Extremal graph theory.

1. What, with proof, is the maximum number of edges in a simple graph $G$ on $n$ vertices with no path of length 3? Which graphs have the maximum number of edges? (Hint: consider a component of $G$ and whether or not it contains a cycle.)

2. Use Problem Set 10, no 5 to re-prove the characterization of equality in Mantel’s theorem: if $G = (V, E)$ is a simple, triangle-free graph with $n$ vertices and $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$ edges, then $G$ is isomorphic to $T^2_n$, the complete bipartite graph on $n$ vertices with parts of nearly equal size.

3. Suppose that $G = (V, E)$ is a simple graph with $n$ vertices and no copy of $K_{2,3}$.
   (a) Prove the inequality
   $$\sum_{v \in V} \binom{\text{deg}(v)}{2} \leq 2 \binom{n}{2}.$$
   (b) Deduce that $|E| \leq 2n^{3/2}$.

Combinatorial geometry.

Suppose that $S$ is a set of $n$ points in the plane $\mathbb{R}^2$.

4. Prove that the number of pairs of points that lie at a unit distance apart is at most $2n^{3/2}$.

5. Prove that the set of distances $\{d(x, y) \mid x, y \in S, x \neq y\}$ contains at least $\frac{1}{4} n^{1/2}$ different values.
   (It is fine if you only manage to prove a lower bound that is a little less than $\frac{1}{4} n^{1/2}$.)

For fun.

(*) Can you find a graph $G$ with more than one vertex and the property that $G$ is isomorphic to its complement? Can you find another? Can you find infinitely many?