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Lifting curves simply

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Let $f_\rho(L)$ indicate the smallest integer so that every curve on a fixed hyperbolic surface (S,ρ) of length at most L lifts to a simple curve on a cover of degree at most $f_\rho(L)$. We provide linear lower bounds for $f_\rho(L)$, improving a recent result of Gupta-Kapovich [6]. When (S,ρ) is without punctures, using work of Patel [9] and Lenzhen-Rafi-Tao [7] we conclude that $f_\rho(L)/L$ grows like the reciprocal of the systole of (S,ρ). When (S,ρ) has a puncture, we obtain lower bounds for f_ρ that are exponential in L.

1 Introduction

Let S be a topological surface of finite type and negative Euler characteristic, and let ρ be a complete hyperbolic metric on S. Let $\mathcal{C}(S)$ indicate the set of *closed curves* on S, i.e. the set of free homotopy classes of the image of immersions of S^1 into S. For $\gamma \in \mathcal{C}(S)$, let $\ell(\gamma,\rho)$ indicate the infimum of the ρ-lengths of representatives of γ, though when clear from context we will also use $\ell(\gamma',\rho)$ to indicate the ρ-length of a chosen representative γ'. Let $\iota(\gamma,\gamma)$ indicate the geometric self-intersection number of γ, and recall that a closed curve $\gamma \in \mathcal{C}(S)$ is *simple* when $\iota(\gamma,\gamma)$ is equal to zero.

It is a corollary of a celebrated theorem of Scott [12] [13] that each closed curve $\gamma \in \mathcal{C}(S)$ lifts to a simple closed curve in some finite-sheeted cover (i.e. γ 'lifts simply'). Recent work has focused on making Scott's result effective [9]. As such, for $\gamma \in \mathcal{C}(S)$, let $\deg(\gamma)$ indicate the minimum degree of a cover to which γ lifts simply.

We focus on two functions f_ρ and f_S. Let $f_S(n)$ be the maximum degree of a curve of self-intersection number at most n, and let $f_\rho(L)$ be the maximum degree of a curve of ρ-length at most L.

Gupta-Kapovich have recently shown:

Theorem 1.1. [6, Thm. C, Cor. 1.1] There are constants $C_1 = C_1(\rho)$ and $C_2 = C_2(S)$ so that

$$f_\rho(L) \geq C_1 \cdot (\log L)^{1/3} \quad \text{and} \quad f_S(n) \geq C_2 \cdot (\log n)^{1/3}.$$

□

Their work analyzed the 'primitivity index' of a 'random' word in the free group, exploiting the many free subgroups of $\pi_1 S$ (e.g. subgroups corresponding to incompressible three-holed spheres, or *pairs of pants*) to obtain the above result. We also exploit the existence of free subgroups of $\pi_1 S$, but instead of following in their delicate analysis of random walks in the free group, we analyze explicit curves on S. The chosen curves are sufficiently uncomplicated to allow a straightforward analysis of the degree of any cover to which the curves lift simply, allowing improved lower bounds.

Recall that a *systole* of (S,ρ) is a shortest non-trivial geodesic on S, and let $\mathrm{sys}(\rho)$ indicate the length of a systole of (S,ρ). We say that a boundary component of (S,ρ) is a *puncture* if the closed curve homotopic to the boundary component has no geodesic representative.

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Theorem 1.2. We have $f_S(n) \geq n+1$. □

Theorem 1.3. Suppose (S,ρ) is a hyperbolic surface without punctures. For any $\epsilon > 0$, there is an $L_0 = L_0(\rho,\epsilon)$ so that for any $L \geq L_0$ we have

$$\frac{f_\rho(L)}{L} \geq \frac{1}{\mathrm{sys}(\rho)+\epsilon}.$$

□

Theorem 1.4. Suppose (S,ρ) is a hyperbolic surface with a puncture. For any $\epsilon > 0$, there is an $L_0 = L_0(\rho,\epsilon)$ so that for any $L \geq L_0$ we have

$$\frac{\log f_\rho(L)}{L} \geq \frac{1}{2+\epsilon}.$$

□

It is interesting to note that the righthand side in Theorem 1.3 above can be replaced in the compact case with a constant that depends only on the topology of S: The systole function is continuous on the moduli space of hyperbolic metrics on S, and Mumford's compactness criterion implies that a global maximum exists. (Quantifying this maximum is a rich area of study, see for example [5], [11], [1]).

The proof of Theorems 1.2, 1.3, and 1.4 follow from an analysis of an explicit sequence of curves $\{\gamma_n\}$ (see Figure 1). These curves are also analyzed by Basmajian [2], where it is shown that they are in some sense the 'shortest' curves of a given intersection number: The infimum of the length function $\ell(\gamma_n,\cdot)$ on the Teichmüller space of S is asymptotically the minimum possible among curves with self-intersection $\iota(\gamma_n,\gamma_n)$ [2, Cor. 1.4].

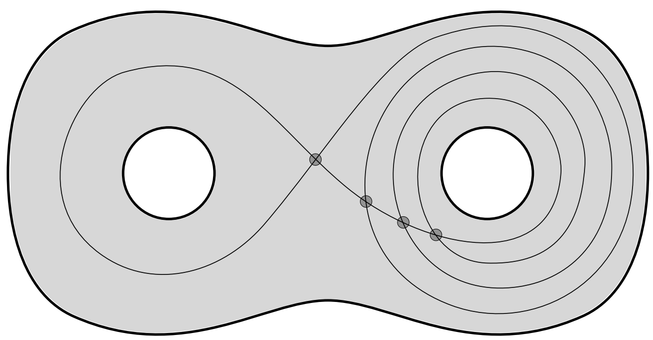

Fig. 1. A minimal position representative of the curve γ_4.

Gupta-Kapovich [6, p. 1] observed that the work of Patel [9] implies linear upper bounds for f_ρ in many cases, so that Theorem 1.3 immediately implies linear growth of f_ρ: When (S,ρ) has no punctures, there is a hyperbolic metric ρ_0 on S and an absolute constant C_1 so that any curve γ on S has degree $\deg(\gamma) \leq C_1 \cdot \ell(\gamma,\rho_0)$ [9, Thm. 1.1]. Since there is a Lipschitz map from (S,ρ) to (S,ρ_0), there is a constant $C(\rho)$ so that $f_\rho(L) \leq C_1 \cdot C(\rho) \cdot L$. In fact, the work of [7] sheds some light on the optimal Lipschitz constant $C(\rho)$ (see Lemma 4.1), and this argument shows that $f_\rho(L)/L$ is roughly $1/\mathrm{sys}(\rho)$ for L large. Precisely:

Corollary (Linear growth of f_ρ)**.** Suppose (S,ρ) is without punctures. There exists a constant $C = C(S)$ and $L_0 = L_0(\rho)$ so that for $L \geq L_0$ we have

$$\frac{1}{C\cdot \mathrm{sys}(\rho)} \leq \frac{f_\rho(L)}{L} \leq \frac{C}{\mathrm{sys}(\rho)}.$$

□

Proof. The lower bound follows from Theorem 1.3. In the compact case, the upper bound follows from [9, Thm. 1.1] together with Lemma 4.1. When S is noncompact but has no punctures, there is a compact convex core S^c of (S,ρ) that contains all closed geodesics. Let $(\hat{S},\hat{\rho})$ be the compact hyperbolic surface obtained by doubling S^c. Any geodesic of length at most L on (S,ρ) has length at most L on $(\hat{S},\hat{\rho})$, and a curve on $S^c \subset \hat{S}$ that lifts simply lifts simply to a cover of S with lesser or equal degree. ■

In the presence of punctures we are not aware of upper bounds for f_ρ. Note that Theorem 1.4 indicates that Patel's upper bounds cannot hold in the punctured setting: If (S,ρ) is a hyperbolic surface with a puncture, the minimal degree of a cover to which a given curve γ lifts to a simple curve cannot be bounded linearly in the curve's length $\ell(\gamma,\rho)$.

There are other avenues for further investigation. It would be natural to seek upper bounds for $f_S(n)$ (cf. [10, p. 15]), since no such bound follows from [9]. Finally, one could explore the set of curves of self-intersection number exactly n. For instance: Among the finitely many mapping class group orbits of curves γ with self-intersection n, which maximize $\deg(\gamma)$?

Outline of the paper

In §2 we introduce a sequence of curves $\{\gamma_n\}$ on a pair of pants and analyze the degrees $\deg(\gamma_n)$, in §3 we deduce Theorems 1.2, 1.3, and 1.4 as straightforward consequences, and in §4 we prove Lemma 4.1 needed for the proof of the corollary.

2 Analysis of a certain curve family

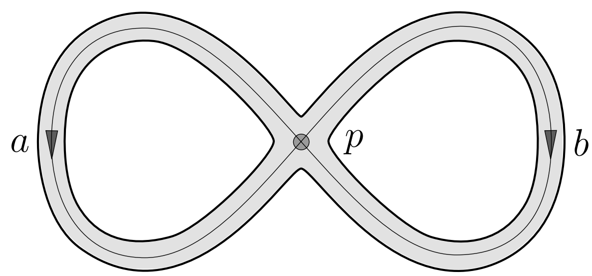

Fig. 2. The pair of pants P_0, with generators a and b.

Let P_0 be a pair of pants. Identify $\pi_1(P_0, p)$ with a rank-2 free group F, with generators a and b as pictured in Figure 2. Let γ_n indicate the closed curve given by the equivalence class of $a \cdot b^n$. The self-intersection of γ_n was analyzed by [2] (see Figure 1). We rely on Basmajian's result below, but the following lemma can also be obtained by invoking a variant of the 'bigon criterion' [3, §1.2.4] [4, Lemma 5.1].

Lemma 2.1. For $n \geq 0$, the curve γ_n has $\iota(\gamma_n, \gamma_n) = n$. □

Proof. Choose any complete hyperbolic structure ρ on P_0, and note that $\iota(\gamma_n, \gamma_n)$ is equal to the number of intersections for the ρ-geodesic representative of γ_n. The conclusion is now a consequence of [2, Lemma 4.1]. ■

We use Lemma 2.1 to estimate $\deg(\gamma_n)$, a calculation reminiscent of [6, Lemma 3.10]. The following proposition is the main tool in our analysis.

Proposition 2.2. We have $\deg(\gamma_n) \geq n + 1$. □

Proof. Towards contradiction, suppose there is a cover $P' \to P_0$ of degree $k' \leq n$, so that γ_n lifts to a simple curve γ. Draw P_0 as a directed ribbon graph with one vertex p and the two edges labeled by a and b, and P' as a directed ribbon graph with vertices $p_1, \ldots, p_{k'}$ and $2k'$ directed edges, k' with a labels and k' with b labels. Choose an orientation for γ so that γ consists of a directed a edge followed by n directed b edges. After relabeling, we may assume that the unique a edge of γ is followed by p_1.

Starting from p_1 there is a cycle of directed b edges of length $k \leq k'$. Relabel if necessary so that the vertices in this cycle are $p_1, \ldots, p_k$, in order. Let the vertex that immediately follows the n consecutive b edges of γ be p_l, where l is equivalent to $n+1$ modulo k. Finally, γ follows an a edge from p_l to p_1. See Figure 3 for a schematic.

This implies that there is an incompressible embedded pair of pants P'' in P' that contains γ (see Figure 4 in the case that $k \nmid n$ – the other case is straightforwardly similar). After identifying P'' with P_0 appropriately, the closed curve γ is given by the equivalence class of $a \cdot b^s$, where $s = \left\lfloor \frac{n}{k} \right\rfloor \geq \left\lfloor \frac{n}{k'} \right\rfloor \geq 1$. By Lemma 2.1 this curve is not simple, a contradiction. ■

Remark. In fact, it is not hard to show that $\deg(\gamma_n) = n + 1$: The subgroup generated by a^2, ba, b^{n+1}, and $b^k a b^{-k}$ for $k = 2, \ldots, n$ is index $n+1$ in F. With a helpful picture, one can check that the curve $ab^n = a^2 \cdot (ba)^{-1} \cdot b^{n+1}$ has a simple lift in the corresponding cover. However, our estimates in Theorems 1.2, 1.3 and 1.4 depend only on the inequality above, so we omit this remaining argument. It seems worthwhile to point out that this computation implies that our estimates cannot be improved without appealing to a different family of curves.

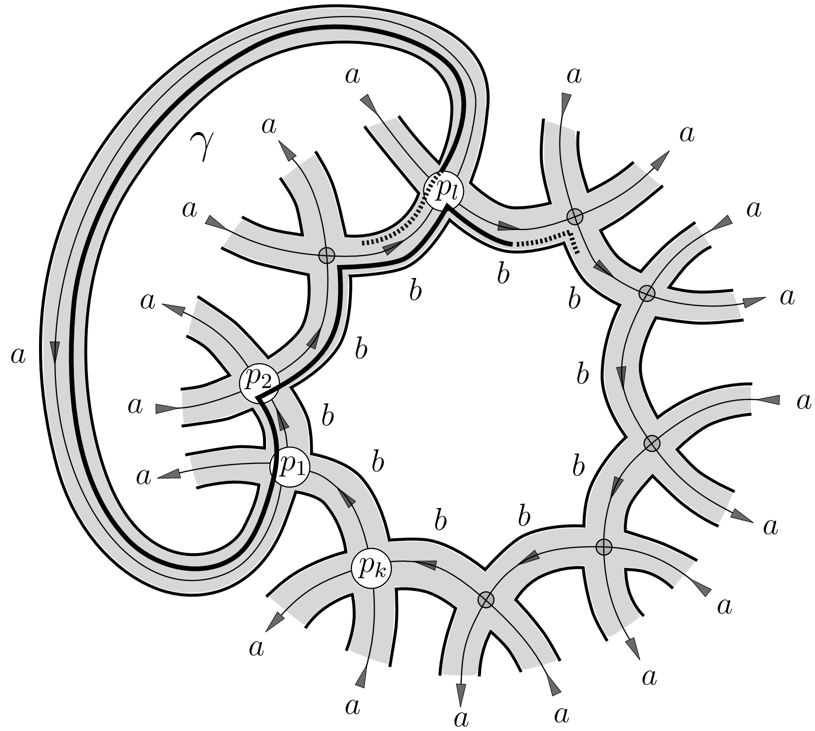

Fig. 3. A supposedly simple lift γ of γ_n to the cover $P' \to P_0$ follows a cycle of directed b edges.

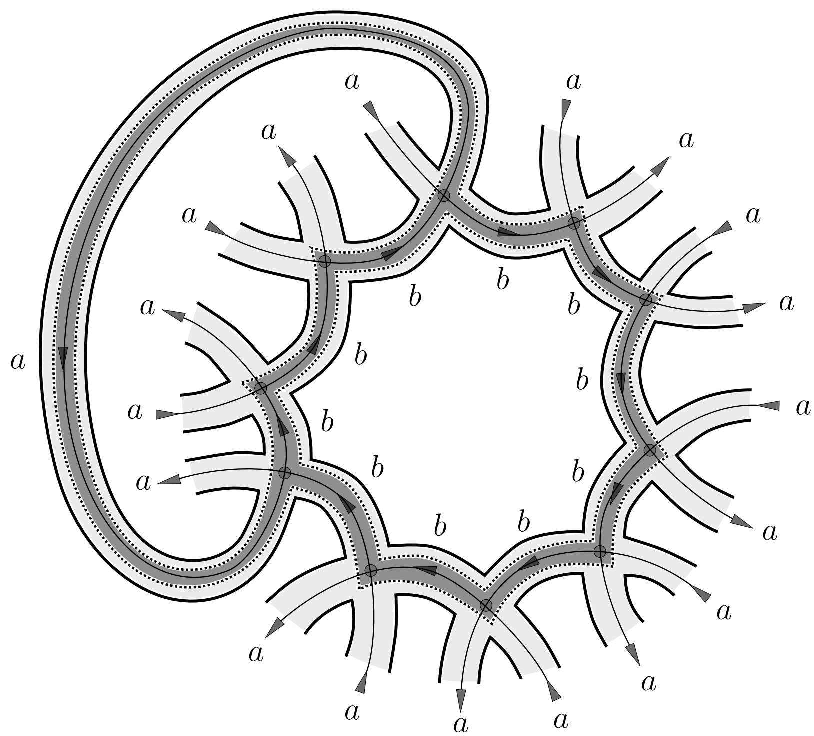

Fig. 4. The darkly shaded incompressible pair of pants $P'' \subset P'$ contains γ.

3 Proofs of Theorems 1.2, 1.3, and 1.4

Proof of Theorem 1.2. Since S has negative Euler characteristic, we may choose a marked embedded pair of pants $P_0 \hookrightarrow S$, so that we may view $\{\gamma_n\}$ as a sequence of closed curves on S. Suppose that $\pi : S' \to S$ is a cover of S so that γ_n lifts to a simple curve γ'. Let P' be the component of $\pi^{-1}(P)$ containing γ'. We obtain a cover $\pi|_{P'} : P' \to P$, so that the degree of $S' \to S$ is at least the degree of $P' \to P$. By Proposition 2.2, the degree of $P' \to P$ is at least $n+1$. Thus $\deg(\gamma_n) \geq n+1$, and the bound for $f_S(n)$ follows immediately from Lemma 2.1. ■

Proof of Theorem 1.3. Since (S, ρ) is without punctures, the systole of (S, ρ) is simple, whose free homotopy class we denote by β. Choose a representative α' of a simple closed curve α—possibly homotopic to a boundary component—disjoint from β, and a simple arc δ connecting α' to β. Let P be the pair of pants homotopic to a regular neighborhood of $\alpha' \cup \beta \cup \delta$.

Identify P with P_0 so that α is in the conjugacy class of a and β is in the conjugacy class of b, and consider the closed curves $\{\gamma_n\}$ in P. Evidently,

$$\ell(\gamma_n, \rho) \leq \ell(\alpha', \rho) + n \cdot \mathrm{sys}(\rho) + 2\ell(\delta, \rho).$$

Given $\epsilon > 0$, for large n we have $\ell(\gamma_n, \rho) \leq n \cdot (\mathrm{sys}(\rho) + \epsilon)$. Let

$$n = n(L) = \left\lfloor \frac{L}{\mathrm{sys}(\rho) + \epsilon} \right\rfloor,$$

so that $\ell(\gamma_n, \rho) \leq L$ for large enough L. Thus, for large enough L, we have

$$f_\rho(L) \geq \deg(\gamma_n) \geq n + 1 \geq \frac{L}{\mathrm{sys}(\rho) + \epsilon}$$

as desired. ■

Proof of Theorem 1.4. Let β indicate a simple closed curve homotopic to a puncture, and as above choose a representative α' of a simple closed curve α disjoint from β, and a pair of pants P with cuffs homotopic to α and β. Identify P with P_0 so that b is homotopic to β and a is homotopic to α, and consider again the sequence of curves $\{\gamma_n\}$ on S.

We assume the upper half plane model for the hyperbolic plane $\mathbb{H}^2$. By conjugating the holonomy representation of ρ appropriately, we may arrange for the holonomy around β to be the transformation $z \mapsto z + 1$, and so that there is a lift of α' to the hyperbolic plane $\mathbb{H}^2$ that contains a point on the imaginary axis, say is.

There is a lift of a curve freely homotopic to b^n that starts at is, travels vertically along the imaginary axis to iy, travels horizontally to $n + iy$, and vertically down to $n + is$. Let β'_y indicate the projection of this curve to P, and note that by construction its starting and ending point are in common, and on α'. We may thus concatenate (a parametrization of) α' with β'_y, and the curve so obtained is homotopic to γ_n.

An elementary computation shows that $\ell(\beta'_y, \rho) = 2\log(y/s) + n/y$. Taking $y = n$ we find

$$\ell(\gamma_n, \rho) \leq \ell(\alpha', \rho) + \ell(\beta'_n, \rho) \leq \ell(\alpha', \rho) - 2\log s + 1 + 2\log n.$$

Given $\epsilon > 0$, for large n the ρ-lengths satisfy $\ell(\gamma_n, \rho) \leq (2 + \epsilon)\log n$, and the result follows as in the proof of Theorem 1.3: Let

$$n = n(L) = \left\lfloor e^{\frac{L}{2+\epsilon}} \right\rfloor,$$

so that $\ell(\gamma_n, \rho) \leq L$ for large enough L. Thus, for large enough L, we have

$$f_\rho(L) \geq \deg(\gamma_n) \geq n + 1 \geq e^{\frac{L}{2+\epsilon}}$$

as desired. ■

4 The minimal Lipschitz constant and the systole function

We rely on the work of [7], rephrasing their results slightly to suit our needs. Let $\mathcal{T}(S)$ indicate the *Teichmüller space* of S, and recall that there is a forgetful map $\pi : \mathcal{T}(S) \to \mathcal{M}(S)$, where $\mathcal{M}(S)$ is the set of isometry classes of hyperbolic metrics on S. A *marking* of S is a set of simple closed curves on S forming a pants decomposition together with a set of transversals ('complete clean marking' in the language of [8, §2.5, p. 17], where this was introduced), and a *short marking* on $X \in \mathcal{T}(S)$ is a marking that is constructed by choosing short curves according to the greedy algorithm (see [7, p. 6] for detail).

Lemma 4.1. Suppose S is compact. For any pair of hyperbolic metrics $\rho, \rho_0 \in \mathcal{M}(S)$, there exists $C = C(\rho_0)$ so that the minimal Lipschitz constant from ρ to ρ_0 is at most $C/\mathrm{sys}(\rho)$. □

Proof. Lift ρ to $X \in \mathcal{T}(S)$, and choose a short marking μ_X of X. There are only finitely many mapping class group-orbits of markings. Each has representatives on (S, ρ_0) of shortest length, and the max over the lengths of these finitely many curves is a constant that depends only on ρ_0. Thus there exists $C_1 = C_1(\rho_0)$ and a lift $X_0 \in \pi^{-1}(\rho_0)$ so that

$$\max_{\alpha \in \mu_X} \ell(\alpha, X_0) \leq C_1.$$

Now [7, Thm. E] shows that there exists a constant C_2 so that there is a Lipschitz map from X to X_0, homotopic to the identity, with Lipschitz constant at most

$$C_2 \cdot \max_{\alpha \in \mu_X} \frac{\ell(\alpha, X_0)}{\ell(\alpha, X)} \leq \frac{C_1 C_2}{\mathrm{sys}(\rho)}.$$

■

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