Degree sequences.

The degree sequence of a graph \( G = (V, E) \) is the sequence of values \((\deg(v))_{v \in V}\).

1. Does there exist a graph with degree sequence \((2, 3, 4, 4, 4, 5)\)? What about \((2, 3, 3, 4, 4, 5)\)? In each case, either provide an example of a graph that works or prove that no such graph exists.

2. For which values \( n \) does there exist a simple graph with \( n \) vertices in which every vertex has a different degree? (A graph is simple if no two edges have the same pair of endpoints, and no edge is a loop, i.e. has both endpoints equal to the same vertex.) In this and all subsequent problems, prove that your answer is correct!

Connectivity.

3. Recall that a super-walk in a graph \( G = (V, E) \) is a walk \( w = (v_1, e_1, \ldots, v_n, e_n, v_{n+1}) \) that visits every vertex, i.e. \( \forall v \in V, \exists i \in \{1, \ldots, n+1\} \) such that \( v_i = v \). Prove that a graph contains a super-walk if and only if it is walk-connected. (Make sure to use precise language and terminology!)

Cycles and walks.

4. A walk \((v_1, e_1, v_2, e_2, \ldots, v_n, e_n, v_{n+1})\) is closed if \( v_1 = v_{n+1} \). Suppose that a graph \( G \) contains a closed walk with an odd number of edges. Prove that \( G \) contains a cycle with an odd number of edges. (Hint: use an extremal argument. Amongst all closed walks in \( G \) with an odd number of edges, consider one whose length is [censored].)

A puzzling party.

5. An eminent mathematician and her husband attended a dinner party also attended by four other couples. During the cocktail hour, some of those present shook hands, but in an unsystematic way, with no attempt to shake everyone’s hand. Of course, no one shook his or her own hand, no one shook hands with his or her spouse, and no one shook hands with the same person more than once. During dinner, the indulgent mathematician asked each of the nine other people present (including her own husband) how many hands that person had shaken. Under the given conditions, the possible answers ranged from zero to eight. She was excited to notice that each person gave a different answer: one person hadn’t shaken anyone’s hand, one person had shaken exactly one other’s hand, one had shaken exactly two hands, and so on, up to one person who had shaken eight hands, that is, all those of the others present, except his or her spouse.

How many hands did the mathematician’s husband shake?

\(\star\) Following problem (3) and our work in class, we have now shown the equivalence of three notions of connectivity: path-connectivity, walk-connectivity, and the existence of a super-walk. Recall that Edward declared that a graph \( G = (V, E) \) is disconnected if its vertex set admits a partition \( V = A \cup B, A, B \neq \emptyset \), with the property that there does not exist any edge \( e = (a, b) \in E \) with \( a \in A \) and \( b \in B \). Complete the theorem from class relating the four notions of connectivity by showing that a graph is not disconnected if and only if it is connected (i.e. it equivalently is path-connected, is walk-connected, or contains a super-walk).

\(\star\) A problem labeled (\(\star\)) indicates a bonus problem.