^p_0 & \\
\text{Next } |Q^p_{mi}| \text{ seats are in } Q^p_0 & \\
\text{Next } |Q^p_{mi}| \text{ seats are in } Q^p_0 & \\
\text{Last } |Q^p_0| \text{ seats are in } Q^p_0 & \\
\end{align*}
\]

Once we define these two implementations of BRCF guidelines, bilateral substitutes property of contracts directly comes from the second proposition of Kominers and Sonmez (2013). Also, since there is only one possible contract for each student to offer to a given program, we can say that contracts satisfy the substitutes condition. Moreover, since each contract is acceptable to all slots, with bigger contract sets chosen set of contract never shrinks. Therefore, $C^{\text{UFMG}}()$ and $C^{\text{UFRGS}}()$ satisfy the Law of Aggregate Demand. Hence, if all programs use one of the
implementations above, the existence of stable allocation is guaranteed by Proposition 1 of Aygun and Sonmez (2013).

5.2 The case against Current Mechanism

The two implementations of guidelines designed by the Brazilian government are instances of choice functions described in Westcamp (2012) and Kominers and Sonmez (2013). Since these choice functions are designed for a single contract for each student, like $C^{MCF}$, contracts are not only bilateral substitutes, a weak version of substitutes condition, as shown in Kominers and Sonmez (2013) but also substitutes for each program. But these choice functions, unlike $C^{MCF}$, fail to satisfy fairness, incentive compatibility and weak-reserves properties. We show, by using examples, how these choice functions violate these three important conditions. We start with incentive compatibility.

Example 1 (Incentive compatibility) For a given program $p$ let $Q_p = 8$, $\gamma^m_p = \frac{1}{2}$ and let set of contracts be $Y = \{x^1, \ldots, x^8\}$ such that $x^1_T = x^2_T = x^3_T = Pr$, $x^5_T = MI$, $x^6_T = mi$, $x^7_T = mI$ and $x^8_T = Mi$. Also let $z_p(x^i_S) > z_p(x^j_S) \iff i < j$. Consider a low-income minority student from public high school $s \notin s(Y)$ with score $z_p(s) > z(x^8_S)$. If she applies with a contract that includes her true type, i.e. $(s,p,mi)$, no matter which example of BCRF program $p$ uses, she will be rejected:

$$(s, p, mi) \notin C_p(Y \cup \{(s, p, mi)\}) = \{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$$

However, if she applies as a low-income non-minority student from public high school, i.e. $(s,p, Mi)$, no matter which implementation of BRCF program $p$ uses, her contract will be accepted:

$$(s, p, Mi) \in C_p(Y \cup \{(s, p, Mi)\}) = \{x^1, x^2, x^3, x^4, x^5, x^6, x^7, (s, p, Mi)\}$$

Therefore, the two examples of BRCF are not incentive compatible.
The example above shows that since the choice function gives priority to low-income non-minority students from public high schools whose type student $s$ can mimic, choice function gives student $s$ incentive to lie about her demographic type. This problem can be solved by $C^{MCF}$. $C^{MCF}$ gives students equal or higher chances to be chosen when their contracts compete with others that students can mimic. Hence students have no incentive to mimic other types.

The second example we give is about fairness property of choice functions.

**Example 2 (Fairness)**  For a given program $p$ let $Q_p = 8$, $r^m_p = \frac{1}{2}$ and let set of contracts be $Y = \{x^1, \ldots, x^9\}$ such that $x^1_T = x^2_T = x^3_T = x^4_T = Pr$, $x^5_T = x^6_T = mi$, $x^7_T = Mi$, $x^8_T = mI$ and $x^9_T = MI$. Also let $z_p(x^i_S) > z_p(x^j_S) \iff i < j$. In this case, no matter which example of BCRF program $p$ uses, chosen set will be:

$$C_p(Y) = \{x^1, x^2, x^3, x^4, x^5, x^7, x^8, x^9\}$$

Let $x^6_S = j$. Since student $j$ can offer $x^6$, we can say that $t_j = mi$ and $MI \in T_j$. Therefore rejecting $x^6$ while accepting $x^9$, violates fairness of choice function.

In this second example, the program $p$, chooses $x^9$, although $mi$ is available to student $j$. This example tells us that the guideline provided by government implicitly tries to provide diversity in the chosen students even when the law does not require it. On the other hand, $C^{MCF}$ only gives priority to students that affirmative action is addressed to. Therefore, $C^{MCF}$ prevents any fairness issues. Next example is about the relationship between choice functions and reserve structure.

**Example 3 (Weak-Reserves)**  For a given program $p$ let $Q_p = 8$, $r^m_p = \frac{1}{2}$ and let set of contracts be $Y = \{x^1, \ldots, x^9\}$ such that $x^1_T = x^2_T = x^3_T = x^4_T = Pr$, $x^5_T = x^6_T = mi$, $x^7_T = Mi$, $x^8_T = mI$ and $x^9_T = MI$. Also let $z_p(x^i_S) > z_p(x^j_S) \iff i < j$. In both applications of BCRF guidelines, numbers of reserved seats for low-income minority students from public high schools are 1. If set of contracts $Y$, no matter which example of BCRF program $p$ uses chosen set will be:

$$C_p(Y) = \{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^9\}$$
Therefore, choice function chooses only one minority student from public high school, although it is possible to choose two, which is the number of reserved seats for minority students from public high schools.

Another problem with BRCF is that choice function considers students with type $MI$ as a first order substitute for students with type $mI$. Therefore, when there is an absence of applications from type $mI$ students, the choice function turns to students with type $MI$ and ignores reserves for minorities. In the example above, one of non-low-income non-minority students from public high school receives the seat reserved for non-low-income minority students from public high schools. Hence, implications of BRCF fail to satisfy weak-reserves.

In the second part, we show that if programs adopt one of the implementations of BRCF, no matter what algorithm one chooses in order to create a stable mechanism, the mechanism violates the properties we defined in previous chapters. Previous papers have shown us that some of the deficiencies of choice functions can be corrected by choosing right algorithms. One example of this is the choice function used by USMA. Sonmez and Switzer (2013) have shown us that USMA priorities may fail to satisfy fairness, but when they use the cumulative offer algorithm the outcome of the mechanism is always fair. However, the following two examples show that violations of incentive compatibility and fairness are carried to any stable mechanism due to the nature of the problem which is choice functions requires one demographic type for each student.

**Example 4 (Strategy Proofness)** There is one program $p$ with capacity of eight and nine students $S = \{s_1, \ldots, s_9\}$. Let $r^m_p = \frac{1}{2}$ and $p$ be more preferred null contract for each student. Score order of students is given as $z_p(s_i) > z_p(s_j) \iff i < j$. Also, demographic types of
students are given as

\[ t_{s_1} = t_{s_2} = t_{s_3} = t_{s_4} = Pr \]
\[ t_{s_5} = t_{s_6} = mi \]
\[ t_{s_7} = MI \]
\[ t_{s_8} = mI \]
\[ t_{s_9} = Mi \]

For this problem, if every student submits her true demographic type, there is only one stable allocation, say \( X' \), that we can achieve if program \( p \) uses one of the implementations of current BRCF and

\[ s(X') = \{s_1, s_2, s_3, s_4, s_5, s_7, s_8, s_9\} \]

Now, given that other students use the same strategy, if \( s_6 \) submits a contract with demographic type \( MI \), i.e. \((s_6, p, MI)\), instead of her true type, there is again only one stable allocation, say \( X'' \), that we can achieve if program \( p \) uses one of the implementations of current BRCF and

\[ s(X'') = \{s_1, s_2, s_3, s_4, s_5, s_6, s_8, s_9\} \]

Therefore, any stable mechanism with these two examples of BRCF are not strategy-proof.

The example above shows that since these choice functions give priority to students of types that student \( s_6 \) can mimic, these choice functions give student \( s_6 \) incentive to lie about her demographic type. Violation of this property not only puts a burden on students to gather more information about their peers and strategize their behavior in order to get better assignments, but also gives some students an unfair advantage in their college applications. Also, violation of this property causes an allocation to be chosen which is actually (with true student types) unstable and that makes it harder to observe the effect of this affirmative action for future policy decisions. The last example we give is about fairness property of mechanisms.

**Example 5 (Fairness)** There is one program \( p \) with capacity of eight and nine students \( S = \)
\( \{s_1, \ldots, s_9\} \). Let \( r^m_p = \frac{1}{2} \) and \( p \) be more preferred null contract for each student. Score order of students is given as \( z_p(s_i) > z_p(s_j) \iff i < j \). Also, demographic types of students are given as

\[
\begin{align*}
t_{s_1} &= t_{s_2} = t_{s_3} = t_{s_4} = Pr \\
t_{s_5} &= t_{s_6} = mi \\
t_{s_7} &= MI \\
t_{s_8} &= mI \\
t_{s_9} &= Mi
\end{align*}
\]

For this problem, if every student submits her true demographic type, there is only one stable allocation, say \( X' \), that we can achieve if program \( p \) uses one of the implementations of current BRCF and

\[
s(X') = \{s_1, s_2, s_3, s_4, s_5, s_7, s_8, s_9\}
\]

Since any demographic type is available to student \( s_6 \) and she has higher score than \( s_7, s_8 \) and \( s_9 \), rejecting \( (s_6, p, mi) \) while accepting \( (s_7, p, MI) \), violates fairness. This result holds no matter what kind of algorithm we use that gives stable allocation with these two implementations of BRCF.

6 Conclusion

In this paper, we presented a new market design application of university program-student matching that emerged as result of the affirmative action policy that was designed by the Brazilian government to aid minority and low-income students from public high schools. This problem is particularly interesting in the sense that the freedom of not declaring one’s actual type during the application process combines the matching problem with the adverse selection problem. Due to this property, we defined incentive compatibility of choice functions for the first time in this literature.
This paper shows that the current guidelines for designing choice functions for programs have unavoidable deficiencies, such as unfair allocations and giving sophisticated students more chance to attend better programs via manipulating the system.

We proposed a new choice function as well as a new mechanism, called the multidimensional Brazil reserves choice function and student optimal stable mechanism. The choice function is incentive compatible and fair unlike current choice functions which are implementation of guidelines designed by the Brazilian government. Moreover, the mechanism we suggest is strategy-proof, fair and gives a stable allocation for any problem.

With a complex reserves structure like we have in this problem, it is hard to satisfy reserves for all cases. We showed that current choice functions used by programs in Brazil not only fail to satisfy reserves when it is possible but also fail to satisfy the weak reserves condition. On the other hand, the choice function we suggest always satisfies weak reserves and if the number of reserves for a certain group is selected right, the diversity target in programs is reachable by our mechanism.

7 References


6. Aygün O. and T. Sönmez (2012b), The Importance of Irrelevance of Rejected Contracts in Matching under Weakened Substitutes Conditions, Boston College working paper.


8 Appendix

Proof of Lemma 1. For any set of contracts $Y$ and any phase $i$, let $Y_i$ be set of contracts that is considered in phase $i$. Think about the procedure:

Phase 1. First observe that $Y'_1 \subseteq Y''_1$. If a contract $x$ in $Y'_1$ is not accepted in the first phase then we have

$$\left| \{ y \in Y'_1 : z_p(y_S) > z_p(x_S) \} \right| \geq q_p.$$ 

Therefore $Y' \subseteq Y''$ implies

$$\left| \{ y \in Y''_1 : z_p(y_S) > z_p(x_S) \} \right| \geq q_p.$$ as well. Hence contract $x$ can not be accepted from $Y''$ in the first phase as well. So we have $Y'_2 \subseteq Y''_2$.

Phase 2. Let $\theta'$ and $\theta''$ be number of unused seats in Phase 1 when we use $Y'$ and $Y''$, respectively. First, we should state that $Y'_1 \subseteq Y''_1$ guarantees $\theta' \geq \theta''$. If a contract $x$ in $Y'_2$ is not accepted in the second phase then we have three cases
Case 1: If $x_T = mi$, we have

$$\min\{|\{y \in Y'_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mI\}|, r_p^m Q_p^2 + \theta' - q_p\} +$$

$$\min\{|\{y \in Y'_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = Mi\}|, \frac{Q_p}{4} + \theta' - q_p\} +$$

$$|\{y \in Y'_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mi\}| \geq r_p^m Q_p^2 + \frac{Q_p}{4} + \theta' - 2q_p$$

Therefore $Y' \subseteq Y''$ implies

$$\min\{|\{y \in Y''_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mI\}|, r_p^m Q_p^2 + \theta'' - q_p\} +$$

$$\min\{|\{y \in Y''_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = Mi\}|, \frac{Q_p}{4} + \theta'' - q_p\} +$$

$$|\{y \in Y''_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mi\}| \geq r_p^m Q_p^2 + \frac{Q_p}{4} + \theta'' - 2q_p$$

as well. Hence contract $x$ can not be accepted from $Y''$ in the second phase as well.

Case 2: If $x_T = mI$, we have either

$$|\{y \in Y'_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mI\}| \geq r_p^m Q_p^2 + \theta' - q_p, \text{ or}$$

$$|\{y \in Y'_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mI\}| +$$

$$\min\{|\{y \in Y'_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = Mi\}|, \frac{q_c}{4} + \theta' - q\} +$$

$$|\{y \in Y'_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mi\}| \geq r_p^m Q_p^2 + \frac{Q_p}{4} + \theta' - 2q_p.$$

Therefore $Y' \subseteq Y''$ implies

$$|\{y \in Y''_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mI\}| \geq r_p^m Q_p^2 + \theta'' - q_p, \text{ or}$$

$$|\{y \in Y''_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mI\}| +$$

$$\min\{|\{y \in Y''_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = Mi\}|, \frac{q_c}{4} + \theta'' - q\} +$$

$$|\{y \in Y''_2 : z_p(y_s) > z_p(x_s) \text{ s.t. } y_T = mi\}| \geq r_p^m Q_p^2 + \frac{Q_p}{4} + \theta'' - 2q_p$$

as well. Hence contract $x$ can not be accepted from $Y''$ in the second phase as well.
Case 3: If $x_T = Mi$, we have either

$$\{|y \in Y'_2 : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = Mi\| \geq \frac{q_c}{4} + \theta' - q, \text{ or}$$

$$\{|y \in Y'_2 : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = Mi\| +$$

$$\min\{|\{y \in Y'_2 : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = mI\|, r_p^m \frac{Q_p}{2} + \theta' - q_p\} +$$

$$\{|y \in Y'_2 : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = mi\| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta' - 2q_p.$$ 

Therefore $Y' \subseteq Y''$ implies

$$\{|y \in Y''_2 : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = Mi\| \geq \frac{q_c}{4} + \theta'' - q, \text{ or}$$

$$\{|y \in Y''_2 : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = Mi\| +$$

$$\min\{|\{y \in Y''_2 : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = mI\|, r_p^m \frac{Q_p}{2} + \theta'' - q_p\} +$$

$$\{|y \in Y''_2 : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = mi\| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta'' - 2q_p$$

as well. Hence contract $x$ can not be accepted from $Y''$ in the second phase as well. So any contract $x$ that is not accepted from $Y'$ in Phase 2, is not accepted from $Y''$ in Phase 2. Moreover that guarantees $Y'_3 \subseteq Y''_3$.

**Phase 3.** Let $\theta'_1$ and $\theta''_1$ be number of unused seats in Phase 2 when we use $Y'$ and $Y''$, respectively. We should also state that $Y'_2 \subseteq Y''_2$ guarantees $\theta'_1 \geq \theta''_1$. If a contract $x$ in $Y'_3$ is not accepted in the third phase then we have

$$\{|y \in Y'_3 : z_p(y_S) > z_p(x_S)\| \geq (1 - r_p^m) \frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta'_1.$$ 

Therefore $Y' \subseteq Y''$ implies

$$\{|y \in Y''_3 : z_p(y_S) > z_p(x_S)\| \geq (1 - r_p^m) \frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta''_1$$

as well. Hence contract $x$ can not be accepted from $Y''$ in the third phase as well. So any contract $x$ that is not accepted from $Y'$ in Phase 3, is not accepted from $Y''$ in Phase 3.
Moreover that guarantees \( Y'_4 \subseteq Y''_4 \).

**Phase 4.** Let \( \theta'_2 \) and \( \theta''_2 \) be number of unused seats in Phase 3 when we use \( Y' \) and \( Y'' \), respectively. We should also state that \( Y'_3 \subseteq Y''_3 \) guarantees \( \theta'_2 \geq \theta''_2 \). If a contract \( x \) in \( Y'_3 \) is not accepted in the fourth phase then we have

\[
|\{ y \in Y'_4 : z_p(y_S) > z_p(x_S) \}| \geq \frac{Q_p}{2} + \theta'_2.
\]

Therefore \( Y'_4 \subseteq Y''_4 \) implies

\[
|\{ y \in Y''_4 : z_p(y_S) > z_p(x_S) \}| \geq \frac{Q_p}{2} + \theta''_2
\]
as well. Hence contract \( x \) can not be accepted from \( Y''_4 \) in the last phase as well. So, any contract \( x \) that is not accepted from \( Y'_4 \) in Phase 4 is not accepted from \( Y''_4 \) in Phase 4.

A contract \( x \) is rejected in set \( Y'_4 \) means that \( x \) must not be accepted in any phase of the procedure. Above, we showed that for any phase if a contract is not accepted from \( Y'_4 \), it can not be accepted from \( Y''_4 \). Therefore, if a contract is rejected from set \( Y'_4 \) it must be rejected from set \( Y''_4 \). Hence contracts are substitutes for any program. ■

**Proof of Lemma 2.** By construction of the choice function \( C^{MCF}(\cdot) \), all contracts of a given student can be rejected from a set only when school reaches full capacity. Hence the size of the chosen set can never shrink as the set of available contracts grows. ■

**Proof of Lemma 3.** The choice function for any program \( p \) satisfies the substitutes condition by Lemma 1 and satisfies the Law of Aggregate Demand by Lemma 2. Hence Lemma 3 is a corollary of Proposition 1 in (Aygun and Sonmez 2013) ■

**Proof of Proposition 1.** The choice function for any program \( p \) satisfies the substitutes condition by Lemma 1 and satisfies Irrelevance of Rejected Contracts by Lemma 3. Hence, as a corollary of Theorem 1 in (Aygun and Sonmez 2013), there is a stable allocation for a problem consists of \( (\succ^t_s)_{s \in S} \) and \( (C^{MCF}_p(\cdot))_{p \in P} \). Because this problem only allows students to offer one contract per school. Let one of possible stable allocations for this problem be \( X' \). We next show that \( X' \) is a stable allocation for matching problem consists of \( (\succ_s)_{s \in S} \) and \( (C^{MCF}_p(\cdot))_{p \in P} \).
Assume this is not true. Then there exists a student-program pair \((s, p)\) and a contract \(x\) such that

\[
x \in X \setminus X', x_S = s \text{ and } x_P = p
\]

\[
x \in C_p((X' \setminus X'_s) \cup \{x\}) \text{ and } x >_s X'_s.
\]

Due to incentive compatibility property of \(C_p^{MCF}\), we can find a contract \(y\) such that

\[
y \in X \setminus X', y_S = s, y_P = p \text{ and } y_T = t_s
\]

\[
y \in C_p((X' \setminus X'_s) \cup \{y\}) \text{ and } y >_s X'_s
\]

which contradicts with the stability of \(X'\) for the problem consists of \((>_s)_{s \in S}\) and \((C_p^{MCF})_{p \in P}\). Hence, \(X'\) is a stable allocation for matching problem consists of \((>_s)_{s \in S}\) and \((C_p^{MCF})_{p \in P}\). ■

**Proof of Proposition 2.** Think about five cases:

**Case 1:** If a student \(j \in S^m \cap S^i\), her available strategies are applying as a low-income minority student from a public high school, a non-low-income minority student from a public high school, a low-income non-minority student from a public high school, a non-low-income non-minority student from a public high school or a student from non-public high school. Assume that student \(j\)’s contract with her true demographic type is rejected and check if she can be accepted by applying with a different contract. For a given program \(p\), let \(x' = (s, p, t_s)\) and \(x = (s, p, t')\) where \(t' \neq t_s\) and let \(Y' = Y \cup \{x'\}\) and \(Y'' = Y \cup \{x\}\). First, observe that if her contract \(x'\), i.e. \(x'_S = j\), is rejected from set \(Y'\), then her contract is not chosen in any phase. Therefore, \(\theta', \theta'_1\) and \(\theta'_2\) are all zero since she is considered in all phases. Assume she offers contract \(x\) instead of \(x'\).

Phase 1: If \(j\) does not apply as a low-income minority student from a public high school then she can not be accepted in the first phase. Moreover, since her contract \(x'\) is rejected from set \(Y'\), there are at least \(q_s\) contracts in \(Y\) with demographic type \(mi\). Therefore, \(\theta' = \theta'' = 0\) and \(Y_2'' = (Y'_2 \setminus \{x'\}) \cup \{x\}\).
Phase 2: Observe that if her contract is rejected from set $Y'$, then we have

$$\min\{|\{y \in Y_0': z_p(y_S) > z_p(j) \text{ s.t. } y_T = mI\}|, r_p^m Q_p + \theta' - q_p\} +$$

$$\min\{|\{y \in Y_0': z_p(y_S) > z_p(j) \text{ s.t. } y_T = Mi\}|, \frac{Q_p}{4} + \theta' - q_p\} +$$

$$|\{y \in Y_0': z_p(y_S) > z_p(j) \text{ s.t. } y_T = mi\}| \geq r_p^m Q_p + \frac{Q_p}{4} + \theta' - 2q_p$$

If $j$ applies as a non-low-income minority student from a public high school, in the second phase we have either

$$|\{y \in Y_0'' : z_p(y_S) > z_p(j) \text{ s.t. } y_T = mI\}| \geq r_p^m Q_p + \theta'' - q_p \text{ or}$$

$$|\{y \in Y_0'' : z_p(y_S) > z_p(j) \text{ s.t. } y_T = mI\}| +$$

$$\min\{|\{y \in Y_0'' : z_p(y_S) > z_p(j) \text{ s.t. } y_T = Mi\}|, \frac{Q_p}{4} + \theta'' - q_p\} +$$

$$|\{y \in Y_0'' : z_p(y_S) > z_p(j) \text{ s.t. } y_T = mi\}| \geq r_p^m Q_p + \frac{Q_p}{4} + \theta'' - 2q_p$$

Therefore, student $j$ can not be accepted in the second phase. If $j$ applies as low-income non-minority student from a public high school, in the second phase we have either

$$|\{y \in Y_0'' : z_p(y_S) > z_p(j) \text{ s.t. } y_T = Mi\}| \geq \frac{Q_p}{4} + \theta'' - q_p \text{ or}$$

$$|\{y \in Y_0'' : z_p(y_S) > z_p(j) \text{ s.t. } y_T = M_i\}| +$$

$$\min\{|\{y \in Y_0'' : z_p(y_S) > z_p(j) \text{ s.t. } y_T = mI\}|, r_p^m Q_p + \theta'' - q_p\} +$$

$$|\{y \in Y_0'' : z_p(y_S) > z_p(j) \text{ s.t. } y_T = mi\}| \geq r_p^m Q_p + \frac{Q_p}{4} + \theta'' - 2q_p$$

Therefore, student $j$ can not be accepted in the second phase. If $j$ applies as a non-low-income non-minority student from a public high school or a student from non-public high school, she will not be considered in the second phase, therefore she cannot be accepted in this phase. Hence, no other available contract of student $j$ can be chosen in this phase. Also $\theta_1' = \theta_1''$ and $Y_0'' = (Y_0' \setminus \{x'\}) \cup \{x\}$. 

37
Phase 3: Observe that if her contract is rejected from set $Y'$, then we have

$$|\{y \in Y_3' : z_p(y_S) > z_p(j)\}| \geq (1 - r_p^m) \frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta'_1$$

If $j$ applies as non-low-income minority student from a public high school, a low-income non-minority student from a public high school or a non-low-income non-minority student from a public high school, in the third phase we have

$$|\{y \in Y_3'' : z_p(y_S) > z_p(j)\}| \geq (1 - r_p^m) \frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta''_1$$

Therefore, student $j$ can not be accepted in the third phase. If $j$ applies as a student from non-public high school, she will not be considered in the third phase, therefore she cannot be accepted in this phase. Hence no other available contract makes student $j$ be chosen. Also $\theta'_2 = \theta''_2$ and $Y_4'' = (Y_4' \setminus \{x'\}) \cup \{x\}$.

Phase 4: First observe that if her contract is rejected from set $Y'$, then we have

$$|\{y \in Y_4' : z_p(y_S) > z_p(j)\}| \geq \frac{Q_p}{2} + \theta'_2$$

If $j$ applies as non-low-income minority student from a public high school, a low-income non-minority student from a public high school, a non-low-income non-minority student from a public high school or a student from non-public high school, in the fourth phase we have

$$|\{y \in Y_4'' : z_p(y_S) > z_p(j)\}| \geq \frac{Q_p}{2} + \theta''_2$$

Therefore, student $j$ can not be accepted in the fourth phase. Hence no other available contract makes student $j$ be chosen. So, applying as a low-income minority student from a public high school makes student $j$ weakly better off.

Case 2: If $j \in S^m \setminus S^i$, her available strategies are applying as a non-low-income minority student from a public high school, a non-low-income non-minority student from a public high school or a student from non-public high school. Like Case 1, assume that her contract $x'$ is
rejected and check if she can do better by offering $x$.

Phase 1: Student $j$ cannot apply as a low-income minority student from a public high school, so she cannot be accepted in the first phase. Also $\theta' = \theta''$ and $Y_2'' = (Y_2' \setminus \{x'\}) \cup \{x\}$.

Phase 2: If she applies as a non-low-income non-minority student from a public high school or a student from a non-public high school she is not considered in the second phase so she cannot be accepted in the second phase. Also $\theta'_1 = \theta''_1$ and $Y_3'' = (Y_3' \setminus \{x'\}) \cup \{x\}$.

Phase 3: First observe that if her contract is rejected from set $Y'$, then we have

$$|\{y \in Y'_3 : z_p(y_S) > z_p(j)\}| \geq (1 - r^m_p) \frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta'_1$$

If $j$ applies as a student from non-public high school, then she is not considered in this phase, so she cannot be accepted in phase 3. If she applies as a non-low-income non-minority student from a public high school, in the third phase we have

$$|\{y \in Y''_3 : z_p(y_S) > z_p(j)\}| \geq (1 - r^m_p) \frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta''_1$$

Therefore, student $j$ cannot be accepted in the third phase. Hence no other available contract makes student $j$ be chosen. Also $\theta'_2 = \theta''_2$ and $Y'_4 = (Y'_4 \setminus \{x'\}) \cup \{x\}$.

Phase 4: First observe that if her contract is rejected from set $Y'$, then we have

$$|\{y \in Y'_4 : z_p(y_S) > z_p(j)\}| \geq \frac{Q_p}{2} + \theta'_2$$

If $j$ applies as non-low-income non-minority student from a public high school or a student from non-public high school, in the fourth phase we have

$$|\{y \in Y''_4 : z_p(y_S) > z_p(j)\}| \geq \frac{Q_p}{2} + \theta''_2$$

Therefore, student $j$ cannot be accepted in the fourth phase. Hence no other available contract makes student $j$ be chosen. So, applying as a non-low-income minority student from a public high school makes student $j$ weakly better off.

39
Case 3: If $j \in S_i \setminus S^m$, her available strategies are applying as a low-income non-minority student from a public high school, a non-low-income non-minority student from a public high school or a student from non-public high school. Like the previous cases, assume that her contract $x'$ is rejected and check if she can do better by offering $x$.

Phase 1: Student $j$ can not apply as a low-income minority student from a public high school, so she can not be accepted in the first phase. Also $\theta' = \theta''$ and $Y_{3}'' = (Y_{2}'' \setminus \{x'\}) \cup \{x\}$.

Phase 2: If she applies as a non-low-income non-minority student from a public high school or a student from a non-public high school she is not considered in the second phase so she can not be accepted in the second phase. Also $\theta_1' = \theta_1''$ and $Y_{3}'' = (Y_{3}'' \setminus \{x'\}) \cup \{x\}$.

Phase 3: First observe that if her contract is rejected from set $X'$, then we have

$$|\{y \in Y_3': z_p(y_S) > z_p(j)\}| \geq (1 - r_p^m) \frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1'$$

If $j$ applies as a student from non-public high school, then she is not considered in this phase, so she can not be accepted in phase 3. If she applies as a non-low-income non-minority student from a public high school, in the third phase we have

$$|\{y \in Y_3'': z_p(y_S) > z_p(j)\}| \geq (1 - r_p^m) \frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1''$$

Therefore, student $j$ can not be accepted in the third phase. Hence no other available contract makes student $j$ be chosen. Also $\theta_2' = \theta_2''$ and $Y_{4}'' = (Y_{4}'' \setminus \{x'\}) \cup \{x\}$.

Phase 4: First observe that if her contract is rejected from set $Y'$, then we have

$$|\{y \in Y_4': z_p(y_S) > z_p(j)\}| \geq \frac{Q_p}{2} + \theta_2'$$

If $j$ applies as non-low-income non-minority student from a public high school or a student from non-public high school, in the fourth phase we have

$$|\{y \in Y_4'': z_p(y_S) > z_p(j)\}| \geq \frac{Q_p}{2} + \theta_2''$$
Therefore, student $j$ cannot be accepted in the fourth phase. Hence no other available contract makes student $j$ be chosen. So, applying as a low-income non-minority student from a public high school makes student $j$ weakly better off.

Case 4: If $j \in S^P \setminus (S^i \cup S^m)$, her available strategies are applying as a non-low-income non-minority student from a public high school or a student from non-public high school. Like the previous cases, assume that her contract $x'$ is rejected and check if she can do better by offering $x$.

Phase 1: Student $j$ cannot apply as a low-income minority student from a public high school, so she cannot be accepted in the first phase. Also $\theta' = \theta''$ and $Y'_0' = (Y'_0 \setminus \{x'\}) \cup \{x\}$.

Phase 2: Student $j$ cannot apply as a low-income or a minority student from a public high school, so she cannot be accepted in the second phase. Also $\theta'_1 = \theta''_1$ and $Y''_3' = (Y'_3 \setminus \{x'\}) \cup \{x\}$.

Phase 3: If $j$ applies as a student from non-public high school, then she is not considered in this phase, so she cannot be accepted in phase 3. Also $\theta'_2 = \theta''_2$ and $Y''_4' = (Y'_4 \setminus \{x'\}) \cup \{x\}$.

Phase 4: First observe that if her contract is rejected from set $Y'$, then we have

$$|\{y \in Y'_4 : z_p(y_S) > z_p(j)\}| \geq \frac{Q_p}{2} + \theta'_2$$

If $j$ applies as non-low-income non-minority student from a public high school or a student from non-public high school, in the fourth phase we have

$$|\{y \in Y''_4 : z_p(y_S) > z_p(j)\}| \geq \frac{Q_p}{2} + \theta''_2$$

Therefore, student $j$ cannot be accepted in the fourth phase. Hence no other available contract makes student $j$ be chosen. So, applying as a non-low-income non-minority student from a public high school makes student $j$ weakly better off.

Case 5: If $j \in S \setminus S^P$, her only available strategy is applying as a student from non-public high school. So, applying as a student from a non-public high school is her best strategy.

Therefore, for all students applying with their true characteristics is their best strategy. Hence Choice function is incentive compatible.
Proof of Proposition 3. For any arbitrary set of contracts Y, any rejected contract \( x \) with type \( mi \), i.e. \( x_T = mi \), has lower score than any chosen contract. So, \( x \notin C_p^{MCF}(Y) \) and \( x_T = mi \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S) \).

For any rejected contract with type \( Mi \), the only possible two types of contracts that is chosen and with lower score than \( x \) are \( mi \) and \( mI \). But since they are not available for the student and other chosen contracts have higher score than \( x \), we have \( x \notin C_p^{MCF}(Y) \) and \( x_T = Mi \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S) \) or \( y_T \notin T_{xs} \).

For any rejected contract with type \( mI \), the only possible two types of contracts that is chosen and with lower score than \( x \) are \( mI \) and \( Mi \). But since they are not available for the student and other chosen contracts have higher score than \( x \), we have \( x \notin C_p^{MCF}(Y) \) and \( x_T = mI \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S) \) or \( y_T \notin T_{xs} \).

For any rejected contract with type \( MI \), the only possible types of contracts that is chosen and with lower score than \( x \) are \( mi,mI \) and \( Mi \). But since they are not available for the student and other chosen contracts have higher score than \( x \), we have \( x \notin C_p^{MCF}(Y) \) and \( x_T = mi \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S) \) or \( y_T \notin T_{xs} \).

For any rejected contract with type \( Pr \), any contract but the one with type \( Pr \) is not available for the student and other chosen contracts have higher score than \( x \), we have \( x \notin C_p^{MCF}(Y) \) and \( x_T = Pr \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S) \) or \( y_T \notin T_{xs} \). Hence for any type of contract, \( x \notin C_p^{MCF}(Y) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S) \) or \( y_T \notin T_{xs} \). \( \blacksquare \)

Proof of Proposition 4. For a given program \( p \) and given set of contracts \( Y \), let

\[
|\{ x \in Y : x_T \in \{mi, Mi, mI, MI\} \}| \geq \frac{Q_p}{2},
\]

\[
|\{ x \in Y : x_T \in \{mi, Mi\} \}| \geq \frac{Q_p}{4},
\]

\[
|\{ x \in Y : x_T \in \{mi, mI\} \}| \geq \frac{r_p Q_p}{2}
\]

and \( |\{ x \in Y : x_T \in \{mi\} \}| \geq q \).

First, consider reserve for minority students from public high schools. In the first phase \( q \) contracts with type \( mi \) will be accepted. In the second phase, a contract will be accepted
whenever it is in top $\frac{Q_p}{2} - q$ among contracts with minority type, i.e. $x_T \in \{m_i, mI\}$, in $Y_2$. Therefore, in the second phase at least $\frac{Q_p}{2} - q$ and in total at least $\frac{r_p^m Q_p}{2}$ contracts with minority type will be accepted, otherwise all contracts with minority type will be accepted. Hence, reserve for minority students from public high schools will be satisfied.

Next, consider reserve for low-income students from public high schools. In the first phase $q$ contracts with type $mi$ will be accepted. In the second phase, a contract will be accepted whenever it is in top $\frac{Q_p}{4} - q$ among contracts with low-income type, i.e. $x_T \in \{mI, Mi\}$, in $Y_2$. Therefore, in the second phase at least $\frac{Q_p}{4} - q$ and in total at least $\frac{Q_p}{4}$ contracts with low-income type will be accepted, otherwise all contracts with low-income type will be accepted. Hence, reserve for low-income students from public high schools will be satisfied.

Finally, consider reserve for students from public high schools. In the first two phases $\frac{r_p^m Q_p}{2} + \frac{Q_p}{4} - q$ contracts with public school type, i.e. $x_T \in \{mI, Mi, MI\}$, will be accepted. In the third phase, a contract will be accepted whenever it is in top $\frac{Q_p}{4} - \frac{r_p^m Q_p}{2} + q$ among contracts with public school type in $Y_3$. Therefore, in the third phase at least $\frac{Q_p}{4} - \frac{r_p^m Q_p}{2} + q$ and in total at least $\frac{Q_p}{2}$ contracts with public school type will be accepted, otherwise all contracts with public school type will be accepted. Hence, reserve for students from public high schools will be satisfied.

**Proof of Proposition 5.** The contracts are substitutes for any program $p$ by Lemma 1. Therefore, as a corollary of Theorem 3 in (Hatfield and Milgrom 2005), SOSM produces a stable allocation for student preferences for a problem consists of $(x^s_t)_{s \in S}$ and $(C^MCF_p())_{p \in P}$. Moreover, as we showed in the proof of Proposition 1, the stable allocation SOSM produces is also stable for the problem consists of $(x^s_t)_{s \in S}$ and $(C^MCF_p())_{p \in P}$. Hence, for any problem, the outcome of SOSM is stable. ■

**Proof of Proposition 6.** Assume that is not true. So, we can find $x, y \in X'$ such that $y_p \succ^*_{x_S} x_p, z_{y_p}(ys) < z_{y_p}(x_S)$ and $y_T \in T_{x_S}$. Since we have $y_p \succ^*_{x_S} x_p$, there exist a contract $x'$ such that $x'_p = (x_S, y_p, t_{x_S})$ and $x' \succ^*_{x_S} x$. By the design of cumulative offer algorithm, $x'$ must be offered by $x_S$ and be rejected before the final step $K$. Therefore at step $K$, we have $y, x' \in A_{y_p}(K)$ and $X'_{y_p} = C^MCF_{y_p}(A_{y_p}(K))$. Since contracts are substitutes for each program and $x'$ is rejected before the final step $K$, $x' \notin C^MCF_{y_p}(A_{y_p}(K))$ must be true. By fairness

43
condition of choice function

\[ x' \notin C^{MCF}_{yp}(A_{yp}(K)) \implies z_{yp}(y_S) > z_{yp}(x'_S) \text{ or } y_T \notin T_{x_S} \]
a contradiction. Hence \( \psi^{SOSM} \) is fair.  

**Proof of Proposition 7.** For an arbitrary student \( s \), assume that \( \delta' = (t', (\succ'_{s})') \neq (t_s, \succ_{s}) \). Let her assigned program from \( \psi^{SOSM}(\delta', \delta_{-s}) \) be \( p^* \). Also let \( \delta'' \) be a strategy with demographic type \( t' \) and preference with only contract \((s, p^*, t')\) is acceptable. Since choice functions satisfies substitutes condition by Lemma 1 and Law of Aggregate Demand by Lemma 2, student \( s \) gets same assignment from \( \psi^{SOSM}(\delta'', \delta_{-s}) \). This part is a corollary of Theorem 10 in (Hatfield and Milgrom 2005).

Now, let \( \delta''' \) be a strategy with demographic type \( t_s \) and preference with only \((s, p^*, t_s)\) is acceptable. Due to incentive compatibility of choice function, her assignment from \( \psi^{SOSM}(\delta''', \delta_{-s}) \) must be \((s, p^*, t_s)\).

Finally, since for any given type profile choice function satisfies substitutes condition by Lemma 1 and Law of Aggregate Demand by Lemma 2, we know that students can not manipulate student proposing cumulative offer algorithm by submitting different preferences, i.e. \( \psi^{SOSM}((t_s, \succ_{s}), \delta_{-s}) \succeq_s \psi^{SOSM}(\delta''', \delta_{-s}) \), by Theorem 11 in (Hatfield and Milgrom 2005). So we have:

\[ \psi^{SOSM}((t_s, \succ_{s}), \delta_{-s}) \succeq_s \psi^{SOSM}(\delta''', \delta_{-s}) \succeq_s \psi^{SOSM}(\delta'', \delta_{-s}) \succeq_s \psi^{SOSM}(\delta', \delta_{-s}) \]

Therefore for any \( \delta' \),

\[ \psi^{SOSM}(\delta', \delta_{-s}) \not\succeq_s \psi^{SOSM}((t_s, \succ_{s}), \delta_{-s}) \]

Hence \( \psi^{SOSM} \) is strategy-proof.  ■