In this lecture, we develop the three pillars supporting Gordon and Luecke’s proof of the Dehn surgery characterization of $S^3$:

**Theorem 0.1.** Non-trivial surgery along a non-trivial knot in $S^3$ is not homeomorphic to $S^3$.

First, we explain a deceptively short proof of this result using Floer homology, dictated in class by Siddhi Krishna.

**Floer homology proof of Theorem 0.1:** Suppose that $K$ is a knot in $S^3$ and $\alpha$ is a non-trivial slope for which $K(\alpha) \approx S^3$. In the coordinates provided by $K$, we write $\alpha$ in the form $p/q$, $q \neq 0$, and $p = |H_1(K(p/q);\mathbb{Z})| = |H_1(S^3;\mathbb{Z})| = 1$. We are saying that $K$ has a non-trivial surgery to $S^3$, which is an L-space. Since $K(p/q) = -\overline{K}(-p/q)$, where $\overline{K}$ denotes the mirror image, it follows that by mirroring $K$, we may assume that $1/q > 0$. Kronheimer-Mrowka-Ozsváth-Szabó proved that if $K$ admits a positive surgery to an L-space (in Seiberg Witten or Heegaard Floer homology), then the surgery slope is at least $2g(K) - 1$, where $g(K)$ denotes the Seifert genus. Hence $1 \geq 1/q \geq 2g(K) - 1$, so either $g(K) = 0$, and $K$ is the unknot, or else $g(K) = 1$ and $q = 1$. Ghiggini proved that a genus-1 knot with an L-space surgery is fibered. Burde-Zieschang (correct me if I am wrong!) proved that the only genus-1 fibered knots in $S^3$ are the figure-eight and the trefoils. However, 1-surgery along each of these knots is distinguished from $S^3$ by the fundamental group. Therefore, if $K$ has a non-trivial surgery to $S^3$, then it is the unknot, as required. \(\square\)

Let us reflect on the argument a little bit. The input from Floer homology relies on the genus bound for knots with L-space surgeries and Ghiggini’s theorem on fibered knot detection. In fact, there is a Floer homology proof along these lines that predates Ghiggini’s result. The necessary facts are the existence of exact triangles in Floer homology and the Dehn surgery characterization of $S^1 \times S^2$ (Property R). We will take a closer look at the argument later in the course. The point I wish to stress now is that the Floer homology proof relies crucially on auxiliary input, namely the proof of Property R. As far as I am aware, there is no proof of Property R solely within the framework of Floer homology. The existing proofs involve heavy lifting, either in the form of sutured manifold hierarchies (Gabai), combinatorics of surface intersections (Gordon and Luecke), or contact and symplectic geometry (Conway and Tosun and their references – thanks to Lev Tovstopyat-Nelip for the reminder).

KMOSz also proved the following conjecture of Gordon in their paper:

**Theorem 0.2 (KMOSz; formerly Gordon’s conjecture).** If $K$ is a knot in $S^3$, $p/q$ is a non-trivial slope, and $K(p/q) \approx U(p/q)$, then $K \approx U$.

The proof of this result is no harder than the proof of the special case $p = 1$, which is the Dehn surgery characterization of $S^3$. It involves an inductive argument, using the exact triangle, building on the base case $p/q = 0/1$ (Property R). Thus, it shows how auxiliary input (Property R) can be bootstrapped using the formal framework of Floer homology (exact triangles) in order to prove results beyond the reach of previous methods.
The Floer homology framework is also very flexible for proving other Dehn surgery characterizations. For instance, using Ghiggini’s theorem, Ozsváth and Szabó proved that if $K_0$ is a genus-1 fibered knot (again, the figure-eight or one of the trefoils), $K$ is a knot in $S^3$, $p/q$ is a non-trivial slope, and $K(p/q) \approx K_0(p/q)$, then $K \simeq K_0$. In addition, it leads to the Dehn surgery characterization of L-space homology spheres (Ravelomanana) and the solution to the knot complement problem within. For instance, knots in the Poincaré homology sphere are determined by their complements.

Onto the pillars. To begin, let us review one of the problems from the first problem set. In that problem, we saw the following pair of faces in the intersection graph $G_P$: 

We were asked to make a deduction about $H_1(K(\beta); \mathbb{Z})$. Here $\beta$ is the boundary slope of the other implied planar surface $Q$. We proceed as with our earlier analysis of the presence of a Scharlemann cycle. We develop the following picture in $K(\beta)$:

We obtain a submanifold $M \subset K(\beta)$ composed of

- one 0-handle ($N(\hat{Q} \smallsetminus \text{a small disk})$)
- two 1-handles (the portions of $V_\beta$ running between disks $D_1, D_2$ and disks $D_3, D_4$)
• two 2-handles \((N(F_1) \text{ and } N(F_2))\)

What is the homology group \(H_1(M; \mathbb{Z})\)? The handle decomposition gives us a presentation. Each 1-handle gives a generator and each 2-handle a relation. Let \(x\) denote the generator coming from the first 1-handle, represented by a loop that travels from 1 to 2 in the 1-handle, and let \(y\) denote the generator coming from the second, oriented from 3 to 4. We obtain relations by examining how \(\partial F_1\) and \(\partial F_2\) run over the 1-handles. We see that \([\partial F_1] = 2x - y\) and \([\partial F_2] = x + 2y\). Said a little differently, the handle decomposition gives a cellular chain complex with ordered bases \(\{F_1, F_2\}\) for \(C_2\) and \(\{x, y\}\) for \(C_1\). In these bases, the boundary operator \(\partial : C_2 \to C_1\) is expressed by the matrix \[
\begin{pmatrix}
2 & -1 \\
1 & 2
\end{pmatrix}.
\]
Since there is a single 0-handle, the boundary operator \(C_1 \to C_0\) vanishes, and \(H_1(M; \mathbb{Z})\) is the cokernel of the matrix. The determinant of the matrix is 5, so its cokernel is the cyclic group of order 5: \(H_1(M; \mathbb{Z}) \approx \mathbb{Z}/5\mathbb{Z}\).

What does this tell us about \(H_1(K(\beta); \mathbb{Z})\)? Note that \(\partial F_1\) and \(\partial F_2\) are linearly independent in \(H_1(\partial M_1)\), where \(M_1\) denotes the 1-skeleton of \(M\), which is a genus-2 handlebody. It follows that \(\partial M \approx S^2\). Therefore, \(M\) is a connected summand of \(K(\beta)\), and it follows that \(\mathbb{Z}/5\mathbb{Z}\) is a subgroup of \(K(\beta)\) by Mayer-Vietoris. However, we did not even have to identify the homeomorphism type of \(\partial M\). It is a closed, orientable surface of some genus, so it has free abelian first homology group. This is enough to deduce from the Mayer-Vietoris sequence that any torsion in \(M\) persists in \(K(\beta)\). (This was the point of another exercise from the first problem set.)

This example, and the example of the Scharlemann cycle from the second lecture, reveals our game plan: given a pair of intersection graphs \(G_P\) and \(G_Q\), we seek to identify a collection of faces in \(G_P\) and use them to certify that \(H_1(K(\beta); \mathbb{Z})\) contains non-trivial torsion: this shows that 
\(K(\beta) \not\approx S^3!\)

Let us codify how to identify homological relations from the faces of an intersection graph. Let \(M_1 \subset K(\beta)\) denote the subspace \(N(\hat{Q}) \cup V_\beta\). (We could equally well discard a small disk \(\hat{Q}\) so as to produce a 0-handle as before, but this is not necessary to do, as it will not influence the homological calculations to follow.) We regard \(M_1\) as the result of attaching 1-handles to \(N(\hat{Q})\), one for each portion of \(V_\beta\). We can thereby label the 1-handles \(V_1, \ldots, V_q\), where \(V_i\) corresponds to the portion of \(V_\beta\) that travels from \(i\) to \(i+1\), indices \((\text{mod } q)\). Thus, 
\(H_1(M_1; \mathbb{Z}) \approx \mathbb{Z}^q = \bigoplus_{i=1}^q \mathbb{Z} \cdot V_i\).

Now let \(F\) denote a disk face of \(G_P\). Its boundary alternately decomposes into arcs of \(P \cap Q\) and corners at the fat vertices of \(P\). Each corner has a pair of edge endpoint labels \(i, i+1\). At such a corner, we record the term \(\pm V_i\), where the sign corresponds to the sign on the vertex of \(P\). The sum of these terms is an element \(r(F) \in \mathbb{Z}^q\). Observe that 
\(r(F) = [\partial F] \in H_1(M_1; \mathbb{Z})\). This is how we read off homological relations directly from the intersection graph. It generalizes the examples we handled above. Note that if \(F\) is not a disk face, so that it has multiple boundary components, then it still introduces a homological relation. However, we will not need to consider such faces.

A disk face \(F\) in \(G_P\) is homogeneous if all of the vertices at which it has an \((i, i+1)\) corner have the same sign, \(i = 1, \ldots, q\). Note that the sign can vary with \(i\), and that there may well exist some values \(i\) for which \(F\) does not have any \((i, i+1)\) corners. For instance, the disk
faces $F_1$ and $F_2$ above are both homogeneous, as is any Scharlemann cycle.

We are en route to our mechanism for identifying non-trivial torsion. Given a fixed orthonormal basis of $\mathbb{R}^q$ (such as $V_1, \ldots, V_q$ above), a **type** is a closed orthant of $\mathbb{R}^q$. Types are naturally labeled by sign vectors $\{\pm\}^q$:

$$
egin{array}{c|c}
++ & ++ \\
-+ & +-
\end{array}
$$

A vector $v \in \mathbb{R}^p$ *represents* a type if $v$ or $-v$ lies in the orthant type. For instance, 0 represents every type, as does any vector lying along a coordinate axis. As you can quickly check, these are precisely the vectors that represent every type. As another example, $(2, 0, -1)$ represents the types $(+, +, -), (+, -, -), (-, -, +), (-, +, +)$. In general, a non-zero vector with $k$ 0-entries represents exactly $2^{k+1}$ types. More generally, a subset of vectors $A \subset \mathbb{R}^q$ represents a type if some vector in $A$ represents that type. For instance, the familiar example $A = \{(2, -1), (1, 2)\}$ represents every type.

Type representation is our key to detecting torsion in homology. Here is the second pillar:

**Theorem 0.3** (Combinatorial optimization theorem). *Suppose that $A \subset \mathbb{Z}^q$ is a collection of vectors that represent all types, and none is a zero vector or a unit vector. Then there exists a subset $A_0 \subset A$ such that the abelian group $\mathbb{Z}^q/\langle A_0 \rangle$ contains non-trivial torsion.*

Theorem 0.3 is due to Walter Parry. In due course, will talk a little more about its history, why we have to pass to a subset $A_0$, and what it has to do with combinatorial optimization.

The third pillar is another result of Gordon and Luecke. It is the most profound of the three pillars:

**Theorem 0.4** (Graph theory theorem). *Suppose that $(G_P, G_Q)$ is an essential pair of intersection graphs. Then either $G_P$ contains a collection $C$ of homogeneous disk faces that represent all types or $G_Q$ contains a Scharlemann cycle.*

Note that the statement is asymmetric in $P$ and $Q$. You can mull over what must occur if one of the two possible outcomes does not occur and we swap the roles of the graphs.

Lastly, recall the first pillar, also due to Gordon and Luecke, and independently to Gabai, stated somewhat differently in the last lecture:

**Theorem 0.5** (Thin position theorem). *Suppose that $K$ is a non-trivial knot in $S^3$ and $X_K$ contains a pair of properly embedded planar surfaces with distinct boundary slopes. Then $X_K$ contains such a pair $P$ and $Q$ with the property that no arc of $P \cap Q$ is boundary-parallel in either $P$ or $Q$.*

In the next lecture, we will explain how the three pillars lead to the original proof of the Dehn surgery characterization of $S^3$. See if you can do so yourself before then – the strategy should be apparent.