Pigeonhole.

1. A $2n \times 2n$ checkerboard is tiled by dominoes.
   a. If $n$ is even, explain how to tile the checkerboard in such a way that exactly half of every row
      and column is covered by dominoes contained entirely within it.
   b. If $n$ is odd, prove that there exists a row or a column such that more than half of its squares
      are covered by dominoes contained entirely within it.
   c. If $n$ is odd, prove that there exists a row or a column such that more than half of its squares
      are covered by dominoes not contained entirely within it.

2. 21 boys and girls line up in 3 rows of 7 children apiece. Prove that some four children of the same
   gender stand at the corners of a rectangle.
   (The sides of the rectangle should be parallel to the rows and the columns of the children.)
   (Hint: consider the dominant gender in each column.)

Ramsey theory.

3. a. Suppose that the edges of $K_{17}$ are colored red, yellow, and blue. Prove that there exists a
    monochromatic triangle.
   b. Let $r(n)$ denote the smallest positive integer – if it exists! – with the property that in any
      $n$-coloring of the edges of $K_{r(n)}$, there exists a monochromatic triangle. Prove that $r(n)$ exists
      for all $n \geq 1$ by showing that $r(n+1) \leq (n+1)(r(n) - 1) + 2$.
      (You can do the two parts out of order, if you like. They are in the order they are in order
      to get you warmed up.)

4. Prove that $R(p,q) \geq (p-1)(q-1) + 1$.

Ramsey number theory.

5. a. Show how to color the numbers from 1 to 8 red and blue so that there does not exist a
    monochromatic three term arithmetic progression.
   b. Show that no matter how the numbers from 1 to 13 are colored red and blue, there exists a
    monochromatic three term arithmetic progression. (Hint: separate two cases by considering
    whether 5 and 9 get the same color or not.)

6. Consider the 2-coloring of the positive integers that begins
   $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, \ldots$
   Prove that there does not exist a monochromatic infinite arithmetic progression.

   ♠ Find, with proof, a 2-coloring of the positive integers with the property that there does not exist
   a monochromatic infinite arithmetic progression and, moreover, no three consecutive numbers get
   the same color.

   Geometric Ramsey theory.

7. Prove that $\chi(\mathbb{R}^2) \leq 7$. 