Vertex coloring.

(1) (a) Suppose that the vertices of the graph $G$ are ordered $v_1, \ldots, v_n$ in such a way that vertex $v_i$ neighbors at most $k$ vertices amongst its predecessors $v_1, \ldots, v_{i-1}$, for $i = 2, \ldots, n$. Prove that $\chi(G) \leq k + 1$.

(b) Suppose that $G$ is a simple graph with maximum degree $\Delta$, and it contains $\leq \Delta$ vertices with degree $\Delta$. Prove that $\chi(G) \leq \Delta$. (Hint: put the vertices of $G$ into a special order.)

(2) Suppose that $\chi(G) = k$ and that $G$ has been properly vertex-colored using $k$ colors.

(a) Prove that for every pair of distinct colors, there exists an edge whose endpoints receive that pair of colors.

(b) What, with proof, is the fewest number of edges in a simple graph of chromatic number $k$?

Kneser graphs.

The Kneser graph $KG(n, 2)$ is defined as follows. Its vertex set consists of all 2-element subsets of the numbers $1, 2, \ldots, n$. Its edge set consists of all pairs of disjoint 2-element subsets.

(3) (a) Determine, with proof, a famous graph that $KG(5, 2)$ is isomorphic to.

(b) Prove that $KG(n, 2)$ is isomorphic to $\overline{L(K_n)}$.

(4) Prove that $\chi(KG(n, 2)) \leq n - 2$.

(♠) Prove that $\chi(KG(n, 2)) = n - 2$.

Ramsey theory.

(5) Suppose that $n \geq 3$ and the edges of $K_n$ are colored red and blue. Prove that there exists a Hamiltonian cycle which is either monochromatic or consists of two monochromatic paths.

Tournaments.

(♠) A round-robin tournament of $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the $n$ games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?