Lifting curves simply

Jonah Gaster

1Department of Mathematics, Boston College

Correspondence to be sent to: gaster@bc.edu

Let \( f_\rho(L) \) indicate the smallest integer so that every curve on a fixed hyperbolic surface \((S, \rho)\) of length at most \( L \) lifts to a simple curve on a cover of degree at most \( f_\rho(L) \). We provide linear lower bounds for \( f_\rho(L) \), improving a recent result of Gupta-Kapovich [6]. When \((S, \rho)\) is without punctures, using work of Patel [9] and Lenzhen-Rafi-Tao [7] we conclude that \( f_\rho(L)/L \) grows like the reciprocal of the systole of \((S, \rho)\). When \((S, \rho)\) has a puncture, we obtain lower bounds for \( f_\rho \) that are exponential in \( L \).

1 Introduction

Let \( S \) be a topological surface of finite type and negative Euler characteristic, and let \( \rho \) be a complete hyperbolic metric on \( S \). Let \( \mathcal{C}(S) \) indicate the set of closed curves on \( S \), i.e. the set of free homotopy classes of the image of immersions of \( S^1 \) into \( S \). For \( \gamma \in \mathcal{C}(S) \), let \( \ell(\gamma, \rho) \) indicate the infimum of the \( \rho \)-lengths of representatives of \( \gamma \), though when clear from context we will also use \( \ell(\gamma', \rho) \) to indicate the \( \rho \)-length of a chosen representative \( \gamma' \). Let \( \iota(\gamma, \gamma) \) indicate the geometric self-intersection number of \( \gamma \), and recall that a closed curve \( \gamma \in \mathcal{C}(S) \) is simple when \( \iota(\gamma, \gamma) \) is equal to zero.

It is a corollary of a celebrated theorem of Scott [12] [13] that each closed curve \( \gamma \in \mathcal{C}(S) \) lifts to a simple closed curve in some finite-sheeted cover (i.e. \( \gamma \) ‘lifts simply’). Recent work has focused on making Scott’s result effective [9]. As such, for \( \gamma \in \mathcal{C}(S) \), let \( \deg(\gamma) \) indicate the minimum degree of a cover to which \( \gamma \) lifts simply.

We focus on two functions \( f_\rho \) and \( f_S \). Let \( f_S(n) \) be the maximum degree of a curve of self-intersection number at most \( n \), and let \( f_\rho(L) \) be the maximum degree of a curve of \( \rho \)-length at most \( L \).

Gupta-Kapovich have recently shown:

Theorem 1.1. [6, Thm. C, Cor. 1.1] There are constants \( C_1 = C_1(\rho) \) and \( C_2 = C_2(S) \) so that

\[
 f_\rho(L) \geq C_1 \cdot (\log L)^{1/3} \quad \text{and} \quad f_S(n) \geq C_2 \cdot (\log n)^{1/3}.
\]

Their work analyzed the ‘primitivity index’ of a ‘random’ word in the free group, exploiting the many free subgroups of \( \pi_1 S \) (e.g. subgroups corresponding to incompressible three-holed spheres, or pairs of pants) to obtain the above result. We also exploit the existence of free subgroups of \( \pi_1 S \), but instead of following in their delicate analysis of random walks in the free group, we analyze explicit curves on \( S \). The chosen curves are sufficiently uncomplicated to allow a straightforward analysis of the degree of any cover to which the curves lift simply, allowing improved lower bounds.

Recall that a systole of \((S, \rho)\) is a shortest non-trivial geodesic on \( S \), and let \( \sys(\rho) \) indicate the length of a systole of \((S, \rho)\). We say that a boundary component of \((S, \rho)\) is a puncture if the closed curve homotopic to the boundary component has no geodesic representative.
Theorem 1.2. We have \( f_S(n) \geq n + 1 \). □

Theorem 1.3. Suppose \((S, \rho)\) is a hyperbolic surface without punctures. For any \( \epsilon > 0 \), there is an \( L_0 = L_0(\rho, \epsilon) \) so that for any \( L \geq L_0 \) we have
\[
\frac{f_\rho(L)}{L} \geq \frac{1}{\sys(\rho) + \epsilon}.
\]

Theorem 1.4. Suppose \((S, \rho)\) is a hyperbolic surface with a puncture. For any \( \epsilon > 0 \), there is an \( L_0 = L_0(\rho, \epsilon) \) so that for any \( L \geq L_0 \) we have
\[
\frac{\log f_\rho(L)}{L} \geq \frac{1}{2 + \epsilon}.
\]

It is interesting to note that the righthand side in Theorem 1.3 above can be replaced in the compact case with a constant that depends only on the topology of \( S \): The systole function is continuous on the moduli space of hyperbolic metrics on \( S \), and Mumford’s compactness criterion implies that a global maximum exists. (Quantifying this maximum is a rich area of study, see for example [5], [11], [1]).

The proof of Theorems 1.2, 1.3, and 1.4 follow from an analysis of an explicit sequence of curves \( \{\gamma_n\} \) (see Figure 1). These curves are also analyzed by Basmajian [2], where it is shown that they are in some sense the ‘shortest’ curves of a given intersection number: The infimum of the length function \( \ell(\gamma_n, \cdot) \) on the Teichmüller space of \( S \) is asymptotically the minimum possible among curves with self-intersection \( \iota(\gamma_n, \gamma_n) \) [2, Cor. 1.4].

Fig. 1. A minimal position representative of the curve \( \gamma_4 \).

Gupta-Kapovich [6, p. 1] observed that the work of Patel [9] implies linear upper bounds for \( f_\rho \) in many cases, so that Theorem 1.3 immediately implies linear growth of \( f_\rho \). When \((S, \rho)\) has no punctures, there is a hyperbolic metric \( \rho_0 \) on \( S \) and an absolute constant \( C_1 \) so that any curve \( \gamma \) on \( S \) has degree \( \deg(\gamma) \leq C_1 \cdot \ell(\gamma, \rho_0) \) [9, Thm. 1.1]. Since there is a Lipschitz map from \((S, \rho)\) to \((S, \rho_0)\), there is a constant \( C(\rho) \) so that \( f_\rho(L) \leq C_1 \cdot C(\rho) \cdot L \). In fact, the work of [7] sheds some light on the optimal Lipschitz constant \( C(\rho) \) (see Lemma 4.1), and this argument shows that \( f_\rho(L)/L \) is roughly \( 1/\sys(\rho) \) for \( L \) large. Precisely:

**Corollary (Linear growth of \( f_\rho \)).** Suppose \((S, \rho)\) is without punctures. There exists a constant \( C = C(S) \) and \( L_0 = L_0(\rho) \) so that for \( L \geq L_0 \) we have
\[
\frac{1}{C \cdot \sys(\rho)} \leq \frac{f_\rho(L)}{L} \leq \frac{C}{\sys(\rho)}.
\]

**Proof.** The lower bound follows from Theorem 1.3. In the compact case, the upper bound follows from [9, Thm. 1.1] together with Lemma 4.1. When \( S \) is noncompact but has no punctures, there is a compact convex core \( S^c \) of \((S, \rho)\) that contains all closed geodesics. Let \((\hat{S}, \hat{\rho})\) be the compact hyperbolic surface obtained by doubling \( S^c \). Any geodesic of length at most \( L \) on \((S, \rho)\) has length at most \( L \) on \((\hat{S}, \hat{\rho})\), and a curve on \( S^c \subset \hat{S} \) that lifts simply lifts simply to a cover of \( S \) with lesser or equal degree. □

In the presence of punctures we are not aware of upper bounds for \( f_\rho \). Note that Theorem 1.4 indicates that Patel’s upper bounds cannot hold in the punctured setting: If \((S, \rho)\) is a hyperbolic surface with a puncture, the minimal degree of a cover to which a given curve \( \gamma \) lifts to a simple curve cannot be bounded linearly in the curve’s length \( \ell(\gamma, \rho) \).
There are other avenues for further investigation. It would be natural to seek upper bounds for \( f_S(n) \) (cf. [10, p. 15]), since no such bound follows from [9]. Finally, one could explore the set of curves of self-intersection number exactly \( n \). For instance:Among the finitely many mapping class group orbits of curves \( \gamma \) with self-intersection \( n \), which maximize \( \deg(\gamma) \)?

Outline of the paper

In \S 2 we introduce a sequence of curves \( \{\gamma_n\} \) on a pair of pants and analyze the degrees \( \deg(\gamma_n) \), in \S 3 we deduce Theorems 1.2, 1.3, and 1.4 as straightforward consequences, and in \S 4 we prove Lemma 4.1 needed for the proof of the corollary.

2 Analysis of a certain curve family

![Fig. 2. The pair of pants \( P_0 \), with generators \( a \) and \( b \).](image)

Let \( P_0 \) be a pair of pants. Identify \( \pi_1(P_0, p) \) with a rank-2 free group \( F \), with generators \( a \) and \( b \) as pictured in Figure 2. Let \( \gamma_n \) indicate the closed curve given by the equivalence class of \( a \cdot b^n \). The self-intersection of \( \gamma_n \) was analyzed by [2] (see Figure 1). We rely on Basmajian’s result below, but the following result can also be obtained by invoking a variant of the ‘bignon criterion’ [3, §1.2.4] [4, Lemma 5.1].

Lemma 2.1. For \( n \geq 0 \), the curve \( \gamma_n \) has \( \iota(\gamma_n, \gamma_n) = n \).

Proof. Choose any complete hyperbolic structure \( \rho \) on \( P_0 \), and note that \( \iota(\gamma_n, \gamma_n) \) is equal to the number of intersections for the \( \rho \)-geodesic representative of \( \gamma_n \). The conclusion is now a consequence of [2, Lemma 4.1].

We use Lemma 2.1 to estimate \( \deg(\gamma_n) \), a calculation reminiscent of [6, Lemma 3.10]. The following proposition is the main tool in our analysis.

Proposition 2.2. We have \( \deg(\gamma_n) \geq n + 1 \).

Proof. Towards contradiction, suppose there is a cover \( P' \to P_0 \) of degree \( k' \leq n \), so that \( \gamma_n \) lifts to a simple curve \( \gamma \). Draw \( P_0 \) as a directed ribbon graph with one vertex \( p \) and the two edges labeled by \( a \) and \( b \), and \( P' \) as a directed ribbon graph with vertices \( p_1, \ldots, p_k \) and \( 2k' \) directed edges, \( k' \) with \( a \) labels and \( k' \) with \( b \) labels. Choose an orientation for \( \gamma \) so that it consists of a directed \( a \) edge followed by \( n \) directed \( b \) edges. After relabeling, we may assume that the unique \( a \) edge of \( \gamma \) is followed by \( p_1 \).

Starting from \( p_1 \) there is a cycle of \( b \) edges of length \( k \leq k' \). Relabel if necessary so that the vertices in this cycle are \( p_1, \ldots, p_k \), in order. Let the vertex that immediately follows the \( n \) consecutive \( b \) edges of \( \gamma \) be \( p_l \), where \( l \) is equivalent to \( n + 1 \) modulo \( k \). Finally, \( \gamma \) follows an \( a \) edge from \( p_l \) to \( p_1 \). See Figure 3 for a schematic.

This implies that there is an incompressible embedded pair of pants \( P'' \) in \( P' \) that contains \( \gamma \) (see Figure 4 in the case that \( k \mid n – \) the other case is straightforwardly similar). After identifying \( P'' \) with \( P_0 \) appropriately, the closed curve \( \gamma \) is given by the equivalence class of \( a \cdot b^s \), where \( s = \left\lfloor \frac{n}{k} \right\rfloor \geq 1 \). By Lemma 2.1 this curve is not simple, a contradiction.

Remark. In fact, it is not hard to show that \( \deg(\gamma_n) = n + 1 \): The subgroup generated by \( a^2 \), \( ba \), \( b^{a+1} \), and \( b^k a b^{-k} \) for \( k = 2, \ldots, n \) is index \( n + 1 \) in \( F \). With a helpful picture, one can check that the curve \( ab^n = a^2 \cdot (ba)^{-1} \cdot b^{a+1} \) has a simple lift in the corresponding cover. However, our estimates in Theorems 1.2, 1.3 and 1.4 depend only on the inequality above, so we omit this remaining argument. It seems worthwhile to point out that this computation implies that our estimates cannot be improved without appealing to a different family of curves.
Fig. 3. A supposedly simple lift $\gamma$ of $\gamma_n$ to the cover $P' \rightarrow P_0$ follows a cycle of directed $b$ edges.

Fig. 4. The darkly shaded incompressible pair of pants $P'' \subset P'$ contains $\gamma$.

3 Proofs of Theorems 1.2, 1.3, and 1.4

Proof of Theorem 1.2. Since $S$ has negative Euler characteristic, we may choose a marked embedded pair of pants $P_0 \hookrightarrow S$, so that we may view $\{\gamma_n\}$ as a sequence of closed curves on $S$. Suppose that $\pi : S' \rightarrow S$ is a cover of $S$ so that $\gamma_n$ lifts to a simple curve $\gamma'$. Let $P'$ be the component of $\pi^{-1}(P)$ containing $\gamma'$. We obtain a cover $\pi|_{P'} : P' \rightarrow P$, so that the degree of $S' \rightarrow S$ is at least the degree of $P' \rightarrow P$. By Proposition 2.2, the degree of $P' \rightarrow P$ is at least $n + 1$. Thus $\deg(\gamma_n) \geq n + 1$, and the bound for $f_S(n)$ follows immediately from Lemma 2.1.
Proof of Theorem 1.3. Since \((S, \rho)\) is without punctures, the systole of \((S, \rho)\) is simple, whose free homotopy class we denote by \(\beta\). Choose a representative \(\alpha'\) of a simple closed curve \(\alpha\)—possibly homotopic to a boundary component—disjoint from \(\beta\), and a simple arc \(\delta\) connecting \(\alpha'\) to \(\beta\). Let \(P\) be the pair of pants homotopic to a regular neighborhood of \(\alpha' \cup \beta \cup \delta\).

Identify \(P\) with \(P_0\) so that \(\alpha\) is in the conjugacy class of \(a\) and \(\beta\) is in the conjugacy class of \(b\), and consider the closed curves \(\{\gamma_n\}\) in \(P\). Evidently,
\[
\ell(\gamma_n, \rho) \leq \ell(\alpha', \rho) + n \cdot \text{sys}(\rho) + 2\ell(\delta, \rho).
\]

Given \(\epsilon > 0\), for large \(n\) we have \(\ell(\gamma_n, \rho) \leq n \cdot (\text{sys}(\rho) + \epsilon)\). Let
\[
n = n(L) = \left\lfloor \frac{L}{\text{sys}(\rho) + \epsilon} \right\rfloor,
\]
so that \(\ell(\gamma_n, \rho) \leq L\) for large enough \(L\). Thus, for large enough \(L\), we have
\[
f_\rho(L) \geq \deg(\gamma_n) \geq n + 1 \geq \frac{L}{\text{sys}(\rho) + \epsilon}
\]
as desired. \(\blacksquare\)

Proof of Theorem 1.4. Let \(\beta\) indicate a simple closed curve homotopic to a puncture, and as above choose a representative \(\alpha'\) of a simple closed curve \(\alpha\) disjoint from \(\beta\), and a pair of pants \(P\) with cuffs homotopic to \(\alpha\) and \(\beta\). Identify \(P\) with \(P_0\) so that \(b\) is homotopic to \(\beta\) and \(a\) is homotopic to \(\alpha\), and consider again the sequence of curves \(\{\gamma_n\}\) on \(S\).

We assume the upper half plane model for the hyperbolic plane \(\mathbb{H}^2\). By conjugating the holonomy representation of \(\rho\) appropriately, we may arrange for the holonomy around \(\beta\) to be the transformation \(z \mapsto z + 1\), and so that there is a lift of \(\alpha'\) to the hyperbolic plane \(\mathbb{H}^2\) that contains a point on the imaginary axis, say \(is\).

There is a lift of a curve freely homotopic to \(b^p\) that starts at \(is\), travels vertically along the imaginary axis to \(iy\), travels horizontally to \(n + iy\), and vertically down to \(n + is\). Let \(\beta_y'\) indicate the projection of this curve to \(P\), and note that by construction its starting and ending point are in common, and on \(\alpha'\). We may thus concatenate (a parametrization of) \(\alpha'\) with \(\beta_y'\), and the curve so obtained is homotopic to \(\gamma_n\).

An elementary computation shows that \(\ell(\beta_y', \rho) = 2\log(y/s) + n/y\). Taking \(y = n\) we find
\[
\ell(\gamma_n, \rho) \leq \ell(\alpha', \rho) + \ell(\beta_n', \rho) \leq \ell(\alpha', \rho) + 2\log s + 1 + 2\log n.
\]

Given \(\epsilon > 0\), for large \(n\) the \(\rho\)-lengths satisfy \(\ell(\gamma_n, \rho) \leq (2 + \epsilon) \log n\), and the result follows as in the proof of Theorem 1.3: Let
\[
n = n(L) = \left\lfloor e^{\frac{L}{2(1+\epsilon)}} \right\rfloor,
\]
so that \(\ell(\gamma_n, \rho) \leq L\) for large enough \(L\). Thus, for large enough \(L\), we have
\[
f_\rho(L) \geq \deg(\gamma_n) \geq n + 1 \geq e^{\frac{L}{2(1+\epsilon)}}
\]
as desired. \(\blacksquare\)

4 The minimal Lipschitz constant and the systole function

We rely on the work of [7], rephrasing their results slightly to suit our needs. Let \(\mathcal{T}(S)\) indicate the Teichmüller space of \(S\), and recall that there is a forgetful map \(\pi : \mathcal{T}(S) \to \mathcal{M}(S)\), where \(\mathcal{M}(S)\) is the set of isometry classes of hyperbolic metrics on \(S\). A marking of \(S\) is a set of simple closed curves on \(S\) forming a pants decomposition together with a set of transversals (‘complete clean marking’ in the language of [8, §2.5, p. 17], where this was introduced), and a short marking on \(X \in \mathcal{T}(S)\) is a marking that is constructed by choosing short curves according to the greedy algorithm (see [7, p. 6] for detail).

Lemma 4.1. Suppose \(S\) is compact. For any pair of hyperbolic metrics \(\rho, \rho_0 \in \mathcal{M}(S)\), there exists \(C = C(\rho_0)\) so that the minimal Lipschitz constant from \(\rho\) to \(\rho_0\) is at most \(C/\text{sys}(\rho)\). \(\square\)
Proof. Lift \( \rho \) to \( X \in T(S) \), and choose a short marking \( \mu_X \) of \( X \). There are only finitely many mapping class group-orbits of markings. Each has representatives on \( (S, \rho_0) \) of shortest length, and the max over the lengths of these finitely many curves is a constant that depends only on \( \rho_0 \). Thus there exists \( C_1 = C_1(\rho_0) \) and a lift \( X_0 \in \pi^{-1}(\rho_0) \) so that 
\[
\max_{\alpha \in \mu_X} \ell(\alpha, X_0) \leq C_1.
\]

Now [7, Thm. E] shows that there exists a constant \( C_2 \) so that there is a Lipschitz map from \( X \) to \( X_0 \), homotopic to the identity, with Lipschitz constant at most 
\[
C_2 \cdot \max_{\alpha \in \mu_X} \frac{\ell(\alpha, X_0)}{\ell(\alpha, X)} \leq \frac{C_1 C_2}{\text{sys}(\rho)}.
\]

□

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References


