Strategic Investments, Trading, and Pricing Under Forecast Updating

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This paper considers two independent firms that invest in resources such as capacity or inventory based on imperfect market forecasts. As time progresses and new information becomes available, the firms update their forecasts and have the option to trade their resources. The trade contract is determined as the bargaining equilibrium or, alternatively, as the price equilibrium. Assuming a fairly general form of the profit functions, we characterize the Nash equilibrium investment levels, which are first-best under the price equilibrium trade contract, but not under the bargaining equilibrium trade contract. To gain additional insights, we then focus on firms that face stochastic demand functions with constant price elasticity and have contingent pricing power. Assuming a general forecast evolution process, we characterize the impact of the option to trade and the firms’ cooperation on equilibrium investments, expected prices, profits, and consumer surplus. Finally, to study the main driving forces of trading, we employ a well-established and empirically tested forecast updating model in which the forecast evolution process follows a two-dimensional geometric Brownian motion. Under this model, we prove that the equilibrium investments, expected prices, profits, and consumer surplus are nondecreasing in the quality and timing of forecast revisions, in market variability, and in foreign exchange volatility, but are nonincreasing in market correlation.

Key words: forecast updating; pricing; risk pooling; subcontracting; transshipment  
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1. Introduction

When making capacity, inventory, and production decisions, firms are typically uncertain about future market conditions such as demand, prices, and exchange rates. The opportunity cost associated with the quantity decisions made under uncertainty can often be mitigated by various resource pooling strategies where initial commitments need to be made only to an aggregate quantity, whereas the allocation among different products, markets, or firms is delayed until a more accurate market forecast becomes available. This paper studies resource pooling that can be achieved by two independent firms trading (subcontracting) capacity\(^1\) or transshipping inventory.\(^2\)

We consider two firms that invest in capacity or inventory in the face of uncertain market conditions. After the investments are made and as the production and selling season approaches, the firms update their forecasts of market conditions based on observed market signals such as market trends, sales of related products, etc. Given the revised forecasts, the firms have the option to trade the resource they invested in. Two alternative ways of determining the quantity and price of the traded resource are considered.\(^3\) The trade can be negotiated cooperatively in a bargaining

\(^1\) Trading or subcontracting capacity is widespread in industries such as telecommunications, pharmaceuticals, or electronics manufacturing with highly variable demand, high investment costs, and low production asset specificity. In the electronics industry, for example, a majority of outsourcing occurs between original equipment manufacturers (OEMs) that use excess capacity to produce for competitors (Plambeck and Taylor 2003).

\(^2\) Inventory trading is an equally widespread practice. According to an AMR Research press release (2000), the global market with excess consumer goods inventory was predicted to reach at least $120 billion in 2000.

\(^3\) Sometimes excess inventory of commodity-like goods is traded by a large number of firms in online auctions or exchanges where the price and quantity traded are dictated by the competitive equilibrium. However, because of the risk of brand dilution, selling to competitors, or purchasing defective or counterfeited goods, the online market for excess inventory plays only a marginal role. According to Pickering (2001), it represented less than 1% of the total excess inventory market. A majority of excess inventory deals are still negotiated privately between firms that know each other well. Similarly, capacity trading and subcontracting typically require face-to-face negotiations between the trading firms because of the inherent asset specificity.
game in which the traded quantity is chosen to maximize the firms’ expected joint profit and the expected benefits of the trade are split by the firms according to their bargaining powers. Alternatively, the firms can trade at the equilibrium price, i.e., the price at which one firm wants to sell the same amount of the resource that the other firm wants to buy. Although for any given resource levels the bargaining game and price equilibrium result in the same (jointly optimal) traded quantity, they lead to different transfer prices and, hence, create different incentives for the firms’ behavior in the investment stage.

Unlike trading decisions, resource investment decisions are typically made unilaterally. To reflect this, we assume that investment decisions result from a noncooperative game. We characterize the Nash equilibrium investment levels for a fairly general functional form of the operating profits and show that the equilibrium is unique when the trade results from the bargaining game, but there may be multiple equilibria when the firms trade at the equilibrium price. The Nash equilibrium investments maximize the firms’ joint profit under the price equilibrium trade contract, but not under the bargaining equilibrium trade contract. While our initial analysis is for generic stochastic profit functions, we develop further results by focusing on firms facing stochastic demand functions with constant price elasticity. Reflecting the relative flexibility of pricing decisions in many applications, we allow the firms to set prices in response to realized demand functions. Given this demand model and an arbitrary forecast evolution, we characterize the effect of trading and centralization on the Nash equilibrium for both types of trade contracts. Under the bargaining-equilibrium trade contract, the option to trade leads to higher investments, expected profits, and consumer surplus, and investment centralization further increases investments, expected profits, and consumer surplus but decreases expected prices. Under the price equilibrium trade contract, the option to trade leads to higher investments, expected profits, and consumer surplus but does not affect expected prices. Because in this case the Nash equilibrium investments are already first-best, investment centralization is inconsequential.

To facilitate further analysis, we assume that the firms’ forecasts follow a two-dimensional geometric Brownian motion. This assumption is supported by statistical studies of actual forecasting systems (Hausman 1969, Heath and Jackson 1994), and its theoretical rationale is derived from the theory of proportional effect. This theory suggests that the change in a forecast from one time period to the next is a random proportion of its current value (see, e.g., Aitchison and Brown 1957). Assuming this model of forecast evolution, we prove that under both types of trade contracts, equilibrium investments and the corresponding expected prices, profits and consumer surplus are (i) nondecreasing in the quality of the forecast revision, (ii) nondecreasing in market variability, (iii) nonincreasing in market correlation, and (iv) nondecreasing in exchange rate volatility considered in addition to market volatility in the case of global trading.

This paper contributes to the literature in three areas: (i) decentralized inventory transshipment and capacity subcontracting, (ii) general resource pooling under forecast updating, and (iii) global capacity management. Almost all the existing literature on decentralized transshipment and subcontracting studies the effect of various resource allocation mechanisms on the cost of decentralization. Among the most significant, Kouvelis and Gutierrez (1997) examine a two-market stochastic inventory system in which the excess inventory from the primary market can be transferred to the secondary market. Van Mieghem (1999) considers a subcontractor and a manufacturer who invest noncooperatively in capacity under demand uncertainty and have the opportunity to trade this capacity on demand revelation. Anupindi et al. (2001), Rudi et al. (2001), and Granot and Sošić (2003) study decentralized distribution systems with multiple retailers who face stochastic demands and have the option to transship inventory. In Lee and Whang (2002), multiple retailers order inventory to satisfy uncertain demand in two periods. After the first-period demand is realized, the retailers trade residual inventory in a secondary market.

All of the above literature assumes given product prices. As a result, the effect of resource pooling on the inventory levels typically depends on the newsvendor’s critical ratio, i.e., on the revenue and cost parameters (see, e.g., Lee and Whang 2002). Similarly, decentralization may lead to understocking or overstocking, depending on these parameters (see, e.g., Rudi et al. 2001). Our paper relaxes the assumption of given prices by reflecting the inverse relationship between price and demand. Resource trading is then driven by the price differential rather than by supply-demand imbalances. We show that in such a scenario, resource levels will be higher due to resource trading and lower due to decentralization.

The extant subcontracting and transshipment literature that considers product pricing is scarce. Plambeck and Taylor (2003) study two OEMs that invest in demand-stimulating innovations and production capacity under uncertain demand curves. Each OEM may invest in its own capacity to meet its own demand, the two OEMs may share capacity to maximize their joint profit, or the OEMs may outsource production to an independent contract manufacturer (CM). The supply contracts with the CM
are negotiated in a bargaining game. The scenario in which the OEMs share capacity to maximize joint profit differs from our model of capacity sharing in that the OEMs compete in innovation investments and cooperate in capacity acquisition and allocation, whereas our firms cooperate in capacity trade but compete in capacity acquisition. Dong and Durbin (2005) analyze secondary market trading among a large number of manufacturers, showing that it may reduce the supply chain efficiency by increasing the pricing power of the supplier. Both Plambeck and Taylor (2003) and Dong and Durbin (2005) assume linear demand curves that are subject to independent and additive random shocks, whereas we consider isoelastic demand functions subject to multiplicative multivariate uncertainty. The different demand models sometimes result in different insights, as we discuss later.

This paper makes four contributions to the literature on decentralized transshipment and subcontracting: (i) Relative to previous treatments, our model of resource trading is characterized by a high degree of generality. Important structural properties of the solution are proved under fairly generic profit functions and an arbitrary uncertainty evolution process. The effects of resource pooling and decentralization are also characterized for an arbitrary uncertainty evolution. (ii) By reflecting the relationship between price and demand, we are able to study expected prices and consumer surplus. (iii) Unlike the current literature, which assumes that the resource reallocation takes place after complete demand realization, we allow the resource trade to be based on improved but still imperfect demand forecasts. The more-realistic information dynamics enable us to examine the effect of forecast updating quality. (iv) Finally, while the comparative statics in most of the existing literature rely primarily on numerical experiments, our innovative modeling approach allows us to obtain most results analytically.

The operations literature on resource pooling under forecast updating is limited to centralized systems. Anand and Mendelson (1998) study the postponement of product differentiation under forecast revisions, considering a multiperiod, infinite-horizon model of a two-product firm that faces linear demand curves subject to binary random shocks. They use a numerical example to show that the value of postponement increases in demand variability and forecast revision accuracy and decreases in demand correlation; their results are consistent with this finding. Their model differs, however, in that profits decrease in demand variability due to the linearity of the demand curves. Furthermore, in the model of Anand and Mendelson (1998), forecast accuracy improves with market correlation because the signal from each market is used to refine the forecasts of both markets. This is not the case in our model of forecast evolution. Lee and Whang (1998) examine the value of postponement in a multiperiod model, distinguishing two sources of uncertainty reduction over time: uncertainty resolution and forecast improvement. We build on their argument documented with data from the Hewlett-Packard Company that the quality of the forecast for a future period improves as one moves closer to that period. Several other articles, including Eppen and Iyer (1997), Aviv and Federgruen (2001), and Petruzzi and Dada (2001), study different models of centralized resource pooling in which demand forecasts are updated in a Bayesian fashion.

Most of the operations literature addressing global capacity management under volatile exchange rates focuses on the impact of exchange rate volatility on the production cost assuming given prices and deterministic demand. For example, Kogut and Kulatilaka (1994), Huchzermeier and Cohen (1996), and Dasu and Li (1997) study the optimal policy and the value of shifting production between countries by a multinational corporation based on differentials in production costs that result from exchange rate volatility. The effect of exchange rate volatility on revenue as well as cost parameters in the case of global capacity trading is considered and examined numerically by Kouvelis and Gutierrez (1997). Our contribution to the global capacity management literature includes an analytical treatment of trading under exchange rate volatility superimposed on demand function uncertainty.

Finally, we must give credit to several methodologically related articles. Our model of a noncooperative investment game in which the payoffs depend on an embedded cooperative trading game is an example of a biform game introduced by Brandenburger and Stuart (2006). Our sensitivity analysis results rely on a forecast updating model initially proposed by Hausman (1969), who provides a theoretical rationale for the hypothesis that the ratios of successive forecasts are independent lognormal random variables. Hausman’s model was later extended by Hausman and Peterson (1972) to a multiproduct case and generalized by Heath and Jackson (1994) in the martingale model of forecast evolution (MMFE), which accommodates the simultaneous evolution of forecasts for demand in many time periods.4

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4 Heath and Jackson (1994) consider two variants of the MMFE. The multiplicative variant is a generalization of Hausman’s (1969) model of independent lognormally distributed forecast ratios. In the additive variant of MMFE, forecast changes due to new information are additive, which leads to independent and normally distributed forecast increments. Heath and Jackson (1994) report that the multiplicative (lognormal) variant of the model fits well with
The rest of this paper is organized as follows. In §2, we formulate the model, characterize the solution, and prove its existence and uniqueness for two alternative trade contracts and fairly general stochastic profit functions. In §3, we introduce a specific form of the profit functions by considering firms that face stochastic demand functions and have contingent pricing power. The impacts of trading and investment centralization are characterized in §3.1. Subsection 3.2 introduces a specific forecast updating model followed by the analysis of the effects of market variability, market correlation, and the quality of forecast revisions. In §4, we discuss some of the model limitations and extensions, including trading under exchange rate uncertainty. We conclude in §5. The appendix includes the proofs of Lemma 10 and Proposition 17. All remaining proofs can be found in the online appendix on the Management Science website at http://mansci.pubs.informs.org/eecompanion.html.

2. Game Formulation

We consider two firms that simultaneously, but independently, invest in resources while facing stochastic profit functions. We let \( q_i \) be the resource investment level of firm \( i \), which is acquired at a unit investment cost \( c \). Throughout the text, we use boldface letters to denote two-dimensional vectors of the corresponding variables, e.g., \( \mathbf{q} = (q_1, q_2) \). The uncertainty associated with the firms’ profit functions results from uncertain market conditions, such as demand or price levels. We use \( \xi_i \) to denote the random shock affecting the operating profit of firm \( i \). After the investment levels have been chosen, the firms receive information \( I \) and use it to revise their forecast of \( \xi \). The information arises from observing market trends, sales of related products, etc. We assume that the distribution of \((I, \xi)\) is common knowledge to both firms and, furthermore, that each firm observes the whole information vector \( I \), i.e., the firms always have the same information. We allow \((I, \xi)\) to have an arbitrary continuous probability distribution until §3.2, in which a specific forecast evolution process is considered. We let \( E \) denote the expectation operator with respect to the joint probability distribution of \((I, \xi)\), and we let \( E(\cdot | I) \) denote the expectation operator with respect to \((I, \xi)\) conditional on \( I \).

Based on the updated forecast of \( \xi \), the firms have the option to engage in a resource trade. We define the traded quantity \( Q \) as the amount of resource sold by Firm 1 to Firm 2 (i.e., if Firm 2 sells to Firm 1, then \( Q < 0 \)) and denote the vector of the firms’ resource levels after trading as \( \bar{\mathbf{q}} = (q_1 - Q, q_2 + Q) \). After the resource trade takes place, the remaining uncertainty is resolved and the firms realize their operating profits. We let \( \pi_i(\xi_i, \bar{\xi}) \) be the operating profit of firm \( i \), which depends on the realization of the random shock \( \xi_i \) and on the firm’s resource level \( \bar{\xi} \). We assume \( \pi \) to be three times differentiable with respect to the second argument, and we denote these first three derivatives as \( \pi_i'(\xi_i, \cdot), \pi_i''(\xi_i, \cdot), \) and \( \pi_i'''(\xi_i, \cdot) \), respectively. Note that the differentiability assumptions restrict the model to applications in which the firms always utilize all of their capacity. To guarantee the existence of the solutions, we assume that the profit is concave in resource level, i.e., \( \pi_i'(\xi_i, \cdot) < 0 \) a.s., \( i = 1, 2 \), and has decreasing curvature, i.e., \( \pi_i''(\xi_i, \cdot) \geq 0 \) a.s., \( i = 1, 2 \). Intuitively, the latter assumption means that the profit is “flattening out” i.e., the marginal profit decreases at a decreasing rate. For the sake of tractability, we assume that \( E[\pi_i'(\xi_i, 0) | I] \geq \pi_i'(\xi_i, \bar{q}_i) | I \) a.s., \( i, j = 1, 2, i \neq j \) for any equilibrium investment vector \( \mathbf{q} \). This assumption precludes the situation in which one firm trades all of its resource. Finally, to rule out the situation in which infinite resource investments are optimal, we assume that at a sufficiently high resource level, the marginal profit drops below the investment cost, i.e., \( \lim_{q_i \to \infty} \pi_i'(\xi_i, \bar{q}_i) < c \) a.s. for \( i = 1, 2 \). All of these assumptions are satisfied, for example, if the profits are of the form \( \pi_i(\xi_i, \bar{q}_i) = \xi_i \bar{q}_i^{a-1} \) with \( a \in (0, 1) \), which is the case that we focus on in §3. We further discuss the level of generality as well as the limitations of this model in §4.2. Next, we formulate and analyze the trading problem.

2.1. The Trading Phase

In this section, we present two alternative mechanisms of determining the quantity and price of the traded resource: first bargaining equilibrium, and then price equilibrium.

2.1.1. Bargaining Equilibrium. To execute a resource trade, the two firms must cooperate. A common way to model cooperative games is to assume that the players maximize their joint payoff and split the benefits of their cooperation according to their bargaining powers.\(^5\) In our context, this means that the firms trade the resource quantity that maximizes their expected joint profit conditional on information \( I \). We let

\[
Q(I, \mathbf{q}) = \arg \max_{Q} \sum_{i=1}^{2} E[\pi_i(\xi_i, \bar{q}_i) | I]
\]

be the optimal traded quantity, and let \( \bar{\mathbf{q}}(I, \mathbf{q}) \) be the vector of the firms’ resource levels given that this opti-

\(^5\) This solution concept was originally introduced by Nash (1950) and has become common in the operations literature (see, e.g., Van Mieghem 1999 or Plambeck and Taylor 2003).
such that the firms’ expected marginal operating profits are equal:

**Lemma 1.** The optimal traded quantity $Q(I, q)$ can be characterized as follows:

$$E[\pi'_1(\xi_1, q_1 - Q) | I] = E[\pi'_2(\xi_2, q_2 + Q) | I].$$  (2)

The price at which the resource is traded is defined implicitly so that the expected increase in the firms’ joint profit resulting from the trade is allocated between the firms in exogenously given proportions that depend on their relative bargaining powers. We let $S(I, q)$ be the expected increase in the firms’ joint profit resulting from the trade, given investments $q$ and information $I$, i.e.,

$$S(I, q) = \sum_{i=1}^2 E[\pi_i(\xi_i, q_i) | I] - \sum_{i=1}^2 E[\pi_i(\xi_i, q_i) | I].$$  (3)

Furthermore, we let $\theta_1 \in (0, 1)$ and $\theta_2 = 1 - \theta_1$ be the indices of the firms’ relative bargaining powers, defined as the proportions in which the firms split the expected benefits of the trade $S(I, q)$. The expected payoff of firm $i$, as a function of the initial investment levels, can then be written as

$$\Pi_i(q) = -cq_i + E[\pi_i(\xi_i, q_i)] + \theta_i S(I, q).$$  (4)

Before analyzing the firms’ investment decisions, we consider an alternative trading contract in which it is price equilibrium rather than bargaining equilibrium that determines the terms of the trade.

### 2.1.1. Price Equilibrium

**The optimal traded quantity** $Q(I, q)$ can be achieved by an integrated firm or by independent firms that cooperate in the same manner as the optimal traded quantity $Q^*$, the equilibrium price $P^*$, and the expected resource allocation $\bar{q}$. Consequently, we focus on the noncooperative investment scenario where each firm maximizes its own expected payoff, taking into account the investment decision of its potential trading partner as benchmarks for the noncooperative investment game with the option to trade, we consider two alternative scenarios: (i) no option to trade, and (ii) the centralized case. We let $q^*$ be the optimal resource investment vector of firms that do not have the option to trade, and let $q^*$ be the resource investment vector that maximizes the expected system profit $\Pi(q) = \Pi_1(q) + \Pi_2(q)$, given the option to trade. The investment levels $q^*$ can be achieved by an integrated firm or by independent firms that cooperate in the investment phase. Because the quantity traded $Q(I, q)$ is jointly optimal under both trade contracts, the resource capacities after trading $\bar{q}(I, q)$ are the same functions of the initial investment vector $q$ in all three investment-trading scenarios (bargaining equilibrium, price equilibrium, and the centralized case). However, the three scenarios lead to different initial resource investments, as Proposition 3 shows.

**Proposition 3.** Without the option to trade, the vector of optimal investment levels $q^*$ is characterized by the following set of necessary and sufficient conditions:

$$E[\pi'_i(\xi_i, q_i)] = c, \quad i = 1, 2.$$  (9)
With the option to trade, the vector of system-optimal investment levels $\mathbf{q}^*$ is characterized by the following set of necessary and sufficient conditions:

$$\mathbb{E}\pi_i(\xi_i, \tilde{q}_i(I, \mathbf{q})) = c, \quad i = 1, 2. \quad (10)$$

Under the bargaining equilibrium trade contract, there exists a unique vector of Nash equilibrium investment levels $\mathbf{q}^b$, which is characterized by the following set of necessary and sufficient conditions:

$$(1 - \theta_i)\mathbb{E}\pi_i(\xi_i, q_i) + \theta_i\mathbb{E}\pi_i(\xi_i, \tilde{q}_i(I, \mathbf{q})) = c, \quad i = 1, 2. \quad (11)$$

Under the price equilibrium trade contract, there exists a vector of Nash equilibrium investment levels $\mathbf{q}^p$, which is characterized by the following set of necessary and sufficient conditions:

$$\mathbb{E}\pi_i(\xi_i, \tilde{q}_i(I, \mathbf{q})) = c, \quad i = 1, 2. \quad (12)$$

In each scenario, the expected marginal payoff from each firm’s investment must be equal to the unit investment cost. Under the bargaining equilibrium contract, each firm receives all of the profit that it would have achieved without trading plus its share of the trade surplus that depends on the system profit. Consequently, in determining its investment level, each firm strikes a balance between maximizing its no-trade profit and the system profit. A firm with relatively high bargaining power expects to receive a larger portion of the trade surplus and, thus, puts more weight on maximizing the system profit. A firm with relatively low bargaining power, on the other hand, places more weight on its no-trade profit, for which it does not have to bargain. The expected marginal investment payoff of each firm (the left-hand side of (11)) is then the weighted average of this firm’s expected marginal investment payoffs in the case without trading (the left-hand side of (9)) and in the system profit maximization scenario (the left-hand side of (10)), with the weights being the bargaining powers $1 - \theta_i$ and $\theta_i$, respectively. As a result, when the trade contract results from the bargaining game, the Nash equilibrium investments deviate from the first-best solution, i.e., $\mathbf{q}^b \neq \mathbf{q}^*$. To better understand the firms’ incentives in the investment game under the price equilibrium trade contract, consider the effect of an infinitesimal increase in the resource investment of one of the firms. This firm will keep some of the additional resource to increase its own operating profit, and it will trade the rest at the equilibrium price, which reflects the marginal operating profit of the other firm. Thus, the expected increase in the firm’s own investment payoff, i.e., its operating profit plus the trade payment, is equal to the expected increase in the system profit, i.e., the sum of the operating profits. (The effect of decreasing the equilibrium price turns out to be zero in expectation because each firm can be a net buyer as well as a net seller.) In other words, the equilibrium price, unlike a price that depends on the firms’ bargaining powers, reflects the marginal value of the resource to the system, and investment decisions based on this price will therefore maximize the system profit, i.e., $\mathbf{q}^p = \mathbf{q}^*$. In §3, we consider a specific form of operating profits and analyze the effects of trading, investment centralization, asymmetry in bargaining power, quality of forecast revisions, market variability, and market correlation on investments, prices, profits, and consumer surplus.

3. An Illustration: Demand Curve Uncertainty

This section focuses on resource trading between two price-setting firms that make investment decisions in the face of uncertain demand curves. In contrast with quantity decisions, which are typically made before demand curves are known with certainty, pricing decisions can often be postponed until accurate information about demand is available. We reflect the relative flexibility of pricing by assuming that the firms set output prices in response to actual demand curves. To model the demand-price relationship, we adopt the isoelastic demand function, which is commonly used among both econometricians and marketing empiricists. Given the isoelastic demand function, the market clearing price at which firm $i$ can sell output quantity $y$ is

$$p_i(\xi_i, y) = \xi_i y^{-a},$$

where $\xi_i > 0$ is a random shock and $a \in (0, 1)$ is a constant. Parameter $a$ is the reciprocal value of the price elasticity of demand

$$\left|\frac{\partial y}{\partial p_i} \right|,$$

which is assumed to be the same in both markets. The effect of responsive pricing in inventory or capacity management is studied for a single-product firm by Van Mieghem and Dada (1999). In the risk-pooling literature, responsive pricing is implicitly assumed by, e.g., Fine and Freund (1990), and, more recently, by Plambeck and Taylor (2003) and Chod and Rudi (2005). Monahan et al. (2004) justify the isoelastic form of demand function by citing its consistency with consumer-utility-maximization theory and its typically good statistical fit with sales data. There is no natural way to incorporate cross-price elasticity of demand. Examples in which the cross-price elasticity would be negligible include firms selling in different geographical markets or different market segments or selling products that are sufficiently differentiated. In all of these cases, demand correlation will stem from external factors affecting both firms, such as market trends, macroeconomic factors, etc.
If we assume that the cost of transforming the resource into sales is negligible, the operating profit of firm \( i \) obtained by selling output \( y \) is \( y p_i(\xi_i, y) \). This operating profit is monotonically increasing in the output level \( y \) for any demand shock \( \xi_i \) and, hence, the firm will always use all of its resource, i.e., \( y = \tilde{y}_i \). Firm \( i \)'s operating profit is thus \( \pi_i(\xi_i, \tilde{y}_i) = \tilde{q}_i p_i(\xi_i, \tilde{y}_i) = \xi_i \tilde{q}_i^{1-a} \), which satisfies all of the assumptions imposed on \( \pi_i \) in \( \S 2 \).

In contrast to newsvendor-like models in which only demand is uncertain, or linear demand curves under which both price and demand are uncertain, only price is uncertain under isoelastic demand curves. As a result, it is the expected price differential rather than a mismatch between supply and demand that drives resource trading. Optimal trading always leads to the same expected prices in the two markets, i.e., \( \mathbb{E}[p_1(\xi_1, \tilde{y}_1)] = \mathbb{E}[p_2(\xi_2, \tilde{y}_2)] \). This is achieved by trading quantity

\[
Q(I, q) = \frac{\mathbb{E}^{1/a}(\xi_1 | I) q_1 - \mathbb{E}^{1/a}(\xi_1 | I) q_2}{\mathbb{E}^{1/a}(\xi_1 | I) + \mathbb{E}^{1/a}(\xi_2 | I)},
\]

where \( \mathbb{E}^{1/a}(\cdot) \) denotes \( \mathbb{E}(\cdot)^{1/a} \). After trading, the total resource will be allocated between the firms independently of the initial allocation, with

\[
\tilde{q}_i(I, q) = \mathbb{E}^{1/a}(\xi_i | I) (q_i + q_2) / \mathbb{E}^{1/a}(\xi_i | I) + \mathbb{E}^{1/a}(\xi_2 | I), \quad i = 1, 2.
\]

To characterize investment levels for the four investment-trading scenarios in Proposition 4, we define

\[
\|q\| = \mathbb{E} \left( \frac{\left( \mathbb{E}^{1/a}(\xi_1 | I) + \mathbb{E}^{1/a}(\xi_2 | I) \right)^{1/a} - 2}{2} \right).
\]

**Proposition 4.** Without the option to trade, the optimal investment levels are

\[
q_i^0 = \left( \mathbb{E} \xi_i \frac{1-a}{c} \right)^{1/a}, \quad i = 1, 2.
\]

With the option to trade, the vector of system-optimal investment levels \( q^* \) is characterized by the following necessary and sufficient condition:

\[
q_1 + q_2 = 2 \left( \|q\| \frac{1-a}{c} \right)^{1/a}.
\]

Under the bargaining equilibrium trade contract, the vector of Nash equilibrium investment levels \( q^* \) is characterized by the following set of necessary and sufficient conditions:

\[
(1 - \theta)(1-a)\mathbb{E}\xi_i q_i + \theta(1-a)\|q\| \left( \frac{q_1 + q_2}{2} \right)^{1-a} = \mathbb{E} p_i(\xi_i, q_i),
\]

\[
\Pi(q^*) = -c(q_1^* + q_2^*) + 2\|q\| \left( \frac{q_1^* + q_2^*}{2} \right)^{1-a},
\]

\[
\mathbb{E} p_i(\xi_i, \tilde{q}_i(I, q^*)) = \frac{c}{1-a},
\]

\[
C(q^*) = \frac{2a}{1-a} \|q\| \left( \frac{q_1^* + q_2^*}{2} \right)^{1-a}.
\]

\[a\] Thus, this illustration captures the best trading inventory of finished goods, even though most of the insights apply to capacity trading as well.

Note from (16) that only the total resource level, not its allocation between the firms, affects the system profit. Similarly, there is a continuum of Nash equilibria and only the total investment can be determined uniquely, when the firms trade at the equilibrium price. Consequently, in the centralized case as well as in the case of the price equilibrium trade contract, we can study resource investments and profits only at the system level.

Before starting a rigorous analysis of the various investment-trading scenarios, it is worthwhile to consider the special case of symmetric mean demand shocks and bargaining powers, which allows us to derive very intuitive insights into the effects of trading and investment centralization on investments, prices, profits, and consumer surplus. Consumer surplus measures, in monetary terms, the net utility gained by the customers purchasing the firms’ output. It is defined as the difference between the price a customer is willing to pay and the price she actually pays, integrated over all customers purchasing the product. Formally, the expected consumer surplus with and without trading can be expressed as

\[
C(q) = \mathbb{E} \sum_{i=1,2} \int_0^{\tilde{q}_i} p_i(\xi_i, x) \, dx - \tilde{q}_i p_i(\xi_i, \tilde{q}_i)
\]

and

\[
C^0(q) = \mathbb{E} \sum_{i=1,2} \int_0^{q_i} p_i(\xi_i, x) \, dx - q_i p_i(\xi_i, q_i),
\]

respectively.

**Corollary 5.** Let \( \theta_1 = \theta_2 = 1/2 \) and \( \mathbb{E}\xi_i = \mathbb{E}\xi_i \). Without the option to trade, the optimal investment levels and the corresponding expected profits, prices, and consumer surplus are

\[
q_i^0 = \left( \frac{1-a}{c} \mathbb{E} \xi_i \right)^{1/a}, \quad \Pi_i(q_i^0) = -c q_i^0 + \mathbb{E} \xi_i (q_i^0)^{1-a},
\]

\[
\mathbb{E} p_i(\xi_i, q_i^0) = \frac{c}{1-a}, \quad C^0(q_i^0) = \frac{2a}{1-a} \mathbb{E} \xi_i (q_i^0)^{1-a}.
\]

With the option to trade, the total system-optimal investment level and the corresponding expected system profit, prices, and consumer surplus are

\[
q_i^0 + q_2^* = 2 \left( \frac{1-a}{c} \|q\| \right)^{1/a},
\]

\[
\Pi(q^*) = -c(q_1^* + q_2^*) + 2\|q\| \left( \frac{q_1^* + q_2^*}{2} \right)^{1-a},
\]

\[
\mathbb{E} p_i(\xi_i, \tilde{q}_i(I, q^*)) = \frac{c}{1-a},
\]

\[
C(q^*) = \frac{2a}{1-a} \|q\| \left( \frac{q_1^* + q_2^*}{2} \right)^{1-a}.
\]
Under the bargaining equilibrium trade contract, the unique Nash equilibrium investment levels and the corresponding expected profits, prices, and consumer surplus are

\[ q^*_i = \left( \frac{1 - a}{c} \cdot \frac{E_x + \|\xi\|}{2} \right)^{\frac{1}{2}} \]

\[ \Pi_i(q^*_i) = -c q^*_i + \|\xi\|(q^*_i)^{1-a} \]

\[ E_p(i, q^*_i) = \frac{c}{1 - a} \frac{2\|\xi\|}{E_x + \|\xi\|} \]

\[ C(q^*_i) = 2a \|\xi\|(q^*_i)^{1-a} \]

Under the price equilibrium trade contract, the total of the Nash equilibrium investment levels and the corresponding expected system profit, prices, and consumer surplus are characterized by (19).

When the firms cannot trade, their optimal investments, profits, and consumer surplus depend only on the expectation of the demand shocks \(E_x\). When the firms have the option to trade, their first-best investments, profits, and consumer surplus depend also on factors such as market variability, market correlation, and the quality of forecast revisions, which are all captured in parameter \(\|\xi\|\). Finally, under the bargaining equilibrium trade contract, the Nash equilibrium investments, profits, and consumer surplus are between the no-trading and the first-best trading cases. The relative performances of the four investment-trading scenarios therefore depend on the relationship between parameter \(\|\xi\|\) and vector \(E_x\). We shed some light on this relationship in Lemma 6, which is valid for any distribution of \((I, \xi)\) such that \(\xi \geq 0\).

**Lemma 6.** If \(\xi \geq 0\) a.s.,

\[ \|\xi\|^{1/a} \geq \frac{E^{1/a}_{\xi_1} + E^{1/a}_{\xi_2}}{2} \]

With Lemma 6 at hand, we can proceed to the formal analysis of the impact of trading and investment centralization for an arbitrary forecast evolution process, i.e., an arbitrary distribution of \((I, \xi)\) and general (asymmetric) problem parameters.

### 3.1 Effects of the Option to Trade and Investment Centralization

The existing subcontracting and transshipment literature focuses on situations in which one firm’s excess supply is used to satisfy another’s unmet demand. In such cases, resource pooling reduces overall uncertainty, inducing the firms to set their total resource level closer to the mean demand. Doing so may translate into increasing or decreasing the resource level, depending on the newsvendor’s fractile, i.e., on the cost and revenue parameters.\(^{10}\) Our model, on the other hand, applies to situations in which it is the price differential that drives trading, i.e., the firms always use all of their total resource level, and trading affects only the prices that they are able to charge. This has critical implications for how trading affects optimal resource levels.

**Proposition 7.** Under the bargaining equilibrium trade contract, the option to trade leads to higher investments, expected prices, profits, and consumer surplus.

Under the price equilibrium trade contract, the option to trade leads to higher total investment, expected system profit, and consumer surplus, but it has no effect on expected prices.

The firms engage in a trade to increase the expected average price of their outputs. Higher expected prices lead to a higher marginal profitability of the resource and induce the firms to invest more. The higher initial investments then result in a higher total output, which moderates—or, in the case of the price equilibrium contract, completely offsets—the increase in expected prices. To be able to sell a given output at a higher average price, the firms must transfer the resource to the market in which consumers value the product more. This and the fact that the option to trade induces larger investments and, hence, larger output, are the reasons why the option to trade increases the expected consumer surplus. Because the no-trading scenario is equivalent to a scenario without forecast revisions, the effect of the option to trade can be alternatively interpreted as the effect of forecast revisions. In particular, the value of the option to trade can be interpreted as the value of information \(I\). In the next proposition, we characterize the implications of centralization in the investment stage.

**Proposition 8.** Under the bargaining equilibrium trade contract, the total investment, expected system profit, and consumer surplus are lower, whereas the expected prices are higher than in the centralized case.

\(^{10}\)For example, Lee and Whang (2002) show that the secondary market may increase or decrease the optimal quantities, depending on the retailers’ critical fractile. If this fractile is small, the order quantities will also be small, which means low leftover inventory after the first-period demand. It will then be expensive to buy leftover inventory in the secondary market, inducing the retailers initially to order more than the newsvendor’s quantity, and vice versa, for a large critical fractile. Similarly, Plambeck and Taylor (2003) show, assuming linear demand curves, that pooling may either increase or decrease the level of capacity depending on whether the optimal capacity level is small or large, i.e., whether pooling increases or decreases the probability that all capacity will be used. If the optimal capacity is small, pooling increases the probability that all of the capacity will be used, leading to a higher marginal revenue from capacity and a higher optimal capacity level. The reverse is true if the optimal capacity is large.
Under the price equilibrium trade contract, the total investment, expected prices, system profit, and consumer surplus are the same as in the centralized case.

We first consider the case of the bargaining equilibrium contract. Proposition 3 established that the option to trade makes firms move from the no-trading case toward the first-best investment-trading case. According to Proposition 7, this means an increase in investment levels. The first-best investments thus represent an upper bound for the Nash equilibrium investments, i.e., decentralization leads to underinvestment. A lower total investment then results in a lower total output, which in turn leads to higher prices and lower consumer surplus. In other words, investment centralization benefits the firms as well as consumers. (Recall that the firms are monopolies in their respective output markets and, thus, centralization increases efficiency without affecting competition.) As for the case of trading at the equilibrium price, the result follows directly from Proposition 3. We end this section by investigating the effect of asymmetry in bargaining power.11

**Proposition 9.** Under the bargaining equilibrium trade contract, the Nash equilibrium investment level of each firm is increasing in this firm’s bargaining power. If, furthermore, \( E\xi_1 = E\xi_2 \), more asymmetry in bargaining powers leads to higher total investment, expected system profit, and consumer surplus, and to lower expected prices.

The more a firm benefits from trading, the more profitable its marginal investment and the higher its investment level. To obtain intuition for the remaining results, consider the extreme situation in which Firm 1 has no bargaining power, i.e., \( \theta = (0, 1) \). Because trading does not affect this firm’s expected profit, it will not affect its investment either—i.e., \( q^*_1 = q^1_1 \). Firm 2 expects, on the other hand, to receive the whole system profit except for the other firm’s reservation (no-trading) profit. This firm will therefore choose the investment level that maximizes the expected system profit, bringing the total investment, \( q^1_1 + q^2_2 \), to its upper bound, \( q^*_1 + q^*2_2 \), i.e., \( q^2_2 = q^*_1 + q^*2_2 - q^1_1 \). In general, as the asymmetry in bargaining powers increases, the Nash equilibrium moves toward the first-best, and the total investment, expected system profit, and consumer surplus increase while expected prices decrease.

### 3.2. Effects of Market Variability, Correlation, and the Quality of Forecast Revision

To analyze the effects of the forecast revision quality, market variability, and market correlation, we apply the forecast updating framework proposed by Hausman (1969). The theoretical rationale of the model is as follows: Let \( \xi \) be a forecast of demand shock \( \xi \) made at time \( t \). If forecast revisions result from a large number of unpredictable information signals, according to the theory of proportional effect (Aitchison and Brown 1957) the change in a forecast from one time period to the next is a random proportion of its current value, i.e., \( \xi_{t+1} - \xi_t = \epsilon \xi_t \), where \( \epsilon \) is a random variable. Therefore, \( \xi_{t+1} = [\prod_{i=0}^{t}(1 + \epsilon_i)]\xi_0 \) and, taking the natural logarithm, \( \ln \xi_{t+1} = \sum_{i=0}^{t} \ln(1 + \epsilon_i) + \ln \xi_0 \). If \( \{\epsilon_i\} \) is a sequence of independent random variables with finite variances, then as \( t \) approaches infinity, \( \sum_{i=0}^{t} \ln(1 + \epsilon_i) \) will be asymptotically normal by the central limit theorem. Hence, \( \xi_{t+1}/\xi_0 \) will be asymptotically lognormal. Furthermore, if ratios of successive forecasts \( \xi_{t+1}/\xi_0 \) and \( \xi_{t+2}/\xi_{t+1} \) were independent, then the forecast change from \( \xi_0 \) to \( \xi_{t+1} \) would contain some information about the probable change from \( \xi_{t+1} \) to \( \xi_{t+2} \), and the forecast \( \xi_{t+1} \) could be improved on the basis of this information. This provides the rationale for the hypothesis that ratios of successive forecasts are independent lognormal random variables, which is equivalent to the forecast evolution process following a geometric Brownian motion.

Let \( X = [X(t), t \geq 0] \) be a symmetric two-dimensional Brownian motion, i.e., \( X(t) \sim N(0, \Sigma) \), where \( \Sigma = \sigma^2 \) and \( \Sigma = \rho \sigma^2 \) if \( i \neq j \). The assumption of symmetric volatilities is made for the sake of simplicity and results in the same relative error (coefficient of variation) in the firms’ forecasts. We let the demand shock \( \xi \) be a forecast of \( \mu, X_i(1) \) for some \( \mu_i, i = 1, 2 \). Finally, we let the information vector \( I = X(\lambda) \) for some \( \lambda \in [0, 1] \). Therefore, the firms’ profits depend on the state of some underlying process \( X \) at time 1, while the information \( I \) indicates the state of this process at time \( \lambda \leq 1 \). If we assume that the investment phase takes place at time 0, we have

\[
\ln \xi \sim N(\mu, \Sigma) \quad \text{and} \quad \ln \xi | I \sim N(\mu + I(1 - \lambda)\Sigma).
\]

Therefore, observing information \( I \) leads to a shift in the mean log-forecast by \( I \) and a reduction of its covariance matrix by fraction \( \lambda \).12 The parameter \( \lambda \) thus measures the quality of forecast revision. In the extreme case of \( \lambda = 0 \), the firms do not receive any information that they could use to trade. In the other extreme case of \( \lambda = 1 \), information \( I \) eliminates all uncertainty about \( \xi \).

An alternative interpretation of parameter \( \lambda \) is that it is the timing of the trade between the investment

---

11 In his seminal work on the bargaining problem, Nash (1950) assumes symmetric bargaining powers.

12 Recall that each of the firms observes the whole information signal \( I = (I, I) \). However, although the markets are correlated, \( I \) does not contain any information about \( \xi \), \( i \neq j \), that would not be contained in \( I \), i.e., \( \xi | I = \xi | I, \ i = 1, 2 \).
stage (time 0) and the selling season (time 1). A higher value of $\lambda$ means that the trade occurs closer to the selling season and, thus, the firms face less uncertainty at the time of trading. Therefore, low values of $\lambda$ will be typical of capacity trading that occurs long before the selling season, i.e., under high uncertainty about future market conditions. Compared to capacity, inventory is typically traded closer to the selling season and, consequently, under relatively less uncertainty. Therefore, inventory trading will typically be characterized by higher values of $\lambda$. Finally, if the trade responds to the actual price differential between the two markets, as in the case of electricity trading, the parameter $\lambda$ will be close to 1.

The updated forecast parameters are

$$E(\xi_i | I) = \exp(\mu_i + I_i + \frac{1}{2}(1 - \lambda)\sigma^2)$$

and

$$\sqrt{\text{var}(\xi_i | I)} = E(\xi_i | I) \frac{\exp((1 - \lambda)\sigma^2) - 1}{\sqrt{\text{exp}(1 - \lambda)\sigma^2}} \text{ for } i = 1, 2.$$  

Note that not only the updated mean but also the updated standard deviation of $\xi_i$ depends on information $I$. In particular, the larger the mean updated forecast $E(\xi_i | I)$, the larger its standard deviation $\sqrt{\text{var}(\xi_i | I)}$, with the updated coefficient of variation being unaffected by information $I$.

Note also that while being independent of information $I$, the updated coefficient of variation of $\xi_i$ is decreasing in the quality of forecast revision $\lambda$. The quality of forecast revisions, together with the market variability and market correlation, is pivotal in determining the level and profitability of resource trading. To formally analyze the effects of these three drivers of trading on equilibrium investments, prices, profits, and consumer surplus, we make use of the following result.

**Lemma 10.** Parameter $\|\xi\|$ is increasing in the quality of forecast revisions and in the market variability, and is decreasing in the market correlation.

We start our sensitivity analysis with the case of the bargaining equilibrium trade contract. In the following, low correlation means low value, not low magnitude, of the correlation coefficient.

**Proposition 11.** Under the bargaining equilibrium trade contract, the Nash equilibrium investment levels, expected prices, profits, and consumer surplus are increasing in the quality of forecast revision and in the market variability, and are decreasing in the market correlation.

13 According to Heath and Jackson (1994), this makes the multiplicative MMFE (lognormal forecast) more appropriate than the additive MMFE (normal forecast). This conclusion is supported by their empirical observation that the standard deviation of the forecast is roughly proportional to the size of the forecast, which is consistent with the multiplicative but not the additive MMFE.

The higher the quality of forecast revisions, the higher the profitability of trading. A higher market variability means more uncertainty in the investment phase and, hence, a higher opportunity cost of under- or overinvestment. At the same time, market variability increases the option value of responsive pricing, which, with isoelastic demand curves, exactly offsets the expected opportunity cost. As a result, without the option to trade, market variability has no impact on either expected profits or expected marginal profits and, hence, on optimal investments, expected prices, and consumer surplus. Similarly, when the firms do not trade, they are not affected by the market correlation.

However, a higher market variability and a lower market correlation result in a higher expected price differential between the markets, increasing the profitability of resource trading. While high market variability and low market correlation create potential trading opportunities, a high quality of forecast revisions allows the firms to take advantage of these opportunities. In other words, market variability increases the value of forecast updating, i.e., the more inherently uncertain the markets are, the more can be learned from the information signal.

By increasing the profitability of trading, a higher quality of forecast revisions, a higher market variability, and a lower market correlation improve profitability as well as marginal profitability of the firms’ investments. This induces higher investment levels and ultimately results in a higher total output and a higher expected consumer surplus. 14 Because more profitable trading results in selling a given output at a higher average price, then higher forecast revision quality, higher market variability, and lower market correlation also lead to higher expected prices, although the price increase is mitigated by the second-order effect of higher investments. In the case of the price equilibrium trade contract, the comparative statics results are similar except for the effects on prices.

**Proposition 12.** Under the price equilibrium trade contract, the total Nash equilibrium investment level, expected system profit, and consumer surplus are increasing in the quality of forecast revision and in the market variability, and are decreasing in the market correlation. The corresponding expected prices are invariant under each of these parameters.

14 Our result that a better quality of forecast revisions stimulates higher investments and output is in the spirit of Anand (1999). Anand studies the effect of information quality on sales and inventory levels in a multiperiod, infinite-horizon model with generic revenue functions and nonparametric information system (IS). In each period, after the production decision is made, the IS generates a signal to which the firm responds by deciding the output quantity brought to the market. Anand proves that more informative IS results in a higher production build-up-to level.
Recall that when the firms trade at the equilibrium price, the option to trade has no effect on expected prices. As a result, the expected prices are not affected by any of the three parameters.

4. Extensions and Limitations
We use the last section of this paper to discuss some extensions and limitations of our model. We first extend our model by considering the effect of exchange rate uncertainty in the case of global trading.

4.1. Effect of Exchange Rate Volatility
This paper considers trading driven by the price differential. Clearly, if the trading firms realize their revenues in different currencies, there may be an additional source of uncertainty affecting the real price differential and, hence, the value of the trading option: an uncertain exchange rate. We consider two firms that operate in two different currency zones and face exchange rate uncertainty in addition to demand curve uncertainty. We define exchange rate $\psi(t)$ as the cost of one unit of currency 1, expressed in currency 2, at time $t$. As is traditional, we assume that the exchange rate process $\psi(t)$ follows a geometric Brownian motion and, furthermore, is independent of $(I, \xi)$.\(^{15}\) If we assume the same nominal interest rates in the two countries, rational expectations, risk neutrality of speculators, and efficient financial markets, the uncovered interest rate parity implies equality between the expected future and the spot exchange rates. To avoid Siegel’s paradox (see, e.g., Baillie and McMahon 1989), we assume that this equality holds in the logarithmic form, i.e., the Brownian motion underlying the exchange rate process has no drift. Therefore, we can write $\psi(t) = \exp(\bar{Y}(t))$, where $\bar{Y}(t) \sim N(\bar{\mu}, t\bar{\sigma}^2)$, $\bar{\mu} = \ln(\psi(0))$, and $\bar{\sigma}$ is a measure of the exchange rate volatility.

The sequence of events is as follows: The investment levels are decided at time 0, the trade takes place at time $\tau \in [0, 1]$, and the sales are realized at time 1. The closer to the selling season the trade occurs, the better forecast of the selling season the firms face at the time of trading. Therefore, parameter $\tau$ has an effect similar to that of parameter $\lambda$ on the residual uncertainty at the time of trading, except that the quality of forecast revision $\lambda$ has a more general interpretation than the timing of the trade $\tau$. We also assume that the firms maximize their expected profits denominated in currency 1 and face the same real investment costs, i.e., their unit nominal investment costs are $c$ and $c\psi(0)$, respectively. Finally, we keep all of the assumptions that have been made up to §3.2, inclusive. We characterize the solution to this extension in Lemma 13.

**Lemma 13.** Under exchange rate uncertainty, all results in Proposition 4 continue to hold with $\xi_2$ replaced by $\xi_2/\psi(1)$ and $I$ replaced by $1/\psi(\tau)$.

The closer to the actual selling season the trade takes place, the better forecast of the selling season exchange rate is available, and the more efficient, in expectation, the trade will be. The effect of the timing of the trade is therefore very similar to the effect of the quality of forecast revisions. At the same time, exchange rate volatility has a similar effect to that of demand curve variability. Without the option to trade, it has no effect on the firms. However, by increasing the expected real price differential between the markets, exchange rate volatility increases the value of the option to trade. We formalize these results in Proposition 14.

**Proposition 14.** Under exchange rate uncertainty and either type of trade contract, the Nash equilibrium investments and expected profits are increasing in the timing of the trade and in the exchange rate volatility.

Note that under the price equilibrium trade contract, the monotonicity results from Proposition 14 relate only to the total investment and the expected system profit. In the next subsection, we discuss some limitations of the generic profit functions introduced at the beginning of §2.

4.2. Generality of Operating Profit Functions $\pi$
The generality of the operating profit functions $\pi_i$, $i = 1, 2$, is somewhat limited by the differentiability assumptions that we made to facilitate analytical tractability. Note that a firm’s operating profit will be smooth in a resource capacity only if the firm always utilizes all of this resource. Therefore, our model will not apply to newsvendor-like settings in which resource utilization is constrained by the realized demand, which creates a kink in the operating profit function. On the other hand, when the firms have contingent pricing power, the applicability of the generic profit functions is not restricted to the case of isoelastic demand curves. The generic profit functions apply, for example, to the case of linear demand curves, provided that demand shocks do not take extreme values. Let $p_i(\xi_i, y) = \xi_i - y$ be the inverse demand function faced by firm $i$, $i = 1, 2$. The firms’

\(^{15}\) Despite stochastic independence between the demand shocks and the exchange rate, the output prices and the exchange rate are linked through the trading mechanism. In particular, depreciation of currency $i$ with respect to currency $j$ stimulates the export of the resource from market $i$ to market $j$, thus increasing the price in market $i$. 
operating profits
\[
\max_{y \in \mathcal{Y}} y p_i(\xi_i, y), \quad i = 1, 2,
\]
fit the generic profit functions as long as the state space of \( \xi \) is bounded, so that (i) it is always optimal for each firm to utilize all of its resource, and (ii) it is never optimal for either firm to trade all of its resource. Formally, condition (i) means that
\[
2\xi_i - E(\xi_i | I) + E(\xi_i | I) > E\xi_1 + E\xi_2 - 2c
\]
as.f. for \( i, j = 1, 2 \) and \( i \neq j \), while condition (ii) means that
\[
|E(\xi_1 | I) - E(\xi_2 | I)| < E\xi_1 + E\xi_2 - 2c \quad \text{a.s.}
\]
To get some intuition for how restrictive these conditions are, consider the case of perfect information updating, i.e., \( E(\xi_i | I) = \xi_i \) a.s., and assume that the first (marginal) unit of each product can be sold, in expectation, at a 50% markup, i.e., \( E p_1(\xi_1, 0) = E p_1(\xi_2, 0) = 2c \). In this case, conditions (i) and (ii) are satisfied, i.e., the generic profit functions fit the model, if \( \xi_i \in (0.5E\xi_i, 1.5E\xi_i) \) a.s., \( i = 1, 2 \), which allows significant demand variability. In conclusion, the generality of the operating profit functions is limited by the requirement that the firms always fully utilize their resources, rather than by the particular shape of demand functions.

Another implicit yet important assumption made throughout this paper is that trading is frictionless. We use the remainder of this section to discuss some implications of possible trading frictions.

### 4.3. Effect of Trading Frictions

In this section, we consider two types of trading frictions: transaction cost and resource specificity. Let \( t \) be a unit transaction cost that can capture, e.g., a unit transportation cost or a quantity-based customs duty. The total transaction cost of trading resource quantity \( Q \) is then \( t|Q| \). To reflect the fact that the firms’ resources may not be perfect substitutes, we assume that buying quantity \( |Q| \) of the other firm’s resource increases the firm’s own resource level by \( \alpha|Q| \), where \( \alpha \in [0, 1] \), i.e.,
\[
\bar{q} = \begin{cases} 
(q_1 - \alpha Q, q_2 + Q) & \text{if } \begin{cases} p_1(\xi_1, q_1) & |I| \geq p_2(\xi_2, q_2) & |I|, \\
q_1 - Q, q_2 + \alpha Q & \text{otherwise}.
\end{cases}
\end{cases}
\]

The parameter \( \alpha \) thus represents the yield from the traded resource, which is assumed to be the same for both firms. A higher resource specificity will presumably result in a lower value of \( \alpha \). We also relax the assumption of equal investment costs for the two firms. To prevent cost arbitrage, it is only required that \( t \geq \alpha c_i - c_j \) for \( i \neq j \). We next incorporate the two trading frictions into the general formulation of §2, starting with the bargaining equilibrium trade contract.

#### 4.3.1. Bargaining Equilibrium

The traded quantity that maximizes the firms’ expected joint profit conditional on information \( I \) is
\[
Q(I, q) = \arg \max_Q \left[ \sum_{i=1}^2 E\{\pi_i(\xi_i, q_i) | I | - t|Q| \} \right].
\]
Before formally characterizing the optimal traded quantity in Lemma 15, we partition the state space of information \( I \) into three events: \( \Omega_-(q), \Omega_+(q), \) and \( \Omega_0(q) \), which result in a negative-, positive-, and zero-traded quantity, respectively. Let
\[
\Omega_-(q) \equiv \{ y : \alpha E(\pi_1(\xi_1, q_1) | I = y) - E(\pi_2(\xi_2, q_2) | I = y) > t \},
\]
\[
\Omega_+(q) \equiv \{ y : \alpha E(\pi_1(\xi_2, q_2) | I = y) - E(\pi_1(\xi_1, q_1) | I = y) > t \}, \quad \text{and}
\]
\[
\Omega_0(q) \equiv \Omega \setminus (\Omega_-(q) \cup \Omega_+(q)).
\]

**Lemma 15.** The optimal traded quantity \( Q(I, q) \) can be characterized as follows:
\[
\alpha \begin{cases} 
E\{\pi_1(\xi_1, q_1 - \alpha Q) | I \} & \text{if } I \in \Omega_-(q), \\
E(\pi_2(\xi_2, q_2 + Q) | I |) & \text{if } I \in \Omega_+(q), \\
E\{\pi_1(\xi_1, q_1 - Q) | I \} & \text{if } I \in \Omega_0(q),
\end{cases}
\]

According to Lemma 15, the firms trade as long as the buyer’s expected marginal operating profit from the acquired resource exceeds the seller’s expected marginal operating profit by more than the transaction cost \( t \). The effect of the trading frictions on the regions \( \Omega_-, \Omega_+, \) and \( \Omega_0 \) for a given investment vector \( q \) is illustrated in Figure 1. The figure shows the case in which the operating profits are exposed to multiplicative random shocks such as the price shocks considered in §3. In Figure 1(a), \( \alpha = 1 \) and \( t \) varies; whereas in Figure 1(b), \( t = 0 \) and \( \alpha \) varies. Clearly, as the transaction cost or the resource specificity increases, the region with no trading becomes larger. Note, however, the difference in the effect of \( t \) and \( \alpha \). If \( \alpha < 1 \) (Figure 1(b)), the difference between \( E(\xi_1 | I) \) and \( E(\xi_2 | I) \) justifying a trade is larger for large values of \( E(\xi | I) \). Intuitively, when higher output prices are expected, losing resource yield due to trading results in a higher expected opportunity cost. Thus, a larger
expected price differential between the two markets is necessary to justify a trade. The expected total value created by the trade, given investment levels \( \mathbf{q} \) and information \( \mathbf{I} \), is 

\[
S(\mathbf{I}, \mathbf{q}) = \sum_{i=1}^{2} \mathbb{E}[\pi_i(\xi_i, \tilde{q}_i(\mathbf{I}, \mathbf{q})) | \mathbf{I}] - t|Q(\mathbf{I}, \mathbf{q})|
\]

so the expected investment payoff of firm \( i \) can be written as 

\[
\Pi_i(\mathbf{q}) = -c_i q_i + \mathbb{E}[\pi_i(\xi_i, \tilde{q}_i(\mathbf{I}, \mathbf{q}))] + \theta_i S(\mathbf{I}, \mathbf{q}).
\]  

We next consider the price equilibrium contract.

4.3.2. Price Equilibrium. Let \( P \) be the price charged per unit of the sold resource, i.e., Firm 1 always receives from Firm 2 the payment of \( Q \). We assume, without any loss of generality, that each firm incurs half of the transaction cost \( t \). Given resource price \( P \), information \( \mathbf{I} \), and resource levels \( \mathbf{q} \), the traded quantity that maximizes the expected profit of firm \( i \) is 

\[
Q_i(P, \mathbf{I}, \mathbf{q}) = \arg \max_{Q} \mathbb{E}[\pi_i(\xi_i, \tilde{q}_i(\mathbf{I}, \mathbf{q})) | \mathbf{I}] - (1)^{3/2}Q - \frac{1}{2}t|Q|.
\]  

The equilibrium price and traded quantity can be characterized as follows:

**Lemma 16.** The equilibrium price is 

\[
P^* = \begin{cases} 
\alpha \mathbb{E}[\pi_i(\xi_i, \tilde{q}_i(\mathbf{I}, \mathbf{q})) | \mathbf{I}] - \frac{1}{2}t & \text{if } \mathbf{I} \in \Omega_-(\mathbf{q}), \\
\alpha \mathbb{E}[\pi_i(\xi_i, \tilde{q}_i(\mathbf{I}, \mathbf{q})) | \mathbf{I}] - \frac{1}{2}t & \text{if } \mathbf{I} \in \Omega_+(\mathbf{q}), \\
\end{cases}
\]

and \( P^* \) does not exist if \( \mathbf{I} \in \Omega_0(\mathbf{q}) \). The price equilibrium traded quantity is \( Q^* = Q(\mathbf{I}, \mathbf{q}) \) as characterized in (22).

Thus, for given investment levels, the price equilibrium traded quantity \( Q^* \) is again equal to the optimal quantity \( Q(\mathbf{I}, \mathbf{q}) \) that results from the bargaining game. Finally, the expected investment payoff of firm \( i \) can be written as 

\[
\Pi_i(\mathbf{q}) = -c_i q_i + \mathbb{E}[\pi_i(\xi_i, \tilde{q}_i(\mathbf{I}, \mathbf{q})) - (1)^{3/2}Q^* - \frac{1}{2}t|Q^*|].
\]  

We are now ready to characterize the investment decisions.

4.3.3. Investment Phase. Proposition 17 characterizes the investment levels for each of the three investment-trading scenarios.

**Proposition 17.** With the option to trade, the vector of system-optimal investment levels \( \mathbf{q}^* \) is characterized by the following set of necessary and sufficient conditions:

\[
\mathbb{E}[\pi_i(\xi_i, \tilde{q}_i(\mathbf{I}, \mathbf{q}))] = c_i, \quad i = 1, 2.
\]  

Under the bargaining equilibrium trade contract, there exists a unique vector of Nash equilibrium investment levels \( \mathbf{q}^c \), which is characterized by the following set of necessary and sufficient conditions:

\[
(1 - \theta_i)\mathbb{E}[\pi_i(\xi_i, q_i)] + \theta_i \mathbb{E}[\pi_i(\xi_i, \tilde{q}_i(\mathbf{I}, \mathbf{q}))] = c_i, \quad i = 1, 2.
\]  

Under the price equilibrium trade contract, the vector of Nash equilibrium investment levels \( \mathbf{q}^e \) is characterized by the following set of necessary conditions:

\[
\mathbb{E}[\pi_i(\xi_i, \tilde{q}_i(\mathbf{I}, \mathbf{q}))] - (1)^{3/2}\mathbb{E}
\left[
\frac{\partial P^*}{\partial q_i}
\right]
= c_i, \quad i = 1, 2.
\]
Although the optimality conditions (26) and (27) appear the same as in Proposition 3, the trading frictions affect trading and, thereby, the post-trading resource level vector \( \mathbf{q}(I, \mathbf{q}) \) embedded in (26)–(28). Therefore, the investment vectors in all three investment-trading scenarios depend, in general, on the trading frictions. The existence and uniqueness of the Nash equilibrium under the bargaining equilibrium trade contract is preserved under trading frictions. As for the price equilibrium trade contract, we expect the equilibrium investments to be unique for nonzero trading frictions (when trading is costly, the initial allocation matters). However, the uniqueness as well as the existence are difficult to prove in this case. Also note that, in general, the equilibrium investments under the price equilibrium trade contract are no longer first-best.

Unfortunately, it is difficult to replicate the analysis of §3 with trading frictions, or to study analytically the effect of trading frictions themselves. The difficulty arises from the fact that with trading frictions, the firms trade if and only if the expected price differential is sufficiently high relative to these frictions. In other words, the functional form of the optimal traded quantity is state dependent. As a result, the firms’ operating profits are not smooth functions of the investment levels, and our solution technique is not applicable.

However, we know from Proposition 7 that the equilibrium investments under frictionless trading are higher than the optimal investments without trading, which can be interpreted as the optimal investments under infinite trading frictions \((t = \infty \text{ or } \alpha = 0)\). It follows from continuity that the firms will invest more under very small trading frictions than they will under very large ones. Our numerical investigation suggests this relationship is monotone, i.e., the investment levels are monotonically decreasing in both frictions and, furthermore, that the main insights derived analytically for frictionless trading continue to hold with trading frictions.

5. Conclusion
This article studies trading or subcontracting of resources such as inventory or capacity between two independently owned firms. The firms invest in resources based on imperfect market forecasts. As time progresses, the firms update their forecasts and have the option to trade resources. The trade contract is determined in a bargaining game or, alternatively, as the price equilibrium. We characterize the Nash equilibrium investment levels and discuss the implications of the alternative trade contracts. As an illustration of the general profit functions, we then consider two firms with contingent pricing power that face stochastic demand functions with constant price elasticity. In this example, trading is driven by the expected price differential between the firms’ respective markets, as is often the case in practical applications. We use this model to study the effect of trading, decentralization, and the firms’ relative bargaining powers on investments, prices, profits, and consumer surplus. Assuming a forecast evolution process that follows a two-dimensional geometric Brownian motion, we also show how investments, prices, profits, and consumer surplus depend on market variability, correlation, and the quality of forecast revisions.

The contribution of this paper to the resource pooling literature is manifold and involves (i) the relatively high degree of generality under which many structural results are derived, (ii) the reflection of the price-demand relationship and the resulting analysis of product prices and consumer surplus, (iii) the natural information dynamics that enable an examination of the effect of forecast updating, and (iv) the multitude of comparative statics with regard to the key drivers of trading derived analytically. Interesting but nontrivial extensions of our work include introducing price competition or information asymmetry between the trading firms.

An online supplement to this paper is available on the Management Science website at http://mansci.pubs.informs.org/e companion.html.

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Appendix
Proof of Lemma 10. Because
\[ E(\xi_i \mid I) = \exp(\mu_i + I_i + \frac{1}{2} \sigma^2(1 - \lambda)), \quad i = 1, 2, \]
we can write
\[
\|\xi\| = E \left\{ \left( \frac{1}{2} \sum_{i=1}^{2} \exp \left( \mu_i + I_i + \frac{1}{2} \sigma^2(1 - \lambda) \right) \right)^a \right\} \\
= \frac{\exp \left( \frac{1}{2} \sigma^2(1 - \lambda) \right)}{2^a} \left\{ \sum_{i=1}^{2} \exp \left( \frac{\mu_i + I_i}{\alpha} \right) \right\}^a.
\]
Using the fact that
\[ I = \sqrt{\lambda} \sigma \begin{pmatrix} \sqrt{(1 - \rho)/2} & \sqrt{(1 + \rho)/2} \\ -\sqrt{(1 - \rho)/2} & \sqrt{(1 + \rho)/2} \end{pmatrix} Z, \]
where \( Z = (Z_1, Z_2)^T \) is a vector of two independent standard normal random variables, we can further rewrite \( \|\xi\| \) as
\[
\|\xi\| = \frac{\exp \left( \frac{1}{2} \sigma^2(1 - \lambda) \right)}{2^a} E \left\{ \exp \left( \sigma \sqrt{\lambda(1 - \rho)/2} Z_1 \right) \right\} \left\{ \sum_{i=1}^{2} \exp \left( \frac{\mu_i - (-1)^i \sigma \sqrt{\lambda(1 - \rho)/2} Z_1}{\alpha} \right) \right\}^a.
\]
Because $Z_1$ and $Z_2$ are independent, we have

\[
\|\xi\| = \sqrt{\frac{\frac{\lambda \sigma^2}{2} + \frac{\lambda \sigma^2}{4} + \frac{1}{2} \lambda \rho^2}{a}}
\]

\[
-\mathcal{E} \left\{ \sum_{i=1}^{2} \exp \left( \mu_i \frac{-\exp(\lambda(1-\rho)/2Z_i)}{a} \right) \right\}.
\]

(29)

We first prove that $\frac{\partial \|\xi\|}{\partial \lambda} \geq 0$. Differentiating (29) with respect to $\lambda$ and applying some algebra gives

\[
\frac{\partial \|\xi\|}{\partial \lambda} = \|\xi\| \left( \frac{-\frac{\lambda \sigma^2}{2} + \frac{\lambda \sigma^2}{4} + \frac{1}{2} \lambda \rho^2}{a} \exp(\frac{\lambda(1-\rho)/2Z_i}{\sqrt{\lambda(1-\rho)/2Z_i}}) \right) 
\]

\[
\cdot \left\{ \left( \exp \left( \mu_1 + \frac{\sigma}{\sqrt{\lambda(1-\rho)/2Z_i}} \right) \right) - \exp \left( \mu_2 - \frac{\sigma}{\sqrt{\lambda(1-\rho)/2Z_i}} \right) \right\}.
\]

Using the fact that for a differentiable function $g$ and a standard normal random variable $Z_r$, $E[g(Z_r)Z_1] = E[g(Z_1)]$ (Rubinstein 1976), and applying some algebra, we obtain

\[
\frac{\partial \|\xi\|}{\partial \lambda} = \frac{1 - a(1 - \rho)}{a} \cdot \left( \frac{1}{2} \right) \cdot 
\]

\[
\cdot \mathcal{E} \left\{ \left( \frac{\alpha}{\sqrt{\lambda(1-\rho)/2Z_i}} \right) \right\}.
\]

(30)

and, therefore,

\[
\frac{d\|\xi\|}{ds_1} = \frac{\partial \|\xi\|}{\partial \sigma} + \frac{\partial \|\xi\|}{\partial \mu_1} \frac{\partial \|\xi\|}{\partial \mu_1} + \frac{\partial \|\xi\|}{\partial \mu_2} \frac{\partial \|\xi\|}{\partial \mu_2} + \frac{\partial \|\xi\|}{\partial \rho} \frac{\partial \|\xi\|}{\partial \rho}.
\]

and

\[
\frac{d\|\xi\|}{dr} = \frac{\partial \|\xi\|}{\partial \rho}.
\]

(31)

From (30), we have

\[
\frac{\partial \sigma}{\partial s_1} = \frac{\exp(\sigma^2) - 1}{\sigma \exp(\sigma^2)} \frac{\exp(\mu_1 + \frac{1}{2} \sigma^2)}{\exp(\mu_1 + \frac{1}{2} \sigma^2)} \geq 0,
\]

\[
\frac{\partial \mu_1}{\partial s_1} = \frac{\partial \mu_2}{\partial s_1} = -\frac{\partial \sigma}{\partial s_1},
\]

\[
\frac{\partial \rho}{\partial s_1} = 2(\rho \exp(\sigma^2) \exp(\sigma^2) + \exp(\sigma^2)) \exp(\sigma^2)
\]

\[
+ \rho \exp(\rho \sigma^2) \exp(\sigma^2) - \exp(\sigma^2)
\]

\[
\cdot (\sigma \exp(\rho \sigma^2)(\exp(\sigma^2) - 1)) \frac{\partial \sigma}{\partial s_1},
\]

and

\[
\frac{\partial \rho}{\partial r} = \frac{\exp(\sigma^2) - 1}{\exp(\rho \sigma^2)} \frac{\exp(\rho \sigma^2)}{\exp(\rho \sigma^2)} \geq 0.
\]

Next, we evaluate $\frac{\partial \|\xi\|}{\partial \mu_1}$, $\frac{\partial \|\xi\|}{\partial \sigma}$, and $\frac{\partial \|\xi\|}{\partial \rho}$. By differentiating (29) with respect to $\mu_1$, we obtain

\[
\frac{\partial \|\xi\|}{\partial \mu_1} = \frac{\exp(\frac{1}{2} \lambda \sigma^2 - \frac{1}{4} \lambda \sigma^2 + \frac{1}{4} \lambda \rho^2 \sigma^2)}{2^e}
\]

\[
\cdot \left\{ \left( \frac{\alpha}{\sqrt{\lambda(1-\rho)/2Z_i}} \right) \right\}.
\]

(31)

It follows that $\frac{\partial \|\xi\|}{\partial \mu_1} + \frac{\partial \|\xi\|}{\partial \mu_2} = \|\xi\|$. By differentiating (29) with respect to $\sigma$, we obtain

\[
\frac{\partial \|\xi\|}{\partial \sigma} = \left( \sigma - \frac{1}{2} \sigma \sigma + \frac{1}{4} \sigma \rho \sigma \right) \|\xi\|
\]

\[
+ \exp(\frac{1}{2} \lambda \sigma^2 - \frac{1}{4} \lambda \sigma^2 + \frac{1}{4} \lambda \rho^2 \sigma^2) \frac{\lambda(1-\rho)}{2^e}
\]

\[
\cdot \left\{ \left( \frac{\alpha}{\sqrt{\lambda(1-\rho)/2Z_i}} \right) \right\}.
\]

(30)
Using again the fact that $E[g(Z_i)Z_i] = E g(Z_i)$, and applying some algebra, leads to
\[
\frac{\partial \| \mathbf{x} \|}{\partial \sigma} = \sigma \| \mathbf{x} \| + 2\sigma^2 \lambda \sigma (1 - \rho) \frac{1 - a}{a} \cdot \exp(\frac{1}{8} \sigma^2 - \frac{1}{4} \lambda \sigma^2 + \frac{1}{4} \lambda \sigma^2 \rho) \\
- \frac{\lambda}{8(1 - \rho)} \sigma \mathbf{x} \left\{ \exp\left(\frac{\mu_1 + \mu_2}{a} \right) + \exp\left(\frac{\mu_2 - \sigma \sqrt{(1 - \rho)/2Z_i}}{a} \right) \right\} + \exp\left(\frac{\mu_2 - \sigma \sqrt{(1 - \rho)/2Z_i}}{a} \right) + 2(-\rho \exp(\sigma^2 \rho) \exp(\sigma^2) + \exp(\sigma^2 \rho) \exp(\sigma^2) + \rho \exp(\sigma^2) - \exp(\sigma^2)) \cdot (\sigma \exp(\sigma^2 \rho)(\exp(\sigma^2) - 1) \frac{\partial \| \mathbf{x} \|}{\partial \rho} + \frac{\partial \| \mathbf{x} \|}{\partial \sigma}
\]

Differentiating (29) with respect to $\rho$ results in
\[
\frac{\partial \| \mathbf{x} \|}{\partial \rho} = \frac{a - 1}{a} \lambda \sigma^2 2^{-a} a \exp(\frac{1}{8} \sigma^2 - \frac{1}{4} \lambda \sigma^2 + \frac{1}{4} \lambda \sigma^2 \rho) \\
- \frac{\lambda}{8(1 - \rho)} \sigma \mathbf{x} \left\{ \exp\left(\frac{\mu_1 + \mu_2}{a} \right) + \exp\left(\frac{\mu_2 - \sigma \sqrt{(1 - \rho)/2Z_i}}{a} \right) \right\} + \exp\left(\frac{\mu_2 - \sigma \sqrt{(1 - \rho)/2Z_i}}{a} \right) + 2(-\rho \exp(\sigma^2 \rho) \exp(\sigma^2) + \exp(\sigma^2 \rho) \exp(\sigma^2) + \rho \exp(\sigma^2) - \exp(\sigma^2)) \cdot (\sigma \exp(\sigma^2 \rho)(\exp(\sigma^2) - 1) \frac{\partial \| \mathbf{x} \|}{\partial \rho} + \frac{\partial \| \mathbf{x} \|}{\partial \sigma}
\]

Using again the fact that $E[g(Z_i)Z_i] = E g(Z_i)$, and applying some algebra, we have
\[
\frac{\partial \| \mathbf{x} \|}{\partial \sigma} = \sigma \| \mathbf{x} \| + 2\sigma^2 \lambda \sigma (1 - \rho) \frac{1 - a}{a} \cdot \exp(\frac{1}{8} \sigma^2 - \frac{1}{4} \lambda \sigma^2 + \frac{1}{4} \lambda \sigma^2 \rho) \\
- \frac{\lambda}{8(1 - \rho)} \sigma \mathbf{x} \left\{ \exp\left(\frac{\mu_1 + \mu_2}{a} \right) + \exp\left(\frac{\mu_2 - \sigma \sqrt{(1 - \rho)/2Z_i}}{a} \right) \right\} + \exp\left(\frac{\mu_2 - \sigma \sqrt{(1 - \rho)/2Z_i}}{a} \right) + 2(-\rho \exp(\sigma^2 \rho) \exp(\sigma^2) + \exp(\sigma^2 \rho) \exp(\sigma^2) + \rho \exp(\sigma^2) - \exp(\sigma^2)) \cdot (\sigma \exp(\sigma^2 \rho)(\exp(\sigma^2) - 1) \frac{\partial \| \mathbf{x} \|}{\partial \rho} + \frac{\partial \| \mathbf{x} \|}{\partial \sigma}
\]

\[
\cdot \lambda \sigma^2 a \exp(\frac{1}{8} \sigma^2 - \frac{1}{4} \lambda \sigma^2 + \frac{1}{4} \lambda \sigma^2 \rho) \\
- \exp(\frac{\mu_1 + \mu_2}{a}) \exp(\sigma^2 \rho)(\exp(\sigma^2 - 1) \geq 0.
\]

It also follows that
\[
\frac{d \| \mathbf{x} \|}{dr} = \frac{\partial \rho}{\partial \sigma} \frac{\partial \| \mathbf{x} \|}{\partial \rho} \leq 0.
\]

Proof of Proposition 17. A Nash equilibrium investment vector must satisfy $\partial \Pi_i(q_i)/\partial q_i = 0$ for $i = 1, 2$. We first consider the bargaining equilibrium trade contract. The expected profit of firm $i$ (23) can be written as
\[
\Pi_i(q_i) = -c_i q_i + E_{\nu_i} (\xi_i, q_i) + \theta_i E_{\nu_i} \left( \sum_{k=1}^{2} \nu_i (\xi_k, q_k | I) - \sum_{k=1}^{2} \nu_i (\xi_k, q_k) - t | Q(I, q) \right).
\]

Taking the derivative of $\Pi_i(q_i)$ with respect to $q_i$ results in
\[
\frac{\partial \Pi_i(q_i)}{\partial q_i} = -c_i + (1 - \theta_i) E_{\nu_i} (\xi_i, q_i) + \theta_i E_{\nu_i} (\xi_i, \tilde{q}_i | I, q_i),
\]

and the equilibrium conditions in (27) follow. In differentiating (32), we used the following two facts: First, the traded quantity $Q(I, q)$—and, therefore, the firm’s realized profits—are continuous in $I$. Thus, the terms from differentiating the boundaries of $\Omega$ (limits of integration) cancel out. Second, the exchange of the derivative and expectation operator is justified by the continuous dominated convergence theorem and the fact that $n_i^*(\xi_i, q_i) < 0$ a.s. for any $q_i > 0$, $i = 1, 2$. The existence of a Nash equilibrium is ensured by the concavity of $\Pi_i$ in $q_i$ for $i = 1, 2$. In particular,
\[
\frac{\partial^2 \Pi_i(q_i)}{\partial q_i^2} = \frac{\partial \Pi_i(q_i)}{\partial q_i} + \theta_i Pr[I \in \Omega_0] E \left[ \frac{\partial \Pi_i(q_i)}{\partial q_i} \right] \left| I \in \Omega_0 \right\}
\]

where $\tilde{q}_i$ is used as a shorthand for $\tilde{q}_i(I, q)$. Similarly, $\partial^2 \Pi_i(q_i)/\partial q_i^2 = 0$. Here, the exchange of the derivative and expectation operator is justified by the continuous dominated convergence theorem and the fact that $n_i^*(\xi_i, q_i) > 0$ a.s. for any $q_i > 0$, $i = 1, 2$. To prove that the equilibrium is unique, we use the approach based on the Gale-Nikaido univalence theorems (see, e.g., Vives 1999, p. 47). Following this approach, sufficient conditions for uniqueness involve the convexity of each player’s strategy space and
\[
\begin{pmatrix}
\frac{\partial^2 \Pi_1}{\partial q_1^2} & \frac{\partial^2 \Pi_1}{\partial q_1 \partial q_2} \\
\frac{\partial^2 \Pi_2}{\partial q_1 \partial q_2} & \frac{\partial^2 \Pi_2}{\partial q_2^2}
\end{pmatrix}
\]
being negative semidefinite. To show the latter, we need
\[
\frac{\partial^2 \Pi_i(\mathbf{q})}{\partial q_i \partial q_j} = \alpha \theta \Pr(I \in \Omega_i) \cdot \left[ \mathbb{E} \left[ \tau_i'(\xi_j, \tilde{q}_i) \mathbb{I} \mathbb{E} \left[ \tau_i'(\xi_j, \tilde{q}_i) \mathbb{I} + \mathbb{E} \left[ \tau_i'(\xi_j, \tilde{q}_i) \mathbb{I} \right] \right] \right]_{i \in \Omega_j} + \alpha \theta \Pr(I \in \Omega_i) \cdot \left[ \mathbb{E} \left[ \tau_i'(\xi_j, \tilde{q}_i) \mathbb{I} \mathbb{E} \left[ \tau_i'(\xi_j, \tilde{q}_i) \mathbb{I} + \mathbb{E} \left[ \tau_i'(\xi_j, \tilde{q}_i) \mathbb{I} \right] \right] \right]_{i \in \Omega_j}
\]
for \(i, j \in \{1, 2\}\) and \(i \neq j\). It can be shown, after some cancellations, that
\[
\frac{\partial^2 \Pi_1}{\partial q_1 \partial q_2} > \frac{\partial^2 \Pi_1}{\partial q_2 \partial q_1} = \frac{\partial^2 \Pi_2}{\partial q_1 \partial q_2} = \frac{\partial^2 \Pi_2}{\partial q_2 \partial q_1},
\]
which completes the proof of uniqueness.

Next, we consider the price equilibrium trade contract. Taking the derivative of \(\Pi_i(\mathbf{q})\) (25) with respect to \(q_i\), \(i = 1, 2\), leaves us with
\[
\frac{\partial \Pi_i(\mathbf{q})}{\partial q_i} = -c_i + \mathbb{E} \tau_i'(\xi_j, \tilde{q}_i)(I, \mathbf{q}) \left\{ \left( Q \frac{\partial \Pi_i}{\partial q_i} \right) \right\},
\]
and the equilibrium conditions in (28) follow.

In the special case of frictionless trading (i.e., \(t = 0\), \(\alpha = 1\), and \(c_1 = c_2 = c\)), (33) simplifies to
\[
\frac{\partial \Pi_i(\mathbf{q})}{\partial q_i} = -c + \mathbb{E} \tau_i'(\xi_j, \tilde{q}_i) \left\{ \left( Q \frac{\partial \Pi_i}{\partial q_i} \right) \right\}.
\]

The Nash equilibrium conditions \(\Pi_i(\mathbf{q})/\partial q_i = 0\), \(i = 1, 2\), together with Lemma 1, imply that at a Nash equilibrium,
\[
\mathbb{E} \left[ \tau_i'(\xi_j, \tilde{q}_i) \mathbb{I} + \mathbb{E} \tau_i'(\xi_j, \tilde{q}_i) \mathbb{I} \right] = 0.
\]
The equilibrium conditions in (12) follow.

Finally, because \(\Pi(\mathbf{q})\) is concave, the necessary and sufficient optimality conditions for the first-best scenario (26) can be obtained by rearranging \(\Pi(\mathbf{q}) = 0\). The optimality conditions for the no-trading scenario are a special case of (26) with \(t = \infty\).

References


