34. Find a power series representation for the function \(1/(1+4x)\) and give the interval of convergence for the new series.

**Solution.** The formula for a geometric series says that
\[
\frac{1}{1-u} = 1 + u + u^2 + \ldots = \sum_{k=0}^{\infty} u^k,
\]
a power series with interval of convergence \((-1, 1)\). Substituting \(u = -4x\) gives
\[
\frac{1}{1+4x} = 1 - 4x + 16x^2 - \ldots = \sum_{k=0}^{\infty} (-4x)^k,
\]
a power series that converges exactly when \(-4x\) is in \((-1, 1)\). Since
\[-1 < -4x < 1\]
is equivalent to
\[-\frac{1}{4} < x < \frac{1}{4},\]
the interval of convergence for the power series above is \((-1/4, 1/4)\).

42. Find the power series representation for \(1/(1-x)^3\) centered at 0 by differentiating or integrating the power series for \(1/(1-x)\) (perhaps more than once). Give the interval of convergence for the resulting series.

**Solution.** We have
\[
\left(\frac{1}{1-x}\right)' = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}
\]
and
\[
\left(\frac{1}{1-x}\right)'' = \left(\frac{1}{(1-x)^2}\right)' = -2(1-x)^{-3}(-1) = \frac{2}{(1-x)^3}.
\]
Since we have
\[
\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k
\]
for \(x\) in \((-1, 1)\), differentiating the power series gives
\[
\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1} = \sum_{l=0}^{\infty} (l-1)x^l,
\]
where we’ve substituted \(l = k - 1\) in the last equality. Differentiating again we have
\[
\frac{2}{(1-x)^3} = \sum_{l=1}^{\infty} l(l-1)x^{l-1} = \sum_{k=0}^{\infty} (k+1)kx^k,
\]
where we’ve substituted \( k = l - 1 \) in the last equality. This implies that
\[
\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{k=0}^{\infty} (k+1)kx^k = \sum_{k=0}^{\infty} \frac{(k+1)k}{2} x^k.
\]
This series converges on \((-1,1)\) just as the series for \(1/(1-x)\), and we reconsider the endpoints. Since neither of the limits
\[
\lim_{k \to \infty} \frac{(k+1)k}{2} \quad \text{and} \quad \lim_{k \to \infty} \frac{(k+1)k}{2} (-1)^k
\]
is equal to 0, the Divergence Test implies that the series diverges at the endpoints \( x = \pm 1 \). Thus the interval of convergence is \((-1,1)\).

52. Find a power series representation centered at 0 for the function \( \tan^{-1}(4x^2) \) using known power series. Give the interval of convergence for the resulting series.

**Solution.** We first find a power series for \( \tan^{-1} x \) by exploiting the derivative of arctangent. Using the formula for the geometric series we have
\[
\frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - \ldots,
\]
which converges exactly \( x^2 \) is in \((-1,1)\), i.e. when \( x \) is in \((-1,1)\). Since we have
\[
(\tan^{-1} x)' = \frac{1}{1 + x^2},
\]
we may compute a power series representation for \( \tan^{-1} x \) by integrating the series for \(1/(1 + x^2)\) term-by-term. That is, an anti-derivative of \(1/(1 + x^2)\) is given by
\[
\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots,
\]
whose value at \( x = 0 \) is 0, and whose interval of convergence contains \((-1,1)\). The endpoints need to be checked separately: When \( x = 1 \) the series becomes
\[
\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}.
\]
The terms \(1/(2k+1)\) are decreasing, positive, and approach 0 as \( k \) goes to \(\infty\), so the series converges by the Alternating Series test. When \( x = -1 \) the series becomes
\[
\sum_{k=0}^{\infty} (-1)^k \frac{(-1)}{2k+1} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{2k+1}
\]
which converges as above by the Alternating Series Test. Thus \([-1,1]\) is the interval of convergence for
\[
\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}.
\]

Since \( \tan^{-1} 0 = 0 \) as well, both \( \tan^{-1} x \) and the series above are anti-derivatives for \(1/(1 + x^2)\) that are equal to 0 at \( x = 0 \), so that
\[
\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1},
\]
with interval of convergence $[-1, 1]$.

54. Find the radius of convergence of $\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^2 x^k$.

**Solution.** We take absolute values and use the Root Test:

$$\lim_{k \to \infty} \left|\left(1 + \frac{1}{k}\right)^2 x^k\right|^{1/k} = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^{2/k} |x|^{k/k} = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right) |x|.$$ 

Since we may compute

$$\log\left(\lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k\right) = \lim_{k \to \infty} k \log\left(1 + \frac{1}{k}\right) = \lim_{k \to \infty} \frac{\log\left(1 + \frac{1}{k}\right)}{\frac{1}{k}},$$

where the limit is indeterminate, we take derivatives of the numerator and denominator to find

$$\lim_{k \to \infty} \frac{\frac{1}{1 + \frac{1}{k}} \cdot \left(-\frac{1}{k^2}\right)}{-\frac{1}{k^2}} = \lim_{k \to \infty} \frac{1}{1 + \frac{1}{k}} = 1.$$

l'Hôpital's Rule applies so that

$$\log\left(\lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k\right) = 1,$$

and the desired limit may be computed:

$$\lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k = e.$$ 

Thus

$$\lim_{k \to \infty} \left|\left(1 + \frac{1}{k}\right)^2 x^k\right|^{1/k} = |x|e,$$

and the series will converge absolutely when $|x|e < 1$ and diverge when $|x|e > 1$, so the interval of convergence is $(-\frac{1}{e}, \frac{1}{e})$ with or without its endpoints. Note that nothing we’ve said above implies either convergence or divergence at the endpoints: Such a determination would require some more subtle analysis. In any case, the radius of convergence must be $1/e$.

55. Find the radius of convergence of $\sum \frac{k!x^k}{k^k}$.

**Solution.** We take absolute values and attempt the Ratio Test:

$$\lim_{k \to \infty} \left|\frac{(k+1)x^{k+1}}{(k+1)^{k+1} \cdot k!x^k}\right| = \lim_{k \to \infty} \left|\frac{(k+1)x}{(k+1)^{k+1}} \cdot \frac{k^k}{k!x^k}\right| = \lim_{k \to \infty} \left|\frac{k^k}{(k+1)^{k+1}}\right| |x| = \lim_{k \to \infty} \left(\frac{k+1}{k}\right)^{-k} |x| = \lim_{k \to \infty} \left(\frac{k+1}{k}\right)^{-1} |x| = |x|/e.$$ 

By the Ratio Test, the series will converge when $|x|/e < 1$ and diverge when $|x|/e > 1$, i.e. the interval of convergence is given by $(-e, e)$ with or without the endpoints. Again, the behavior at the endpoints is unclear, but in any case the radius of convergence is $e$. 
62. Find the function represented by \( \sum_{k=0}^{\infty} (x^2 + 1)^{2k} \) and find the interval of convergence.

**Solution.** This series can be rewritten as \( \sum_{k=0}^{\infty} ((x^2 + 1)^2)^k \) which is a geometric series with common ratio given by \( (x^2 + 1)^2 \). The formula for the geometric series applies, so that the series converges exactly when \( |x^2 + 1| < 1 \). On the other hand, \( x^2 + 1 \geq 1 \) for all values of \( x \), and thus the function has empty interval of convergence, and is not a well-defined function anywhere.

64. Find the function represented by \( \sum_{k=1}^{\infty} \frac{x^{2k}}{4k} \) and find the interval of convergence.

**Solution.** We compute the derivative of this power series by differentiating term-by-term:

\[
\left( \sum_{k=1}^{\infty} \frac{x^{2k}}{4k} \right)' = \sum_{k=1}^{\infty} \frac{2kx^{2k-1}}{4k} = \sum_{k=1}^{\infty} \frac{1}{2}x^{2k-1} = \sum_{k=0}^{\infty} \frac{1}{2}x^{2k+1} = \frac{x}{2} \sum_{k=0}^{\infty} x^{2k} = \frac{x}{2} \left( \frac{1}{1-x^2} \right),
\]

where we’ve computed the sum by summing a geometric series with common ratio \( x^2 \), so that the last series converges whenever \( |x^2| < 1 \) and diverges when \( |x^2| > 1 \). That means that the interval of convergence of the last series is \((-1, 1)\) with or without the endpoints. Since we are after the function defining the original series, we integrate the derivative function:

\[
\int \frac{x}{2(1-x^2)} \, dx = \int \frac{-du}{4u} = -\ln |4u| + C = -\ln |4(1-x^2)| + C.
\]

This means that

\[
\sum_{k=1}^{\infty} \frac{x^{2k}}{4k} = -\ln |4(1-x^2)| + C
\]

for some \( C \). Plugging in \( x = 0 \) we find

\[
0 = -\ln 4 + C
\]

so that \( C = \ln 4 \). We obtain

\[
\sum_{k=1}^{\infty} \frac{x^{2k}}{4k} = -\ln |4(1-x^2)| + \ln 4 = \ln \left| \frac{4}{4(1-x^2)} \right| = \ln \frac{1}{|1-x^2|},
\]

with interval of convergence given by \((-1, 1)\) with or without endpoints. We examine the endpoints:

When \( x = \pm 1 \), we have \( x^{2k} = 1 \) for each \( k \), so that the series becomes \( \sum_{k=1}^{\infty} \frac{1}{4k} \). This series diverges, for instance by the Limit Comparison Test: Since

\[
\lim_{k \to \infty} \frac{\frac{1}{k}}{\frac{1}{4k}} = \frac{1}{4}
\]

and the harmonic series diverges, the series \( \sum_{k=1}^{\infty} \frac{1}{4k} \) also diverges. This implies that the interval of convergence is \((-1, 1)\).

66. Find the function represented by \( \sum_{k=1}^{\infty} \frac{(x-2)^k}{3^{2k}} \) and find the interval of convergence.
Solution. We rearrange and use the formula for a geometric series:

\[
\sum_{k=1}^{\infty} \frac{(x-2)^k}{3^{2k}} = \sum_{k=1}^{\infty} \left( \frac{x-2}{9} \right)^k = \frac{x-2}{9} = \frac{9-(x-2)}{9} = x - \frac{2}{11} - x
\]

for any \( x \) so that \( |x-2|/9 < 1 \). This set is the set of \( x \) so that \( |x-2| < 9 \), i.e. the interval of convergence is \((-7,11)\).

68. Replace \( x \) with \( x-1 \) in the series \( \ln(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \) to obtain a power series for \( \ln x \) centered at \( x = 1 \). What is the interval of convergence for the new power series?

Solution. We have the power series representation

\[
\ln(x + 1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}
\]

with interval of convergence \((-1,1]\). Replacing \( x \) with \( x-1 \) we obtain

\[
\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x - 1)^k,
\]

which converges exactly when \( x-1 \) is in \((-1,1]\), so that its interval of convergence is \((0,2]\).

75. Let

\[
f(x) = \sum_{k=0}^{\infty} c_k x^k \text{ and } g(x) = \sum_{k=0}^{\infty} d_k x^k.
\]

(a) Multiply the power series together as if they were polynomials, collecting all terms that are multiples of \( 1, x, \text{ and } x^2 \). Write the first three terms of the product \( f(x)g(x) \).

(b) Find a general expression for the coefficient of \( x^n \) in the product series, for \( n = 0, 1, 2, \ldots \).

Solution (a). We perform the multiplication as recommended, ignoring terms with powers of \( x \) greater than 2:

\[
\left( \sum_{k=0}^{\infty} c_k x^k \right) \left( \sum_{k=0}^{\infty} d_k x^k \right) = (c_0 + c_1 x + c_2 x^2 + \ldots)(d_0 + d_1 x + d_2 x^2 + \ldots)
\]

\[
= \begin{array}{c}
    c_0 (d_0 + d_1 x + d_2 x^2 + \ldots) + c_1 x (d_0 + d_1 x + d_2 x^2 + \ldots) \\
    + c_2 x^2 (d_0 + d_1 x + d_2 x^2 + \ldots) + \ldots
\end{array}
\]

\[
= \begin{array}{c}
    (c_0d_0 + c_0d_1 x + c_0 d_2 x^2 + \ldots) + (c_1d_0 x + c_1d_1 x^2 + \ldots) + (c_2d_0 x^2 + \ldots) + \ldots
\end{array}
\]

\[
= (c_0d_0) + (c_0d_1 + c_1d_0) x + (c_0d_2 + c_1d_1 + c_2d_0) x^2 + \ldots
\]

Solution (b). In order to multiply two powers of \( x \) and obtain \( x^n \), say \( x^k \) and \( x^l \), we must have \( k + l = n \). Both powers must be non-negative, so that it is clear that the possibilities for pairs \((k,l)\) are \((0,n),(1,n-1),(2,n-2), \ldots, (n-1,1),(n,0)\). Multiplying the terms \( c_k x^k \) and \( d_l x^l \) we obtain \( c_k d_l x^{k+l} \), so that the listed pairs of \( k \) and \( l \) give the sum of terms

\[
c_0d_n x^n + c_1d_{n-1} x^n + c_2d_{n-2} x^n + \ldots + c_{n-1}d_1 x^n + c_n d_0 x^n = (c_0d_n + c_1d_{n-1} + c_2d_{n-2} + \ldots + c_{n-1}d_1 + c_n d_0) x^n.
\]
We may write this formula for the coefficient of $x^n$ more concisely as $\sum_{k=0}^{n} c_k d_{n-k}$, so that a formula for the product is given by

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} c_k d_{n-k} \right) x^n.$$