First, here is a list of a dozen important results from class that you should know how to prove directly. You may expect two of them to appear more or less verbatim on the final exam.

**Trees.**

1. Prove that every tree besides the stump contains a leaf.
2. Prove that every tree with \( n \) vertices contains \( n - 1 \) edges. (You may use a result we proved about trees that does not rely on this result.)

**Eulerian circuits.**

3. Prove that if \( G \) is a graph in which every vertex has even degree, then there exists a collection of edge-disjoint cycles that uses every edge in \( G \) exactly once. (You may not assume any part of Euler’s theorem. You may assume any results that we have proven about connectivity and trees.)

**Hamiltonian cycles.**

4. Prove that if \( G = (V,E) \) is a simple graph with \( n \) vertices and \( \deg(v) \geq n/2 \) for all \( v \in V \), then \( G \) is connected. (You may not use Dirac’s theorem. You may use any notion of connectivity.)

**Vertex coloring.**

5. Prove that a connected graph is bipartite if it does not contain any odd cycles. (You may assume a result from homework relating the existence of an odd closed walk to an odd cycle.)

**Perfect matching.**

6. State Hall’s theorem. Suppose that \( G = (V,E) \) has bipartition \( V = A \sqcup B \) with \( |A| = |B| \). Prove that \( |N(S)| \geq |S| \) for all \( S \subset A \) if and only if \( |N(T)| \geq |T| \) for all \( T \subset B \).

**Stable assignments.**

7. Define what a stable matching is, stating all of the necessary hypotheses. State the Gale-Shapley algorithm. Prove that the matching output by the Gale-Shapley algorithm is a stable matching.

**Chromatic index.**

8. Define chromatic index. Prove that \( \chi'(G) \geq \frac{|E(G)|}{m(G)} \). Determine \( \chi'(K_{2n+1}) \), with proof. (Your coloring can be defined with the same level of rigor as we gave in class, but please explain it clearly.)

**Ramsey theory.**

9. Define the Ramsey number \( R(s, t) \). Prove that \( R(3, 4) = 9 \). (You may assume the value of \( R(3, 3) \).)

10. Prove Schur’s theorem: in any \( n \)-coloring of the positive integers \( 1, \ldots, 3n! - 1 \), there exists a monochromatic solution to the equation \( x + y = z \). (You may assume that in any \( n \)-coloring of the edges of \( K_r \), \( r = 3n! \), there exists a monochromatic triangle.)

**Extremal graph theory.**

11. Define the Turán graph \( T_n^3 \). Prove that \( |E(T_n^3)| = \binom{n}{2} + 2(n - 3) + |E(T_{n-3}^3)| \) for all \( n \geq 4 \). Suppose that \( G = (V,E) \) is a simple graph, \( |V| = n \), and \( G \) does not contain a copy of \( K_4 \). Prove that \( |E| \leq |E(T_n^3)| \).

12. Suppose that \( G = (V,E) \) is a simple graph.
   (a) Prove that \( \sum_{(x,y) \in E} \deg(x) + \deg(y) = \sum_{v \in V} \deg(v)^2 \).
   (b) Prove that \( \sum_{v \in V} \deg(v)^2 \geq 4|E|^2/|V| \).
   (c) Prove that \( \sum_{v \in V} \left( \frac{\deg(v)}{2} \right) \) equals the number of copies of the two-edge path in \( G \).
Second, here is a list of a half dozen additional practice problems. You may or may not see them on the final exam, but they are good practice, nevertheless.

(1) A square is removed from a 5 \times 5 board. Determine, with proof, when the remaining squares can be tiled by non-overlapping 1 \times 3 trominoes.

(2) Suppose that \( G = (V, E) \) is a simple graph and \( A \subseteq V \) is an independent set with cardinality \( \alpha(G) \).
   (a) Let \( G - A \) denote the subgraph of \( G \) induced on \( V - A \). Prove that \( \Delta(G - A) < \Delta(G) \).
   (b) Using part (a), prove that \( \chi(G) \leq \Delta(G) + 1 \) by induction on \( \Delta(G) \).

(3) Suppose that \( G \) is a simple graph with the property that every induced subgraph of \( G \) contains a vertex of degree 5 or less. Prove that \( \chi(G) \leq 6 \). Give an example where \( \chi'(G) = 10^{100} \).

(4) (a) What, with proof, is the maximum number of edges in a simple graph \( G \) on \( n \) vertices with no copy of \( K_{1,3} \)?
   (b) For each value \( n \), what are all of the simple graphs on \( n \) vertices with no copy of \( K_{1,3} \) and the maximum number of edges?
   (c) What, with proof, is the maximum number of edges in a simple graph \( G \) on \( n \) vertices with no copy of \( K_{1,4} \)?

(5) Suppose that \( G = (V, E) \) is a simple graph that does not contain a copy of \( K_{17,29} \). Prove that \( \sum_{v \in V} \binom{\deg(v)}{17} \leq 28 \binom{n}{17} \) and \( \sum_{v \in V} \binom{\deg(v)}{29} \leq 16 \binom{n}{29} \). Which one do you think leads to a better upper bound on \( |E| \), and why? (You do not need to give a rigorous argument for this last part; if you are curious, then you might look up Jensen’s inequality.)

(6) A magician takes all of the twiddle cards from a SET deck, for a total of 27 distinct cards. An audience member selects four cards at random and hands them to the magician. The magician then conceals one of the cards and displays the other three in a carefully chosen order. The magician’s assistant, who has been sequestered, then views the three ordered cards, and from them – and them alone! – she is able to identify the concealed card.
   (a) Explain why such a trick exists, without actually devising one.
   (♣) Devise one!
   (b) Prove that no such trick exists (without mind-reading!) if the magician instead uses 28 cards.