We turn now to the third pillar in the proof of the knot complement problem:

**Theorem 0.1** (Graph theory theorem). Suppose that \((G_P, G_Q)\) is a pair of graphs of intersection. Then either \(G_P\) represents all types, or else \(G_Q\) contains a schycle.

In a sense, this is the heart of Gordon and Luecke’s argument. It will take us several lectures to prove it, acquiring some useful tools along the way. With any luck, by the time you see the Distinguished Lecturer, you will more or less know the solution of the knot complement problem.

First, let us recall the meaning of the terms. \(G_P\) and \(G_Q\) are both fat labeled graphs on a sphere, \(\hat{P}\) and \(\hat{Q}\), respectively. The spheres are oriented, and \(P\) and \(Q\) are the complements of the fat vertices on their respective spheres. They inherit orientations from their spheres, and the boundaries of fat vertices inherit orientations from \(P\) and \(Q\) by the outward normal convention. The fat vertices of \(G_P\) are labeled \(1, \ldots, p\) and the fat vertices of \(G_Q\) are labeled \(1, \ldots, q\). The edge endpoints at a fat vertex of \(G_P\) are labeled \(1, \ldots, q\), repeated \(\Delta\) times around. A vertex is positive if these labels appear in reverse order in the orientation of the boundary of the vertex, oriented as the boundary of the planar surface \(P\). It is negative if they occur in forward order. For the pictures that we will draw, the surface \(P\) will be drawn in the plane of the page, and it will always be oriented counterclockwise. When we examine the boundary of a fat vertex, it will be oriented clockwise. Thus, a positive vertex has its labels increase (mod \(q\)) in counterclockwise order, and vice versa at a negative vertex. Similar remarks pertain to \(G_Q\), with \(p\) and \(q\) switched. We have an (endpoint, label) reversing bijection \(f : E(G_P) \to E(G_Q)\), and it obeys the parity rule: an edge \(e \in E(G_P)\) joins vertices of the same sign if and only if \(f(e)\) joins vertices of opposite signs. The pair is essential if there is no inessential loop in either graph; this assumption implies that any schycle has more than one side, i.e. it is not a unischycle.

Suppose that \(F\) is a face of \(G_P\), oriented by the orientation on \(P\). A component of the boundary \(\partial F\) alternately cycles through corners of fat vertices and edges of \(G_P\). Each corner is labeled \((i, i + 1)\) for some \(i = 1, \ldots, q\), indexing (mod \(q\), and at such a corner, \(\partial F\) either runs from it from \(i\) to \(i + 1\) or vice versa. The face \(F\) is coherent (or homogeneous) if, for all \(i\), every \((i, i + 1)\) corner that \(\partial F\) runs over it does so with the same sign. Recall that, in any event, we record the vector \(r(F) \in \mathbb{Z}^q\) that records, in the \(i\)-th coordinate, the signed number of \((i, i + 1)\) corners that \(\partial F\) runs across. As we have seen, when \(G_P\) and \(G_Q\) come from planar surfaces in a knot exterior, we can identify \(\mathbb{Z}^q\) with \(H_1(V_\beta \cup Q)\), and it has a preferred basis coming from the given handle decomposition of this space. If \(F\) is a face, then \(r(F)\) is the homology class of \(\partial F\) expressed in this basis. To say that \(G_P\) represents a type means that it has a homogeneous disk face such that \(r(F)\) represents that type in \(\mathbb{Z}^q\). As we have seen, if \(G_P\) represents all types, then \(H_1(K(\beta))\) contains non-trivial torsion: this is a combination of the second pillar and a little argument involving homology and 3-manifold topology.

We have seen two interesting examples of essential pairs of intersection graphs so far. One comes from the cabling construction:
In this example, we see four 3-sided schycles in $G_P$. All faces of $G_Q$ are disks and are homogeneous. We know that $G_Q$ does not represent all types because $K(\alpha) \approx S^3$ does not contain torsion in $H_1$. Let us certify this directly. The graph $G_Q$ represents the types that are the cyclic shifts of the sign vector $+000-00000000$ and their negatives. It does not represent all types: it avoids the $2^4$ types that have the same sign in the $i$-th, $(i + 4)$-th, and $(i + 8)$-th coordinates, $i = 1, 2, 3, 4$. In particular, it avoids the trivial type $++++++++++++++$.

Here is another example, which corresponds to a bireducible surgery:

Neither graph contains a schycle. The graph theory theorem implies that both must represent all types. Do you see that this is the case?
En route to the graph theory theorem, we will begin by proving the following variant, originally proven in the context of the cyclic surgery theorem. It makes a stronger hypothesis and generates a correspondingly stronger conclusion:

**Theorem 0.2 (Culler-Gordon-Luecke-Shalen).** Suppose that \((G_P, G_Q)\) is a pair of intersection graphs and \(\Delta \geq 2\). Then one of \(G_P\) and \(G_Q\) contains a schycle.

It follows from this result that if \(K\) is a non-trivial knot in \(S^3\), then it has at most one non-trivial surgery to \(S^3\). For otherwise we have distinct slopes \(1/0, 1/m, 1/n\) (numerators all 1, by homology) such that surgery along \(K\) with any of these slopes gives \(S^3\). Two of 0, \(m, n\) are at least two apart, and the corresponding slopes have distance at least two. We get a pair of essential intersection graphs corresponding to these slopes, by the first pillar. One contains a schycle, by Theorem 0.2, but this schycle has \(s \geq 2\) sides, leading to a lens space summand of order \(s\) in \(S^3\), a contradiction.

The proof we give of Theorem 0.2 is based on a way to assign an index to certain faces of an intersection graph. It is different from the way that CGLS originally did it, which is arguably more elementary. However, I think the argument we will give here is a nice precursor to some of the ideas to come in the proof of Theorem 0.1.

We call an edge in an intersection graph \(G\) pure if its endpoints have the same sign and mixed otherwise. Similarly, we call a face of \(G\) pure if all of its vertices have the same sign. These are the faces to which we will assign an index. Note that faces do not have to be simply connected, so a pure face with disconnected boundary has to have all vertices the same sign, even on different boundary components. Note that by the parity rule, \(e \in E(G_P)\) is pure if and only if \(f(e) \in E(G_Q)\) is mixed, and vice versa.

Consider a pure face \(F\) of \(G = G_P\). Assume that all of the vertices of \(F\) are positive. Select a boundary component \(\gamma\) of \(F\), oriented as \(\partial F\). Thus, \(\gamma\) will appear oriented counterclockwise in the plane of the page when we draw pictures. It alternately consists of corners of fat vertices and edges. We will define a map from \(\gamma\) to the circle \(S^1\). It is built out of a map from the boundary components of \(P\) and oriented edges of \(G\) to \(S^1\), so we describe these first. First, we place a counterclockwise orientation on \(S^1\), and we affix labels 1,...,\(q\) to it in counterclockwise order. For emphasis, as we traverse our copy of \(S^1\) according to its orientation, then its labels cycle through 1,...,\(q\) in that order, whereas if we traverse the boundary of a positive vertex according to the orientation on \(\partial P\), then its labels cycle through the order 1,...,\(q\) (\(\Delta\) times) in reverse. Consider an oriented arc of the boundary of a positive vertex that runs from label \(i+1\) to label \(i\) and avoids all other labels in its interior (i.e. a corner; the second constraint is redundant if \(\Delta = 1\)). We map each endpoint of the arc to the point of \(S^1\) with the same label, and we map the arc between them to the arc of \(S^1\) that runs clockwise between these labels (thereby avoiding all other labels in its interior). This map therefore reverses orientation. Next, for each positive edge of \(G\) (i.e. pure with positive endpoints), we orient it somehow, and then we map it to \(S^1\) by a one-to-one, orientation-preserving map that sends its endpoints onto the points of \(S^1\) with the corresponding labels. Notice that the labels at the endpoints of a pure edge are different, by a special case of the parity rule, so this map is actually one-to-one and not degenerate. To stress, an edge of \(G\) needs an orientation before we can say how to map it to \(S^1\). Moreover, if we fix a positive edge of \(G\) and consider its two different orientations,
then they map onto complementary arcs of $S^1$. The map from $\gamma$ to $S^1$ is now the concatenation of the maps from its corners and (oriented!) edges into $S^1$. As such, it has a degree, or winding number. We denote this value $\deg(\gamma)$, and we let $\deg(\partial F) \in \mathbb{Z}$ denote the sum of $\deg(\gamma)$ over all boundary components $\gamma$ of $F$. The index of $F$ is $\text{ind}(F) = \chi(F) - \deg(\partial F) \in \mathbb{Z}$.

Next time we will see the proof of the following result, which you will do for homework:

**Proposition 0.3** (Schycle Detection). For a positive face $F$, we have $\text{ind}(F) \in \mathbb{Z}$ and $\text{ind}(F) \leq 1$, with equality iff $F$ is a schycle.