Geometric Ramsey Theory.

(1) An equilateral triangular array with base consisting of 13 points has its \( \binom{13+1}{2} \) points colored red and blue. Prove that it is possible to locate three points, all the same color, that form the vertices of an equilateral triangle oriented the same way as the full array.

(Hint: examine the base. Note that the 13 here is the same 13 as on the previous problem set.)

This equilateral triangular array of \( \binom{4+1}{2} \) points contains three blue points at the vertices of an equilateral triangle oriented the same way as the full array, as well as three blue points at the vertices of an equilateral triangle not oriented the same way as the full array.

(♠) What is the smallest value \( n \) that you can substitute for 13 and generate the same conclusion? What if we do not require the equilateral triangle to be oriented the same way as the full array?

Infinite Ramsey Theory.

Each part to the following problem should only require one to two sentences of explanation.

(2) Suppose that the edges of a complete graph on a countable vertex set \( V \) are colored red and blue. (For concreteness, you could imagine that \( V \) consists of the positive integers.)

(a) PROARTYMHSIC\(^1\), that for every \( n \geq 2 \), there exists a monochromatic copy of \( K_n \).

Now consider the following algorithm. Follow its steps in cyclic order, unless you are told to STOP.

- Suppose that \( V_k \) has been defined for some positive integer \( k \).
- If \( V_k = \emptyset \), then STOP.
- Select any vertex \( v_k \in V_k \).
- Let \( c_k \) denote the dominant color on the edges incident with \( v_k \) to the other vertices of \( V_k \).
- Let \( V_{k+1} \subset V_k \) denote the vertices connected to \( v_k \) by an edge of color \( c_k \).

(b) Explain why this algorithm never STOPs, hence returns an infinite sequence of distinct vertices.

(c) What is the color on edge \((v_i, v_j)\) for distinct vertices \( v_i, v_j \) in this sequence, and why?

(d) Conclude that there exists a monochromatic infinite complete subgraph.

(e) Contemplate the conclusions of parts (a) and (d). Does one imply the other?

Extremal graph theory.

(3) Suppose that \( t \) is a positive integer and \( G \) is a simple graph with \( n \) vertices \( \geq tn \) edges.

(a) Prove that the average vertex degree of \( G \) is \( \geq 2t \).

(b) Prove that \( n \geq 2t+1 \).

(c) Prove by induction on \( n \) that \( G \) contains an induced subgraph with minimum degree \( \geq t \).

(Hint: either \( G \) has a vertex of degree \( < t \) or it does not; in one case you are done, and in the other case you can induct.)

(4) Suppose that \( G \) is a simple graph with minimum degree \( t \). Prove that \( G \) contains a path of length \( (= \text{number of edges}) \geq t \). Prove, by way of example, that \( G \) need not contain a path of length \( t+1 \).

(♠) Suppose that \( G \) is a simple graph with minimum degree \( t \). Suppose that \( T \) is a tree with \( t \) edges. Prove that \( G \) contains a subgraph isomorphic to \( T \).

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\(^{1}\)Prove, relying on any result that you might have seen in class.
(5) Suppose that $G$ is a triangle-free simple graph with $n$ vertices. Let $v$ denote a vertex of $G$ of maximum degree, let $A = N(v)$ denote its set of neighbors, and let $B = V - A$.

(a) Explain why every edge in $G$ has an endpoint in $B$.

(b) Explain each step in the following sequence of (in)equalities:
\[
|E| \leq \sum_{v \in B} \deg(v) \leq |B| \cdot \Delta(G) = |B| \cdot |A|.
\]

(c) Using part (b) and one more step, give another proof of Mantel’s theorem.

**Ramsey number theory.**

(♠) Color the positive integer multiples of $5^0$ but not $5^1$ red, blue, yellow, and green in cyclic order; color the positive integer multiples of $5^1$ but not $5^2$ red, blue, yellow, and green in cyclic order; color the positive integer multiples of $5^2$ but not $5^3$ red, blue, yellow, and green in cyclic order; etc.:

\[
\begin{array}{cccccccccccccccc}
\hline
5 & 10 & 15 & 20 & 25 & 30 & \ldots
\end{array}
\]

Prove that there does not exist a monochromatic solution to $x + y = 3z$ in this 4-coloring of $\mathbb{Z}^+$. 