MT 832: Combinatorial methods in low-dimensional topology
Lecture 11: Combinatorial Optimization, part 2

We are about halfway to establishing the second pillar in the solution to the knot complement problem. Recall that we are working with the following hypothesis:

\[(\dagger) \ A \subset \mathbb{Z}^n, \text{ and for every subset } A_0 \subset A, \text{ the quotient } \mathbb{Z}^n / \langle A_0 \rangle \text{ is torsion-free.}\]

We produced from \(A\) a coefficient matrix \(M = (c_{ij})\) by writing \(A = \{a_1, \ldots, a_m, \ldots\}\), where \(a_1, \ldots, a_m\) is a maximum linearly independent set of elements in \(A\), and expressing each \(a_i\) in terms of the basis \(a_1, \ldots, a_m\): 

\[a_i = \sum_{j=1}^m c_{ij} a_j, \ c_{ij} \in \mathbb{Z}. \]

As we showed, condition \((\dagger)\) translates into the total unimodularity of \(M\):

\[(\diamond) \text{ every square submatrix of } M \text{ has determinant } \pm 1 \text{ or } 0.\]

Here is a simple example of a class of TU matrices, just to get some experience with them. Take any directed graph \(G = (V,E)\). Form a matrix \(M\) with columns indexed by vertices \(V\) and rows indexed by edges \(E\). For a vertex \(v \in V\) and edge \(e \in E\), the \((v,e)\) entry of \(M\) is \(+1\) if \(e\) orients out of \(v\), \(-1\) if \(e\) orients into \(v\), and \(0\) otherwise. Thus, \(M\) is the vertex-edge incidence matrix of \(G\).

For instance, consider the graph displayed here:

We obtain the matrix displayed here:

\[
\begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & -1 & 0 & 1 \\
1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0
\end{pmatrix}
\]

We claim that such a matrix \(M\) is TU. We prove by induction on \(k\) that all \(k \times k\) minors of \(M\) are \(\pm 1\) or \(0\). For \(k = 1\), this is immediate, so suppose that \(k > 1\) and the assertion holds for values less than \(k\). Consider a \(k \times k\) submatrix. If it has a zero row, then its determinant is \(0\). If it has a row with a single non-zero entry, then its determinant is the same, up to sign, as that of the \((k-1) \times (k-1)\) minor obtained by striking out that row and the column containing its single non-zero entry; and this is \(\pm 1\) or \(0\) by induction. Otherwise, every row has a single \(+1\) entry, a single \(-1\) entry, and every other entry \(0\). Then the all-1’s vector is in the null-space of the submatrix, so its determinant is \(0\). As an exercise, you can try to
characterize which submatrices have non-zero determinant in this class of examples. There is
a beautiful structure theorem for TU matrices due to Seymour, but it is beyond our scope to
pursue.

Totally unimodular matrices admit several “cryptomorphic” definitions. Some of them are
contained in the following theorem; the last condition is the one that leads to the desired end.

**Theorem 0.1.** The following conditions on an integral matrix $M$ are equivalent:

1. $M$ is totally unimodular;
2. the vertices of $\{0 \leq x, Mx \leq b\}$ are integral for every integral $b$;
3. the vertices of $\{a \leq x \leq b, c \leq Mx \leq d\}$ are integral for every integral $a, b, c, d$; and
4. for every set of the columns of $M$, it is possible to partition them into two parts so
that the sum of the columns in one part is nearly equal to the sum of the columns in
the other part, i.e. their difference is a vector with entries $\pm 1$ and 0.

We will just prove that total unimodularity implies the other three conditions, as that is all
we need for our application. We more or less proved $(1) \implies (2)$ in the last lecture. We explain
below how to derive it using what we did. The equivalence $(1) \iff (2) \iff (3)$ is due to
Hoffman and Kruskal (1956). The equivalence $(1) \iff (4)$ is originally due to Ghouila-Houri
(1960). The implication $(1) \implies (4)$ was rediscovered by Parry (1990). The proof we give
of $(1) \implies (4)$ follows Chapter 19 of Schrijver’s excellent book Theory of Linear and Integer
Programming. I find it very illuminating, using the link through polyhedral combinatorics
implicit in (2) and (3).

Let us reflect for a moment on what (4) says about the simple class of TU matrices we
introduced earlier. Given the full set of columns of $M$, how do we split them into two parts
with the desired property? In this case, it is easy, since the total column sum is 0: take one of
the parts to be empty. What about an arbitrary subset of columns of $M$? Again, it is easy,
since the sum of any subset of columns of $M$ is a vector with entries $\pm 1$ and 0: once again,
take one of the parts to be empty.

For a more interesting example, note that the transpose of a TU matrix is again TU. Thus,
condition (4) can be altered into condition $(4)'$ by replacing “columns” with “rows”. Given
the full set of rows of $M$, how do we split them into two parts with the desired property?
For the full row sum, the entry corresponding to a given vertex is the sum of the number of
edges oriented out of it minus the number of edges oriented into it. That is, the total row sum
is the signed vertex degree sequence of the graph $G$. What $(4)'$ asserts is that it is possible
to partition the edges of $G$ into two parts in such a way that the subgraph on the edges in
either part has signed vertex degree sequence equal to nearly half of the signed vertex degree
sequence of $G$. For the example given above, we get the signed degree sequence $(3, -2, -1, 0)$.
Thus, we are looking for a subset of the edges with signed degree sequence $(1$ or 2, $-1$, $-1$ or
0, 0). By inspection, the subgraph on the edges $e_1, e_4$ has the desired property. Thus, we can
take one part of the partition to be $\{e_1, e_4\}$ and the other to be $\{e_2, e_3, e_5\}$. As an exercise,
you might consider how to do this in general in graph theoretic terms.

*Proof of parts of Theorem 0.1.* $(1) \implies (2)$ The two conditions $0 \leq x$ and $Mx \leq b$ can be
collected into one: \( \begin{pmatrix} M \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix} \). Here \( I \) denotes the identity matrix with the same number of columns as \( M \). The matrix \( \begin{pmatrix} M \\ -I \end{pmatrix} \) is TU, assuming that \( M \) is: given any \( k \times k \) submatrix of it, it has the same determinant, up to sign, as the sub-submatrix gotten by striking out its rows and columns that contain a \(-1\) entry of the matrix \(-I\); this sub-submatrix is a submatrix of \( M \), possibly augmented by 0 rows, so it has determinant \( \pm 1 \) or 0. Now we can apply the theorem from the last lecture to the matrix \( \begin{pmatrix} M \\ -I \end{pmatrix} \) to deduce (2).

(1) \( \implies \) (3) This falls to a similar trick as above. We form a matrix \( M' \) by stacking \( M \), \(-M \), \( I \), and \(-I \) on top of one another, in order, and a vector \( b' \) by stacking \( d \), \(-c \), \( b \), and \(-a \) on top of one another, in order. The polyhedron of (3) is then expressed as \( \{ M'x \leq b' \} \). The matrix \( M' \) is TU, assuming \( M \) is, by a similar argument as above. Therefore, the vertices of the polyhedron are integral, by the theorem from the last lecture.

(1) \( \implies \) (4) Choose a subset of the columns of \( M \). Let \( b \) denote the \( 0/1 \) indicator vector whose \( j \)-th entry is 1 if the \( j \)-th column of \( M \) is in the subset and 0 if not. Then the sum of the columns in the subset is simply \( Mb \). To obtain the desired partition, we seek to write \( b = x + y \), where each of \( x \) and \( y \) is a \( 0/1 \) vector, such that \( Mx \) and \( My \) are each nearly half of \( Md \): that is, \( \lfloor \frac{1}{2} Mb \rfloor \leq Mx \leq \lceil \frac{1}{2} Mb \rceil \) (note that the corresponding condition on \( My \) follows from this one). Then the partition of the subset of columns is given by \( \{ \text{supp}(x), \text{supp}(y) \} \). Thus, we seek an integer vector in \( P = \{ 0 \leq x \leq b, |\lfloor \frac{1}{2} Mb \rfloor| \leq Mx \leq |\lceil \frac{1}{2} Mb \rceil| \} \). This is nearly guaranteed by condition (3). However, we must verify that \( P \) contains a vertex! To show this, first observe that \( P \) is non-empty, since it contains \( \frac{1}{2} b \). Second, \( P \) is compact, thanks to the constraint \( 0 \leq x \leq b \). This vertex is integral by (3), so it leads to the desired point \( x \). \( \square \)

Using Theorem 0.1(1) \( \implies \) (4), we shall deduce the following lemma of Parry:

**Theorem 0.2.** Suppose that \( \mathbb{Z}^n \) has a preferred basis, \( A \subset \mathbb{Z}^n \), and for every subset \( A_0 \subset A \), the quotient \( \mathbb{Z}^n/\langle A_0 \rangle \) is torsion-free. Then \( \mathbb{Z}^n \) contains a balancing characteristic element \( \chi \) for \( A \): \( \chi \in \text{Char}(\mathbb{Z}^n) \), and \( |\chi \cdot a| \leq 1 \) for all \( a \in A \).

This is not Parry’s language: I am just trying to dress it up a bit, to give it context. Here \( \text{Char}(\mathbb{Z}^n) \) denotes the set of characteristic elements for \( \mathbb{Z}^n \) with respect to the inner product coming from the preferred basis \( e_1, \ldots, e_n \) for \( \mathbb{Z}^n \). In prosaic terms, this is the set of elements \( \sum_{i=1}^n \chi_i e_i \), where \( \chi_i \in \mathbb{Z} \) is odd for \( i = 1, \ldots, n \). The terminology comes from lattice theory. If \( L \) is an integral lattice, then it has a dual lattice \( L^* = \{ x \in L \otimes \mathbb{Q} \mid x \cdot \lambda \in \mathbb{Z}, \forall \lambda \in L \} \). Within the dual lattice sits a distinguished coset of \( 2L \), the set of characteristic elements: \( \text{Char}(L) = \{ \chi \in L^* \mid \chi \cdot \lambda \equiv \lambda \cdot \lambda \text{ (mod 2)}, \forall \lambda \in L \} \). For instance, for the \( E_8 \) lattice, every element has an even self-pairing, so the 0 vector is a characteristic element. (In general, a lattice is even if and only if 0 is a characteristic element.) For the case of \( L = \mathbb{Z}^n \) equipped with the standard Euclidean inner product that we have been using, \( \text{Char}(\mathbb{Z}^n) \) takes the form described above, as you can check.

Characteristic elements of lattices frequently arise in low-dimensional topology, essentially as incarnations of integral lifts of second Stiefel-Whitney classes. Their self-pairings come
about in the grading shift formulas in Floer homology. It is curious that they appear here, and I am not sure what significance to ascribe to them. I just mention all this to give a little surrounding context and an air of mystery.

The term balancing is used to mean that \( \chi \) “balances” \( A \), in some sense: the elements of \( A \) are nearly orthogonal to \( \chi \), meaning that they are contained in the slab \( \{ \lambda \in L \mid |\chi \cdot \lambda| \leq 1 \} \) concentrated nearby the hyperplane \( \chi^\perp \) (it is the union of this hyperplane and its two closest translates that contain points of \( L \)).

Let us see how Theorem 0.2 implies our target theorem: “all types implies torsion”.

**Proof that all types implies torsion.** We establish the contrapositive. Recall that the contrapositive asserts that under the hypotheses of Theorem 0.2, either \( A \) avoids some type, or else \( A \) contains 0 or a unit vector (i.e. \( \pm \) a preferred basis element). Apply Theorem 0.2 to produce a balancing characteristic covector \( \chi \). Note that each entry of \( \chi \) is non-zero.

Let \( \tau \) denote the sign vector of \( \chi \), noting that it is well-defined: \( \tau = \sum_{i=1}^{n} \tau_i e_i \), where \( \tau_i = \chi_i/|\chi_i| \in \{\pm 1\} \). We claim that \( \tau \) is an avoided type, or else \( A \) contains 0 or a unit vector. Thus, suppose that \( a \in A \) represents the type \( \tau \); we seek to show that \( a \) is 0 or a unit vector. Write \( a = \sum_{i=1}^{n} a_i e_i \). Then \( a_i \tau_i = \epsilon |a_i| \), where \( \epsilon \) is a fixed sign \( \pm 1 \) independent of \( i \). Thus, \( a \cdot \chi = \sum_{i=1}^{n} a_i \chi_i = \sum_{i=1}^{n} a_i \tau_i |\chi_i| = \epsilon \cdot \sum_{i=1}^{n} |a_i||\chi_i| \). Taking the absolute value, \( |a \cdot \chi| = \sum_{i=1}^{n} |a_i||\chi_i| \). Since \( |\chi_i| \geq 1 \) for all \( i \), we obtain \( |a \cdot \chi| \geq \sum_{i=1}^{n} |a_i| \). By the balancing condition, we obtain \( 1 \geq \sum_{i=1}^{n} |a_i| \). Hence \( a \) is either 0 or a unit vector. This completes the proof. \( \square \)

We will review a bit and quickly derive Theorem 0.2 from Theorem 0.1 in the next lecture, rounding out the proof of the second pillar.